Proceedings of CONTROLO'98
3rd Portugese Conference on Automatic Control
Coimbra, Portugal, 9–11 September, 1998

STABILIZATION AND TRACKING OF A NONHOLONOMIC MOBILE ROBOT WITH SATURATING ACTUATORS

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Abstract: The stabilization and tracking problem for the kinematic model and simplified dynamic model of a wheeled mobile robot with input saturations is considered. The results form a direct extension of the case where no actuator constraints are involved. Simulations illustrate the proposed control strategy.

Keywords: Nonholonomic mobile robot; Stabilization; Tracking; Backstepping; Bounded feedback control.

1. INTRODUCTION

In recent years a lot of interest has been devoted to the stabilization and tracking of a wheeled mobile robot, see e.g. (Bloch and Drakunov, 1996; Canudas de Wit *et al.*, 1996; Escobar *et al.*, 1998; Fliess *et al.*, 1995; Jiang and Nijmeijer, 1997; Kanayama *et al.*, 1990; Kolmanovsky and McClamroch, 1995; Samson and Ait-Abderrahim, 1991; Walsh *et al.*, 1994). One of the reasons for this is, undoubtedly, that no smooth time-invariant stabilizing controller for this system exists, which is a corollary from the fact that Brockett's necessary condition for smooth stabilization is not met, see (Brockett, 1983). Many of the above references, as well as (Coron, 1992; Escobar *et al.*, 1998; Jiang, 1996; Jiang and Pomet, 1996; Lin, 1996; Pomet, 1992) therefore aim at developing suitable time varying stabilizing (tracking) controllers for mobile robots or more general chained form nonholonomic systems.

In the present note we want to study the stabilization and tracking problem for a wheeled mobile robot under saturation constraints on the inputs. At this point we exploit the normalization technique known from adaptive control, see e.g. (Jiang and Praly, 1992; Krstić *et al.*, 1995), for solving the stabilization and tracking problem under saturating velocities. Also, for a simplified dynamic model of the mobile robot (cf. (Jiang and Nijmeijer, 1997)) the same technique turns out to be of great value. The proposed controller design is inspired by previous work of Pomet, (Pomet, 1992) (see also (Lin, 1996)) where for general driftless systems time-varying stabilizing controllers are developed. Our stabilizing and tracking controllers for the kinematic model of the robot globally fulfill the given input constraints, and for the dynamic extended model semi-globally fulfill the given input constraints, thus given the initial condition to belong to some compact set, appropriate parameter tuning for the bounded controller is possible.

A different approach using cascaded ideas for the same problems can be found in (Panteley *et al.*, 1998).

The paper is organized as follows. In section 2 the bounded state feedback stabilization problem for the wheeled mobile robot is addressed, while in section 3 the bounded state feedback tracking problem is investigated. Section 4 contains the conclusions.

2. STABILIZATION VIA BOUNDED STATE FEEDBACK

The purpose of this section is to show that it is not difficult to extend Pomet's method (Pomet, 1992) to the kinematic model of a wheeled mobile robot under saturation constraints on the control inputs. Then, we employ the integrator backstepping idea to establish a similar result for a simplified dynamic model of the robot.

2.1 *Kinematic model*

The benchmark wheeled mobile robot considered by many researchers (see, e.g., (Kolmanovsky and McClamroch, 1995; Canudas de Wit *et al.*, 1996) and references therein) is described by the following kinematic model:

$$\begin{aligned} \dot{x}_c &= v\cos\theta \\ \dot{y}_c &= v\sin\theta \\ \dot{\theta} &= \omega \end{aligned} \tag{1}$$

where v is the forward velocity, ω is the steering velocity. (x_c, y_c) is the position of the mass center of the robot moving in the plane and θ denotes its heading angle from the horizontal axis. Here, the velocities v and ω are subject to the following constraints:

$$|\omega| \le \omega_{max}, \quad |v| \le v_{max} \tag{2}$$

where ω_{max} and v_{max} are arbitrary positive constants.
The stabilization problem to be addressed, is to construct a time-varying state-feedback law of the form

$$\omega = \alpha_1(t,\theta,x_c,y_c), \quad v = \alpha_2(t,\theta,x_c,y_c) \tag{3}$$

in such a way that (2) holds and the zero solution of the robot system (1) in closed-loop with (3) is globally uniformly asymptotically stable (GUAS).
We follow (Pomet, 1992) to achieve our control objective. First, define a set BF_r of continuous and bounded functions indexed by a parameter $r > 0$, i.e.

$$BF_r = \{\phi : \mathbb{R} \to \mathbb{R} \mid \phi \text{ is continuous and } -r \le \phi(x) \le r \quad \forall x \in \mathbb{R}\} \tag{4}$$

and a corresponding set of saturation functions S_r, i.e

$$S_r = \{\phi : \mathbb{R} \to \mathbb{R} \in BF_r \mid s\phi(s) > 0 \text{ for all } s \neq 0\} \tag{5}$$

Examples of nontrivial functions in S_r include for instance

$$\phi(x) = \tfrac{2rx}{1+x^2}, \quad \phi(x) = \tfrac{rx^2}{1+x^2}, \quad \phi(x) = \tfrac{2r}{\pi}\arctan(x) \tag{6}$$

Denote

$$x = (\theta, x_c, y_c)^T \tag{7}$$

Introduce a Lyapunov function candidate

$$V_1(t,x) = \tfrac{1}{2}\left(\theta + \varepsilon_1 g_1(x_c^2+y_c^2)\cos t\right)^2 + \tfrac{1}{2}x_c^2 + \tfrac{1}{2}y_c^2 \tag{8}$$

for (1) in closed-loop with (3) which is written in more compact form

$$\dot{x} = f_1(x)\omega + f_2(x)v \tag{9}$$

In (8) $\varepsilon_1 > 0$ is a design parameter to be chosen later and g_1 is a smooth (i.e., of class C^∞) function in BF_1 with the property that $g_1(s) = 0$ if and only if $s = 0$.
It is direct to verify that the conditions of (Pomet, 1992, Theorem 2) hold for such choice of Lyapunov function V_1 in (8). Using the controller design scheme proposed in (Pomet, 1992), we obtain the time-varying state feedback laws

$$\begin{aligned} \omega &= \varepsilon_1 g_1(x_c^2+y_c^2)\sin t - \\ &\quad -h_{\varepsilon_2}\left(\theta + \varepsilon_1 g_1(x_c^2+y_c^2)\cos t\right) \qquad (10) \\ &:= \alpha_1(t,\theta,x_c,y_c) \\ v &= -h_{\varepsilon_3}\Big([x_c\cos\theta + y_c\sin\theta] \times \\ &\quad \times [1 + 2\varepsilon_1(\theta + \varepsilon_1 g_1 \cos t) g_1' \cos t]\Big) \qquad (11) \\ &:= \alpha_2(t,\theta,x_c,y_c) \end{aligned}$$

where ε_2 and ε_3 are two positive design parameters, $h_{\varepsilon_2} \in S_{\varepsilon_2}$, $h_{\varepsilon_3} \in S_{\varepsilon_3}$ and $g_1' := \frac{dg_1}{ds}(x_c^2+y_c^2)$.
We establish the following result.

Proposition 1. The equilibrium $x = 0$ of the closed-loop system (1), (10) and (11) is globally uniformly asymptotically stable (GUAS) for any positive ε_1, ε_2 and ε_3. In particular, given any saturation levels $\omega_{max} > 0$, $v_{max} > 0$ as in (2), we can always tune ε_1, ε_2 and ε_3 so that (2) holds while $x = 0$ is GUAS.

PROOF. Noticing that

$$\alpha_1(t,\theta,x_c,y_c) = \varepsilon_1 g_1(x_c^2+y_c^2)\sin t - h_{\varepsilon_2}\Big(L_{f_1}V_1(t,x)\Big),$$

$$\alpha_2(t,\theta,x_c,y_c) = -h_{\varepsilon_3}\Big(L_{f_2}V_1(t,x)\Big),$$

the time derivative of V_1 as defined in (8) satisfies:

$$\begin{aligned} \dot{V}_1(t,x) = &-\Big(L_{f_1}V_1(t,x)\Big)h_{\varepsilon_2}\Big(L_{f_1}V_1(t,x)\Big) - \\ &-\Big(L_{f_2}V_1(t,x)\Big)h_{\varepsilon_3}\Big(L_{f_2}V_1(t,x)\Big) \end{aligned} \tag{12}$$

The proof is completed along the same lines of (Pomet, 1992, Proof of Theorem 1) using LaSalle's invariance principle. We can meet (2) choosing $\varepsilon_1 + \varepsilon_2 \le \omega_{max}$ and $\varepsilon_3 \le v_{max}$. □

2.2 *Dynamic model*

In the preceding subsection we have solved the stabilization problem for the kinematic model (1) of the benchmark wheeled robot with saturating velocities. In this subsection, we demonstrate that the same control task can be achieved for a simplified dynamic model of the robot with saturation on the control torques.

More precisely, we consider the following dynamic extension of the robot (1), see also (Jiang and Nijmeijer, 1997):

$$\begin{aligned} \dot{x}_c &= v\cos\theta \\ \dot{y}_c &= v\sin\theta \\ \dot{\theta} &= \omega \\ \dot{\omega} &= u_1 \\ \dot{v} &= u_2 \end{aligned} \tag{13}$$

where u_1 and u_2 are generalized torque-inputs subject to the constraints:

$$|u_1| \leq u_{1,max}\,, \quad |u_2| \leq u_{2,max} \tag{14}$$

with $u_{1,max} > 0$ and $u_{2,max} > 0$ arbitrary positive constants.

Introduce two new variables $\overline{\omega}$ and $\overline{v}$ as

$$\overline{\omega} = \omega - \alpha_1(t,\theta,x_c,y_c), \quad \overline{v} = v - \alpha_2(t,\theta,x_c,y_c) \tag{15}$$

with $\alpha_1(t,\theta,x_c,y_c)$ and $\alpha_2(t,\theta,x_c,y_c)$ as defined in (10) and (11).

Consider the positive definite proper Lyapunov function candidate for system (13)

$$V_2(t,X) = \varepsilon_4\log(1+V_1(t,\theta,x_c,y_c)) + \tfrac{1}{2}\overline{\omega}^2 + \tfrac{1}{2}\overline{v}^2 \tag{16}$$

where $X := (x^T,\omega,v)^T = (x_c,y_c,\theta,\omega,v)^T$ and $\varepsilon_4 > 0$ is a design parameter to be chosen later.

In view of (12) and (15), differentiating V_2 along the solutions of system (13) yields

$$\begin{aligned} \dot{V}_2(t,X) = &-\Big[\Big(L_{f_1}V_1(t,x)\Big)h_{\varepsilon_2}\Big(L_{f_1}V_1(t,x)\Big) + \\ &+\Big(L_{f_2}V_1(t,x)\Big)h_{\varepsilon_3}\Big(L_{f_2}V_1(t,x)\Big)\Big] + \frac{\varepsilon_4}{1+V_1(t,x)} \\ &+ \frac{\varepsilon_4(\theta+\varepsilon_1 g_1\cos t)}{1+V_1(t,x)}\overline{\omega} + \\ &+ \frac{\varepsilon_4(x_c\cos\theta+y_c\sin\theta)(1+2\varepsilon_1(\theta+\varepsilon_1 g_1\cos t)g_1'\cos t)}{1+V_1(t,x)}\overline{v} + \\ &+ \overline{\omega}(u_1-\dot{\alpha}_1) + \overline{v}(u_2-\dot{\alpha}_2) \end{aligned} \tag{17}$$

where

$$\dot{\alpha}_1 = \frac{\partial\alpha_1}{\partial t} + \frac{\partial\alpha_1}{\partial\theta}\omega + \left(\frac{\partial\alpha_1}{\partial x_c}\cos\theta + \frac{\partial\alpha_1}{\partial y_c}\sin\theta\right)v$$

$$\dot{\alpha}_2 = \frac{\partial\alpha_2}{\partial t} + \frac{\partial\alpha_2}{\partial\theta}\omega + \left(\frac{\partial\alpha_2}{\partial x_c}\cos\theta + \frac{\partial\alpha_2}{\partial y_c}\sin\theta\right)v$$

Therefore, we choose the time-varying control laws as

$$u_1 = -h_{\varepsilon_5}(\overline{\omega}) + \dot{\alpha}_1 - \frac{\varepsilon_4(\theta+\varepsilon_1 g_1\cos t)}{1+V_1(t,x)} \tag{18}$$

$$\begin{aligned} u_2 = &-h_{\varepsilon_6}(\overline{v}) + \dot{\alpha}_2 - \\ &- \frac{\varepsilon_4(x_c\cos\theta+y_c\sin\theta)(1+2\varepsilon_1(\theta+\varepsilon_1 g_1\cos t)g_1'\cos t)}{1+V_1(t,x)} \end{aligned} \tag{19}$$

where $\varepsilon_6 > 0$ and $\varepsilon_6 > 0$ are design parameters and $h_{\varepsilon_5} \in S_{\varepsilon_5}$, $h_{\varepsilon_6} \in S_{\varepsilon_6}$.

We are now ready to state the result.

Proposition 2. The equilibrium $X = 0$ of the closed-loop system (13), (18) and (19) is GUAS for any

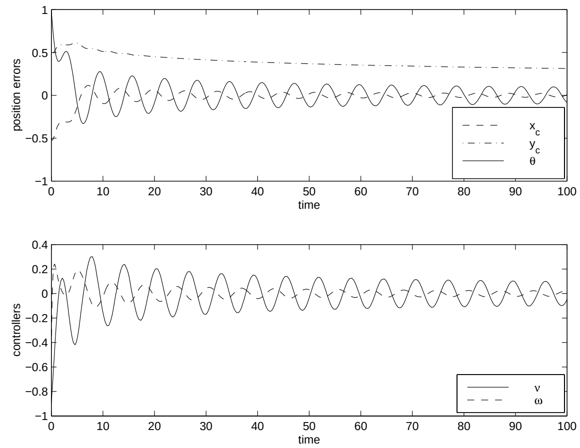

Fig. 1. Stabilization of the kinematic model with initial conditions $[x_c(0), y_c(0), \theta(0)]^T = [-0.5, 0.5, 1]^T$.

positive values of ε_i, $1 \leq i \leq 5$. In particular, given any saturation levels $u_{1,max} > 0$, $u_{2,max} > 0$ as in (14) and any compact set Ω_1 in $\mathbb{R}^5$, we can always tune our design constants ε_i $(1 \leq i \leq 6)$ so that (14) holds for all trajectories starting in Ω_1.

PROOF. Under the choice of (18) and (19) for the torques inputs, it holds

$$\begin{aligned} \dot{V}_2(t,X) = &-\Big[\Big(L_{f_1}V_1(t,x)\Big)h_{\varepsilon_2}\Big(L_{f_1}V_1(t,x)\Big) + \\ &+\Big(L_{f_2}V_1(t,x)\Big)h_{\varepsilon_3}\Big(L_{f_2}V_1(t,x)\Big)\Big]\frac{\varepsilon_4}{1+V_1(t,x)} \\ &-\overline{\omega}h_{\varepsilon_5}(\overline{\omega}) - \overline{v}h_{\varepsilon_6}(\overline{v}) \end{aligned} \tag{20}$$

The first part of Proposition 2 readily follows from LaSalle's invariance principle as in the proof of Proposition 1.

The second statement is more or less direct from the expressions (18) and (19) of the control laws u_1 and u_2. □

2.3 *Simulations*

To support our results, we simulated with MATLAB™ the wheeled mobile robot (1) in closed-loop with the controller (10, 11) with $\varepsilon_1 = 1$ and $g_1(s) = h_{\varepsilon_2}(s) = h_{\varepsilon_3} = \tanh(s)$, which guarantees that $|\omega(t)| \leq 2$ and $|v(t)| \leq 1$ for all $t \geq 0$. The resulting performance is depicted in Figure 1.

From the initial condition $[x_c(0), y_c(0), \theta(0)]^T = [-0.5, 0.5, 1]^T$ we see a very slow convergence to the origin, which is a well known consequence from using Pomet's method (cf. (M'Closkey and Murray, 1997)).

If we then consider the simple dynamic extension (13) in closed-loop with the controller (18, 19) where we additionally use $\varepsilon_4 = 1$, and $h_{\varepsilon_5}(s) = h_{\varepsilon_6} = \tanh(s)$ the resulting performance if we start from the initial condition $[x_c(0), y_c(0), \theta(0), \omega(0), v(0)]^T = [-0.5, 0.5, 1, 0, 0]^T$ is depicted in Figure 2.

Again we see a very slow convergence to the origin.

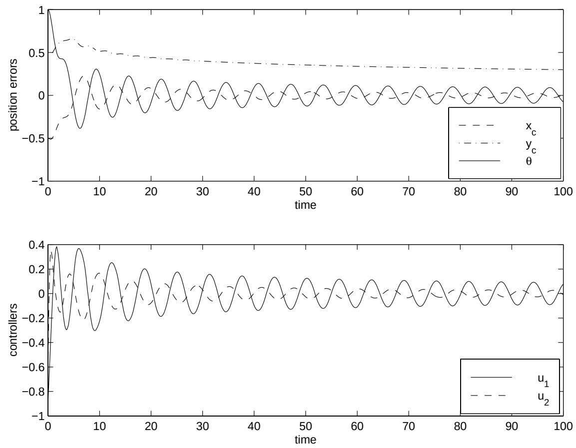

Fig. 2. Stabilization of the dynamic model with initial conditions $[x_c(0), y_c(0), \theta(0), \omega(0), \nu(0)]^T = [-0.5, 0.5, 1, 0, 0]^T$.

3. TRACKING VIA BOUNDED STATE FEEDBACK

3.1 *Kinematic model*

In this section, we address the tracking problem for the robot (1) under a constraint on the velocities. To quantify the saturation level, it is assumed that the reference trajectory (x_r, y_r, θ_r) satisfies

$$\begin{aligned} \dot{x}_r &= \nu_r \cos\theta_r \\ \dot{y}_r &= \nu_r \sin\theta_r \\ \dot{\theta}_r &= \omega_r \end{aligned} \tag{21}$$

where ω_r and ν_r are bounded reference velocities.
The objective is to find time-varying state-feedback controllers of the form

$$\omega = \omega^*(t, \theta, x_c, y_c) , \quad \nu = \nu^*(t, \theta, x_c, y_c) \tag{22}$$

such that $x_c(t) - x_r(t)$, $y_c(t) - y_r(t)$ and $\theta(t) - \theta_r(t)$ tend to zero as $t \to +\infty$ while guaranteeing the following property:

$$|\omega(t)| \leq \omega_{max} , \quad |\nu(t)| \leq \nu_{max} \quad \text{for all } t \geq 0 \tag{23}$$

where $\omega_{max} > \sup_{t\geq 0} |\omega_r(t)|$ and $\nu_{max} > \sup_{t\geq 0} |\nu_r(t)|$ are arbitrary.
As in (Jiang and Nijmeijer, 1997) (see also (Kanayama *et al.*, 1990)), consider the following tracking errors

$$\begin{bmatrix} x_e \\ y_e \\ \theta_e \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_r - x_c \\ y_r - y_c \\ \theta_r - \theta \end{bmatrix} \tag{24}$$

Obviously, for any value of θ, $(x_e, y_e, \theta_e) = 0$ if and only if $(x_c, y_c, \theta) = (x_r, y_r, \theta_r)$.
It can be directly checked, the tracking error dynamics of the robot satisfy

$$\begin{aligned} \dot{x}_e &= \omega y_e - \nu + \nu_r \cos\theta_e \\ \dot{y}_e &= -\omega x_e + \nu_r \sin\theta_e \\ \dot{\theta}_e &= \omega_r - \omega . \end{aligned} \tag{25}$$

We show next that the following control laws solve our tracking problem:

$$\begin{aligned} \omega &= \omega_r + \frac{\lambda_1 \nu_r y_e}{1 + x_e^2 + y_e^2} \int_0^1 \cos(s\theta_e(t))ds + h_{\lambda_2}(\theta_e) \\ &:= \beta_1(t, \theta_e, x_e, y_e) \end{aligned} \tag{26}$$

$$\nu = \nu_r \cos\theta_e + h(\lambda_3)(x_e) := \beta_2(t, \theta_e, x_e) \tag{27}$$

where λ_1, λ_2, λ_3 are positive design parameters and $h_{\lambda_1} \in S_{\lambda_1}$, $h_{\lambda_2} \in S_{\lambda_2}$.

Proposition 3. Assume that ω_r and ν_r are bounded and uniformly continuous over $[0, \infty)$. If either $\omega_r(t)$ or $\nu_r(t)$ does not converge to zero, then the zero equilibrium of the closed-loop system (25), (26) and (27) is globally asymptotically stable. In particular, given any $\omega_{max} > \sup_{t\geq 0} |\omega_r(t)|$ and $\nu_{max} > \sup_{t\geq 0} |\nu_r(t)|$, we can always tune our design parameters λ_1, λ_2 and λ_3 so that the condition (23) is met.

PROOF. Consider the positive definite and proper Lyapunov function candidate

$$W_1(x_e, y_e, \theta_e) = \frac{\lambda_1}{2} \log(1 + x_e^2 + y_e^2) + \frac{1}{2}\theta_e^2 \tag{28}$$

Differentiating W_1 along the solutions of the closed-loop system (25), (26) and (27) yields:

$$\dot{W}_1(x_e, y_e, \theta_e) = -\frac{\lambda_1 x_e h_{\lambda_3}(x_e)}{1 + x_e^2 + y_e^2} - \theta_e h_{\lambda_2}(\theta_e) \leq 0 \tag{29}$$

Therefore, the trajectories $(x_e(t), y_e(t), \theta_e(t))$ are uniformly bounded on $[0, \infty)$. It follows, as in (Jiang and Nijmeijer, 1997), by direct application of Barbălat's lemma (Khalil, 1996) that

$$\lim_{t\to\infty} [x_e(t) h_{\lambda_3}(x_e(t)) + \theta_e(t) h_{\lambda_2}(\theta_e(t))] = 0 \tag{30}$$

which, in turn, gives

$$\lim_{t\to\infty} (|x_e(t)| + |\theta_e(t)|) = 0 \tag{31}$$

It remains to prove that $y_e(t)$ goes to zero as $t \to \infty$. Indeed, this fact can be established by mimicking the arguments used in the proof of (Jiang and Nijmeijer, 1997, Proposition 2).
The last statement of Proposition 3 is more or less direct. □

3.2 *Dynamic model*

We extend the tracking result from subsection 3.1 to the simplified dynamic model (13) of the robot. The tracking error dynamics are described by

$$\begin{aligned} \dot{x}_e &= \omega y_e - \nu + \nu_r \cos\theta_e \\ \dot{y}_e &= -\omega x_e + \nu_r \sin\theta_e \\ \dot{\theta}_e &= \omega_r - \omega \\ \dot{\omega} &= u_1 \\ \dot{\nu} &= u_2 \end{aligned} \tag{32}$$

where u_1 and u_2 are torque-inputs subject to the contraint:

$$|u_1| \le u_{1,max}, \quad |u_2| \le u_{2,max} \tag{33}$$

where $u_{1,max}$ and $u_{2,max}$ are two arbitrary saturation levels satisfying the property

$$u_{1,max} > \sup_{t\ge 0} |\dot{\omega}_r(t)|, \quad u_{2,max} > \sup_{t\ge 0} |\dot{\nu}_r(t)|. \tag{34}$$

Contrary to the kinematic model (25) considered in the subsection 3.1, ω and ν are not the actual control inputs to the dynamic model (32) of the robot. Consequently, the tracking control laws obtained in (26) and (27) cannot be implemented in the present situation. To invoke integrator backstepping (see (Krstić *et al.*, 1995)) for the purpose of designing our true tracking controllers subject to (33), we introduce two new variables

$$\omega_e = \omega - \beta_1(t, \theta_e, x_e, y_e), \quad \nu_e = \nu - \beta_2(t, \theta_e, x_e) \tag{35}$$

where β_1 and β_2 are defined as in (26) and (27), respectively.

Consider the positive definite and proper Lyapunov function candidate for system (32)

$$W_2(t, X_e) = \lambda_4 \log(1 + W_1(t, x_e, y_e, \theta_e)) + \frac{1}{2}\omega_e^2 + \frac{\lambda_5}{2}\log(1+\nu_e^2) \tag{36}$$

where $X_e := (x_e, y_e, \theta_e, \omega_e, \nu_e)$ and λ_4, $\lambda_5 > 0$ are two design parameters to be chosen later.

Using (29), the time derivative of W_2 along the solutions of (32) satisfies

$$\begin{aligned}\dot{W}_2(t, X_e) = &-\left(\frac{\lambda_1 x_e h_{\lambda_3}(x_e)}{1 + x_e^2 + y_e^2} + \theta_e h_{\lambda_2}(\theta_e)\right)\frac{\lambda_4}{1+W_1} \\ &+\left(\frac{-\lambda_1 x_e}{1 + x_e^2 + y_e^2}\nu_e - \theta_e\omega_e\right)\frac{\lambda_4}{1+W_1} + \\ &+\omega_e(u_1 - \dot{\beta}_1) + \frac{\lambda_5 \nu_e}{1+\nu_e^2}(u_2 - \dot{\beta}_2)\end{aligned} \tag{37}$$

where

$$\begin{aligned}\dot{\beta}_1 &= \frac{\partial\beta_1}{\partial t} + \frac{\partial\beta_1}{\partial x_e}(\omega y_e - \nu + \nu_r\cos\theta_e) + \\ &\quad + \frac{\partial\beta_1}{\partial y_e}(-\omega x_e + \nu_r \sin\theta_e) + \frac{\partial\beta_1}{\partial\theta_e}(\omega_r - \omega) \\ \dot{\beta}_2 &= \frac{\partial\beta_2}{\partial t} + \frac{\partial\beta_2}{\partial x_e}(\omega y_e - \nu + \nu_r\cos\theta_e) + \frac{\partial\beta_2}{\partial\theta_e}(\omega_r - \omega) \\ &= \nu_r\omega_e\sin\theta_e + \dot{\nu}_r\cos\theta_e + \\ &\quad + \left(\tfrac{\lambda_1\nu_r y_e}{1+x_e^2+y_e^2}\int_0^1 \cos(s\theta_e(t))ds + h_{\lambda_2}(\theta_e)\right)\nu_r\sin\theta_e + \\ &\quad + h'_{\lambda_3}(x_e)\,(\omega y_e - \nu + \nu_r\cos\theta_e)\end{aligned}$$

Let $\lambda_6 > 0$, $\lambda_7 > 0$ be design parameters. By making the following choice of tracking control laws for the torques u_1 and u_2

$$u_1 = -h_{\lambda_6}(\omega_e) + \dot{\beta}_1 + \frac{\lambda_4\theta_e}{1+W_1} + \frac{\lambda_5\nu_e}{1+\nu_e^2}\nu_r\sin\theta_e \tag{38}$$

$$\begin{aligned}u_2 = &-h_{\lambda_7}(\nu_e) + \tfrac{\lambda_1\lambda_4 x_e(1+\nu_e^2)}{\lambda_5(1+W_1)(1+x_e^2+y_e^2)} + \dot{\nu}_r\cos\theta_e + \\ &+ h'_{\lambda_3}(x_e)\,(\omega y_e - \nu + \nu_r\cos\theta_e) \\ &+ \left[\tfrac{\lambda_1\nu_r y_e}{1+x_e^2+y_e^2}\int_0^1\cos(s\theta_e(t))ds + h_{\lambda_2}(\theta_e)\right]\nu_r\sin\theta_e\end{aligned} \tag{39}$$

with $h_{\lambda_6} \in S_{\lambda_6}$, $h_{\lambda_7} \in S_{\lambda_7}$, it follows from (37) that

$$\begin{aligned}\dot{W}_2(t, X_e) = &-\left(\frac{\lambda_1 x_e h_{\lambda_3}(x_e)}{1+x_e^2+y_e^2} + \theta_e h_{\lambda_2}(\theta_e)\right)\frac{\lambda_4}{1+W_1} - \\ &-\omega_e h_{\lambda_6}(\omega_e) - \nu_e h_{\lambda_7}(\nu_e)\end{aligned} \tag{40}$$

We are now in a position to state our tracking result for the dynamic model (32).

Proposition 4. Assume that ω_r, $\dot{\omega}_r$, ν_r and $\dot{\nu}_r$ are bounded over $[0, \infty)$. If either $\omega_r(t)$ or $\nu_r(t)$ does not converge to zero, then the zero equilibrium $X_e = 0$ of the closed-loop system (32), (38) and (39) is globally asymptotically stable. In particular, given any $u_{1,max} > \sup_{t\ge 0}|\dot{\omega}_r(t)|$ and $u_{2,max} > \sup_{t\ge 0}|\dot{\nu}_r(t)|$ and any compact set Ω_2 in $\mathbb{R}^5$, we can always tune our design parameters λ_1 to λ_7 so that the condition (33) is also met for all trajectories starting from Ω_2.

PROOF. As in the proof of Proposition 3, the first part of Proposition 4 follows from (40) together with a straightforward application of Barbălat's lemma (Khalil, 1996).

The second part of Proposition 4 is more or less direct from the expressions of the time-varying feedbacks (38) and (39). □

3.3 *Simulations*

To support our results, we simulated the closed-loop system (25, 26, 27). The desired trajectory has been given to be $\omega_r(t) = 1$, $\nu_r(t) = 1$, i.e. a circle. Using $\lambda_1 = 1$ and $h_{\lambda_2}(s) = h_{\lambda_3} = \tanh(s)$, which guarantees us that $|\omega(t)| \le 3$ and $|\nu(t)| \le 2$ for all $t \ge 0$, we obtained starting from the initial condition $[x_e(0), y_e(0), \theta_e(0)]^T = [-0.5, 0.5, 1]^T$ the performance as depicted in Figure 3.

We see that the control inputs obviously remain within their bounds and yield a quick convergence to the desired trajectory.

Next, we simulated the closed-loop system (32, 38, 39) where $\lambda_4 = \lambda_5 = 1$ and $h_{\lambda_6}(s) = h_{\lambda_7} = \tanh(s)$, where we want to track the same desired trajectory again. The resulting performance if we start from the initial condition $[x_e(0), y_e(0), \theta_e(0), \omega_e(0), \nu_e(0)]^T = [-0.5, 0.5, 1, 1, 1]^T$ is depicted in Figure 4.

We see an even quicker convergence of the tracking errors than in the previous case for the kinematic model.

4. CONCLUSIONS

(Semi-)global solutions for the stabilization and tracking problem for the kinematic and simplified dynamic

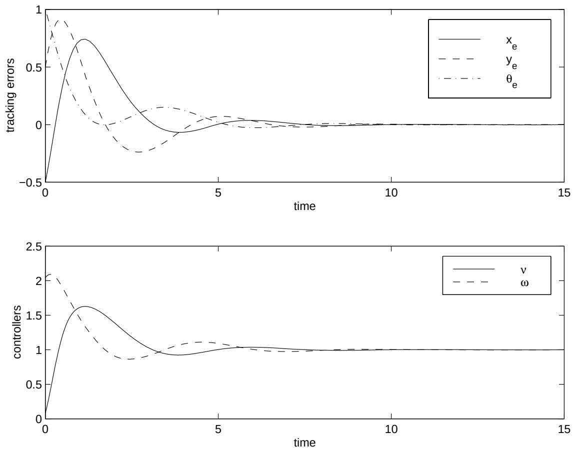

Fig. 3. Tracking of the kinematic model with initial errors $[x_e(0), y_e(0), \theta_e(0)]^T = [-0.5, 0.5, 1]^T$.

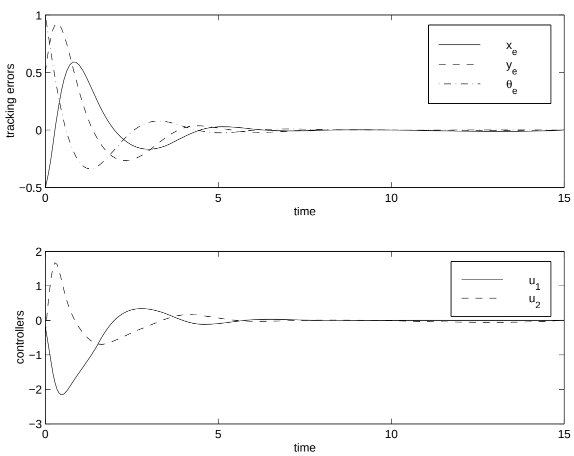

Fig. 4. Tracking of the dynamic model with initial errors $[x_e(0), y_e(0), \theta_e(0), \omega_e(0), v_e(0)]^T = [-0.5, 0.5, 1, 1, 1]^T$.

model of a wheeled mobile robot with input saturations are derived. On the basis of these results it becomes plausible that the same problems admit similar solutions if a complete dynamic model for the mobile robot is considered. Further research in this direction is however, still needed.

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