# Investigation of nonlinear dynamic effects in production systems

$2^{nd}$ international symposium
25.–26. February 1999, ADITEC Aachen

## Saturated stabilization and tracking of a nonholonomic mobile robot

# Saturated Stabilization and Tracking of a Nonholonomic Mobile Robot


Zhong-Ping Jiang
Dept of Electrical Engineering
College of Engineering
University of California
Riverside, CA 92521
USA
Emails: zjiang@ee.ucr.edu

Erjen Lefeber and Henk Nijmeijer*
Faculty of Mathematical Sciences
Dept of Systems, Signals and Control
University of Twente
P.O. Box 217, 7500 AE Enschede
The Netherlands
{A.A.J.Lefeber,H.Nijmeijer}@math.utwente.nl


Submitted to *Systems & Control Letters*


**Abstract**

The stabilization and tracking problem for the kinematic model and simplified dynamic model of a wheeled mobile robot with input saturations are considered. A model-based control design strategy is developed via a simple application of backstepping and normalization. While the problems with input saturations are globally solved for the kinematic model using Lipschitz continuous time-varying feedback laws, the same tasks are semiglobally fulfilled for the simplified dynamic model of the nonholonomic mobile robot. Simulations illustrate the effectiveness of the proposed controllers.


**Keyword:** Nonholonomic mobile robot; Stabilization; Tracking; Backstepping; Input saturation.

## 1 Introduction

In recent years a lot of interest has been devoted to the stabilization and tracking of nonholonomic dynamical systems. A wheeled mobile robot under nonholonomic constraints, or its feedback equivalent chained form system, has served as a benchmark mechanical example in several papers – see, e.g., [1, 3, 5, 6, 8, 12, 14, 18, 20, 21]. In addition to practical motivations, one of the technical reasons for this is, undoubtedly, that no smooth time-invariant stabilizing controller for this system exists, which is a corollary from the fact that Brockett's necessary condition for smooth stabilization [2] is not met. Many of the above references, as well as [4, 5, 7, 10, 16, 19] therefore aim at developing suitable time varying stabilizing (tracking) controllers for mobile robots or more general chained form nonholonomic systems.

In the present note, we want to study the stabilization and tracking problem for a wheeled mobile robot under saturation constraints on the inputs. The stabilization and tracking of nonholonomic systems with input saturations have been addressed rarely in the literature. In [16], the stabilization problem using bounded time-varying state-feedback was dealt with for a class of driftless controllable systems. The results of [16] are a direct application of ideas from passivity theory together with Pomet's method [19]. Unfortunately, there is still no result in the literature on the bounded

*Corresponding author. This author is also with the Faculty of Mechanical Engineering, Eindhoven University of Technology, PO Box 513, 5600 MB Eindhoven, The Netherlands.

state-feedback stabilization of nonholonomic systems with drift. In this paper, we first show that Pomet's method can be directly extended to the saturated feedback stabilization of the mobile robot in kinematic model. Then, we extend the result to the simplified dynamic model which is a nonholonomic system with drift. We do it via backstepping and normalization techniques known from adaptive control, see e.g. [11, 15]. Similar to the bounded feedback stabilization, the normalization technique is employed to treat the global tracking of the nonholonomic mobile robot under saturation constraints on the velocities. This complements the results in [1, 5, 8] where no saturation constraint was imposed on the inputs. We then use integrator backstepping to extend our result to establish a saturated tracking result for the nonholonomic mobile robot in simplified dynamic model.

When dealing with dynamic models and using Lipschitz continuous time-varying state-feedback, the input saturation requirement is achieved only semiglobally, that is, for the solutions starting from a given (but arbitrary) compact set. To the best of our knowledge, it seems that integrator backstepping under input saturation is missing for both linear and nonlinear systems in the literature.

It should be mentioned that our work follows a different approach from those of recent papers [1, 5] in that we base the control design on the physical model rather than its abstract, though mathematically feedback equivalent, three-dimensional chained form system or double integrator.

The paper is organized as follows. In Section 2, the bounded state feedback stabilization problem for the wheeled mobile robot is addressed, while in Section 3 the bounded state feedback tracking problem is investigated. Section 4 contains the conclusions.

# 2 Stabilization via bounded state feedback

The purpose of this section is to show that it is not difficult to extend Pomet's method [19] to the kinematic model of a wheeled mobile robot under saturation constraints on the control inputs. Then, we employ the integrator backstepping idea to establish a similar result for a simplified dynamic model of the robot.

## 2.1 Kinematic model

The benchmark wheeled mobile robot considered by many researchers (see, e.g., [14, 3] and references therein) is described by the following kinematic model:

$$\begin{aligned} \dot{x}_c &= \nu \cos\theta \\ \dot{y}_c &= \nu \sin\theta \\ \dot{\theta} &= \omega \end{aligned} \tag{1}$$

where $\nu$ is the forward velocity, $\omega$ is the steering velocity, $(x_c, y_c)$ is the position of the mass center of the robot moving in the plane and $\theta$ denotes its heading angle from the horizontal axis. Here, the velocities $\nu$ and $\omega$ are subject to the following constraints:

$$|\omega(t)| \;\leq\; \omega_{\max}\;, \quad |\nu(t)| \;\leq\; \nu_{\max} \quad \forall\, t \geq 0 \tag{2}$$

where $\omega_{\max}$ and $\nu_{\max}$ are arbitrary positive constants.

The stabilization problem to be addressed, is to construct a time-varying state-feedback law of the form

$$\omega \;=\; \alpha_1(t, \theta, x_c, y_c)\;, \quad \nu \;=\; \alpha_2(t, \theta, x_c, y_c) \tag{3}$$

in such a way that (2) holds and the zero solution of the robot system (1) in closed-loop with (3) is globally uniformly asymptotically stable (GUAS).

We follow [19] to achieve this control objective. First, define a set $\mathcal{BF}_r$ of continuous and bounded functions indexed by a parameter $r > 0$, i.e.

$$\mathcal{BF}_r \;=\; \{\phi : \mathbb{R} \to \mathbb{R} \mid \phi \text{ is continuous and } \; -r \le \phi(x) \le r \quad \forall x \in \mathbb{R}\} \tag{4}$$

and a corresponding set of saturation functions $\mathcal{S}_r$, i.e

$$\mathcal{S}_r = \{\phi : \mathbb{R} \to \mathbb{R} \mid \phi \in \mathcal{BF}_r,\; s\phi(s) > 0 \text{ for all } s \neq 0\} \tag{5}$$

Examples of nontrivial functions in $\mathcal{S}_r$ include for instance

$$\phi(x) = \tfrac{2rx}{1+x^2}, \quad \phi(x) = \tfrac{2r}{\pi}\arctan(x) \tag{6}$$

Denote

$$x \;=\; (\theta,\; x_c,\; y_c)^T \tag{7}$$

Introduce a Lyapunov function candidate

$$V_1(t,x) = \tfrac{1}{2}\left(\theta + \varepsilon_1 g_1(x_c^2 + y_c^2)\cos t\right)^2 + \tfrac{1}{2}x_c^2 + \tfrac{1}{2}y_c^2 \tag{8}$$

For simplicity, we rewrite system (1) in more compact form

$$\dot{x} \;=\; f_1(x)\omega \;+\; f_2(x)\nu \tag{9}$$

In (8), $\varepsilon_1 > 0$ is a design parameter to be chosen later and $g_1$ is a smooth (i.e., of class $C^\infty$) function in $\mathcal{BF}_1$ with the property that $g_1(s) = 0$ if and only if $s = 0$.

It is direct to verify that the conditions of [19, Theorem 2] hold for such choice of Lyapunov function $V_1$ in (8). Using the controller design scheme proposed in [19], we obtain the time-varying state feedback laws

$$\begin{aligned}
\omega &= \varepsilon_1 g_1(x_c^2 + y_c^2)\sin t \;-\; h_{\varepsilon_2}\Big(\theta + \varepsilon_1 g_1(x_c^2 + y_c^2)\cos t\Big) && (10)\\
&:= \alpha_1(t,\theta,x_c,y_c)\\
\nu &= -h_{\varepsilon_3}\Big((x_c\cos\theta + y_c\sin\theta)\left(1 + 2\varepsilon_1(\theta + \varepsilon_1 g_1\cos t)g_1'\cos t\right)\Big) && (11)\\
&:= \alpha_2(t,\theta,x_c,y_c)
\end{aligned}$$

where $\varepsilon_2$ and $\varepsilon_3$ are two positive design parameters, $h_{\varepsilon_2} \in \mathcal{S}_{\varepsilon_2}$, $h_{\varepsilon_3} \in \mathcal{S}_{\varepsilon_3}$ and $g_1' := \frac{dg_1}{ds}(x_c^2 + y_c^2)$.

We establish the following result.

**Proposition 1** *The equilibrium $x = 0$ of the closed-loop system (1), (10) and (11) is globally uniformly asymptotically stable (GUAS) for any positive $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon_3$. In particular, given any saturation levels $\omega_{\max} > 0$, $\nu_{\max} > 0$ as in (2), we can always tune $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon_3$ so that (2) holds while $x = 0$ is GUAS.*

**Proof.** Noticing that

$$\begin{aligned}
\alpha_1(t,\theta,x_c,y_c) &= \varepsilon_1 g_1(x_c^2 + y_c^2)\sin t - h_{\varepsilon_2}\Big(L_{f_1}V_1(t,x)\Big),\\
\alpha_2(t,\theta,x_c,y_c) &= -h_{\varepsilon_3}\Big(L_{f_2}V_1(t,x)\Big)\;,
\end{aligned}$$

the time derivative of $V_1$ as defined in (8) satisfies:

$$\dot{V}_1(t,x) \;=\; -\Big(L_{f_1}V_1(t,x)\Big)h_{\varepsilon_2}\Big(L_{f_1}V_1(t,x)\Big) - \Big(L_{f_2}V_1(t,x)\Big)h_{\varepsilon_3}\Big(L_{f_2}V_1(t,x)\Big) \tag{12}$$

The proof is completed along the same lines of [19, Proof of Theorem 1] using LaSalle's invariance principle. We can meet (2) by choosing $\varepsilon_1 + \varepsilon_2 \le \omega_{\max}$ and $\varepsilon_3 \le \nu_{\max}$. □

## 2.2 Dynamic model

In the preceding subsection we have solved the stabilization problem for the kinematic model (1) of the benchmark wheeled robot with saturating velocities. In this subsection, we demonstrate that the same control task can be achieved for a simplified dynamic model of the robot with saturation on the control torques.

More precisely, like in [8], we consider the dynamic model of the robot which is composed of (1) and two integrators

$$\begin{aligned} \dot{\omega} &= u_1 \\ \dot{\nu} &= u_2 \end{aligned} \tag{13}$$

where $u_1$ and $u_2$ are generalized torque-inputs subject to the constraints:

$$|u_1(t)| \le u_{1,\max}\ , \quad |u_2(t)| \le u_{2,\max} \quad \forall\, t \ge 0 \tag{14}$$

with $u_{1,\max} > 0$ and $u_{2,\max} > 0$ arbitrary positive constants.

Introduce two new variables $\overline{\omega}$ and $\overline{\nu}$ as

$$\overline{\omega} = \omega - \alpha_1(t,\theta,x_c,y_c)\ , \quad \overline{\nu} = \nu - \alpha_2(t,\theta,x_c,y_c) \tag{15}$$

with $\alpha_1(t,\theta,x_c,y_c)$ and $\alpha_2(t,\theta,x_c,y_c)$ as defined in (10) and (11).

Consider the positive definite proper Lyapunov function candidate for system (1),(13)

$$V_2(t,X) = \varepsilon_4 \log(1+V_1(t,\theta,x_c,y_c)) + \tfrac{1}{2}\overline{\omega}^2 + \tfrac{1}{2}\overline{\nu}^2 \tag{16}$$

where $X := (x^T,\omega,\nu)^T = (x_c,y_c,\theta,\omega,\nu)^T$ and $\varepsilon_4 > 0$ is a design parameter to be chosen later.

In view of (12) and (15), differentiating $V_2$ along the solutions of system (1), (13) yields

$$\begin{aligned} \dot{V}_2(t,X) &= -\Big[\Big(L_{f_1}V_1(t,x)\Big)h_{\varepsilon_2}\Big(L_{f_1}V_1(t,x)\Big) + \Big(L_{f_2}V_1(t,x)\Big)h_{\varepsilon_3}\Big(L_{f_2}V_1(t,x)\Big)\Big] + \frac{\varepsilon_4}{1+V_1(t,x)} \\ &\quad + \frac{\varepsilon_4(\theta+\varepsilon_1 g_1\cos t)}{1+V_1(t,x)}\overline{\omega} + \frac{\varepsilon_4(x_c\cos\theta + y_c\sin\theta)(1+2\varepsilon_1(\theta+\varepsilon_1 g_1\cos t)g_1'\cos t)}{1+V_1(t,x)}\overline{\nu} \\ &\quad + \overline{\omega}(u_1-\dot{\alpha}_1) + \overline{\nu}(u_2-\dot{\alpha}_2) \end{aligned} \tag{17}$$

where

$$\begin{aligned} \dot{\alpha}_1 &= \frac{\partial\alpha_1}{\partial t} + \frac{\partial\alpha_1}{\partial\theta}\omega + \left(\frac{\partial\alpha_1}{\partial x_c}\cos\theta + \frac{\partial\alpha_1}{\partial y_c}\sin\theta\right)\nu\ , \\ \dot{\alpha}_2 &= \frac{\partial\alpha_2}{\partial t} + \frac{\partial\alpha_2}{\partial\theta}\omega + \left(\frac{\partial\alpha_2}{\partial x_c}\cos\theta + \frac{\partial\alpha_2}{\partial y_c}\sin\theta\right)\nu\ . \end{aligned}$$

Therefore, we choose the time-varying control laws as

$$u_1 = -h_{\varepsilon_5}(\overline{\omega}) + \dot{\alpha}_1 - \frac{\varepsilon_4(\theta+\varepsilon_1 g_1\cos t)}{1+V_1(t,x)} \tag{18}$$

$$u_2 = -h_{\varepsilon_6}(\overline{\nu}) + \dot{\alpha}_2 - \frac{\varepsilon_4(x_c\cos\theta + y_c\sin\theta)(1+2\varepsilon_1(\theta+\varepsilon_1 g_1\cos t)g_1'\cos t)}{1+V_1(t,x)} \tag{19}$$

where $\varepsilon_6 > 0$ and $\varepsilon_6 > 0$ are design parameters and $h_{\varepsilon_5} \in \mathcal{S}_{\varepsilon_5}$, $h_{\varepsilon_6} \in \mathcal{S}_{\varepsilon_6}$.

We are now ready to state the result.

**Proposition 2** *The equilibrium $X = 0$ of the closed-loop system (1), (13), (18) and (19) is GUAS for any positive values of $\varepsilon_i$, $1 \le i \le 5$. In particular, given any saturation levels $u_{1,\max} > 0$, $u_{2,\max} > 0$ as in (14) and any compact set $\Omega_1$ in $\mathbb{R}^5$, we can always tune our design constants $\varepsilon_i$ $(1 \le i \le 6)$ so that (14) holds for all trajectories starting in $\Omega_1$.*

**Proof.** Under the choice of (18) and (19) for the torques inputs, it holds

$$\begin{aligned}\dot{V}_2(t,X) &= -\Big[\Big(L_{f_1}V_1(t,x)\Big)h_{\varepsilon_2}\Big(L_{f_1}V_1(t,x)\Big) + \Big(L_{f_2}V_1(t,x)\Big)h_{\varepsilon_3}\Big(L_{f_2}V_1(t,x)\Big)\Big]\frac{\varepsilon_4}{1+V_1(t,x)} \\ &\quad - \overline{\omega}h_{\varepsilon_5}(\overline{\omega}) - \overline{\nu}h_{\varepsilon_6}(\overline{\nu}) \qquad (20)\end{aligned}$$

The first part of Proposition 2 readily follows from LaSalle's invariance principle as in the proof of Proposition 1.

The second statement is more or less direct from the expressions (18) and (19) of the control laws $u_1$ and $u_2$. □

## 2.3 Simulations

To support our results, we simulated with MATLAB™ the wheeled mobile robot (1) in closed-loop with the controller (10, 11) with $\varepsilon_1 = 1$ and $g_1(s) = h_{\varepsilon_2}(s) = h_{\varepsilon_3} = \tanh(s)$, which guarantees that $|\omega(t)| \le \omega_{\max} = 2$ and $|\nu(t)| \le \nu_{\max} = 1$ for all $t \ge 0$. The resulting performance is depicted in Figure 1.

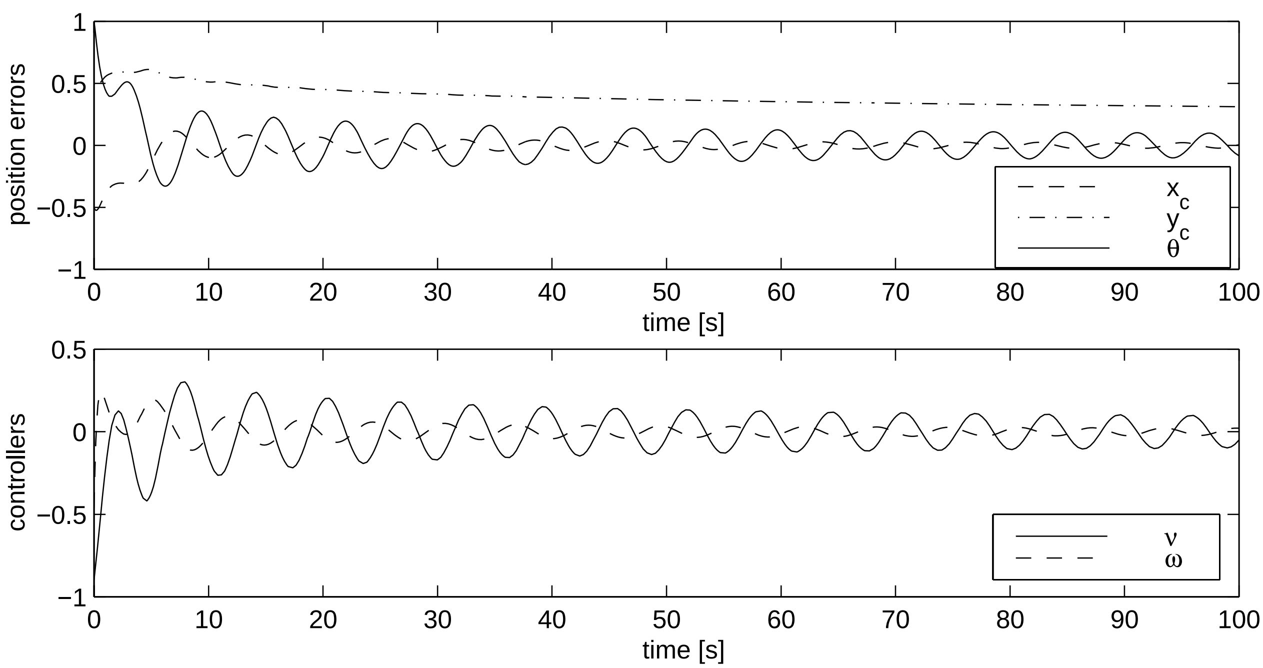


Figure 1: Stabilization of the kinematic model with initial conditions $[x_c(0), y_c(0), \theta(0)]^T = [-0.5, 0.5, 1]^T$.

From the initial condition $[x_c(0), y_c(0), \theta(0)]^T = [-0.5, 0.5, 1]^T$ we see a very slow convergence to the origin, which is a well known effect when using Pomet's method (cf. [17]).

If we consider the dynamic extension (13) in closed-loop with the controller (18, 19) where we additionally use $\varepsilon_4 = 1$, and $h_{\varepsilon_5}(s) = h_{\varepsilon_6} = \tanh(s)$ the resulting performance if we start from the initial condition $[x_c(0), y_c(0), \theta(0), \omega(0), \nu(0)]^T = [-0.5, 0.5, 1, 1, 1]^T$ is depicted in Figure 2.

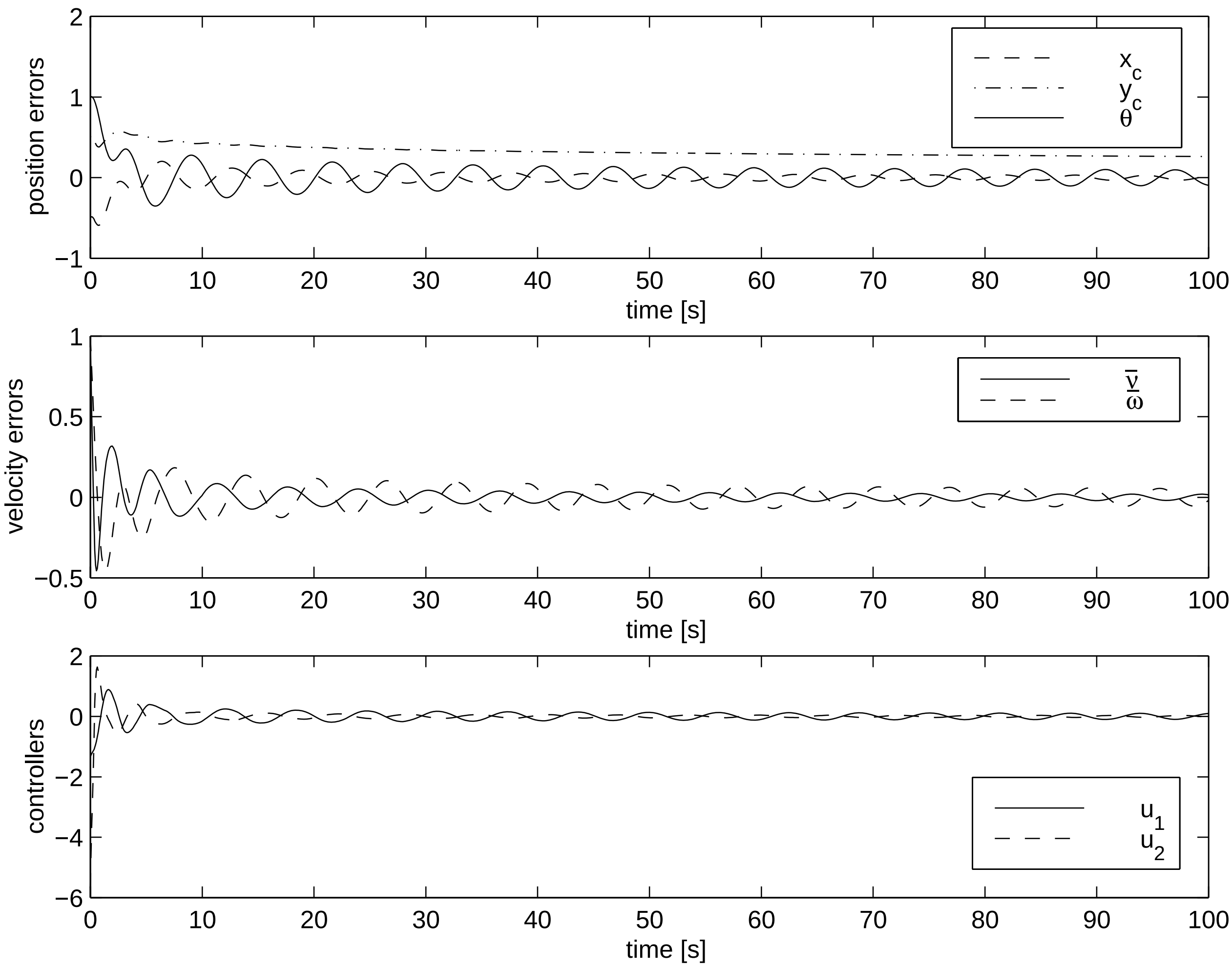


Figure 2: Stabilization of the dynamic model with initial conditions $[x_c(0), y_c(0), \theta(0), \omega(0), \nu(0)]^T = [-0.5, 0.5, 1, 0, 0]^T$.

We again see a very slow convergence to the origin.

## 3 Tracking via bounded state feedback

### 3.1 Kinematic model

In this section, we address the tracking problem for the robot (1) under a constraint on the velocities. To quantify the saturation level, it is assumed that the reference trajectory $(x_r, y_r, \theta_r)$ satisfies

$$\begin{aligned} \dot{x}_r &= \nu_r \cos\theta_r \\ \dot{y}_r &= \nu_r \sin\theta_r \\ \dot{\theta}_r &= \omega_r \end{aligned} \tag{21}$$

where $\omega_r$ and $\nu_r$ are bounded reference velocities.

The objective is to find time-varying state-feedback controllers of the form

$$\omega \;=\; \omega^*(t, \theta, x_c, y_c)\;, \quad \nu \;=\; \nu^*(t, \theta, x_c, y_c) \tag{22}$$

such that $x_c(t) - x_r(t)$, $y_c(t) - y_r(t)$ and $\theta(t) - \theta_r(t)$ tend to zero as $t \to +\infty$ while guaranteeing the following property:

$$|\omega(t)| \;\leq\; \omega_{\max}\;, \quad |\nu(t)| \;\leq\; \nu_{\max} \quad \forall\, t \geq 0 \tag{23}$$

where $\omega_{\max} > \sup_{t\geq 0} |\omega_r(t)|$ and $\nu_{\max} > \sup_{t\geq 0} |\nu_r(t)|$ are arbitrary constants.

As in [8] (see also [12]), consider the following tracking errors

$$\begin{bmatrix} x_e \\ y_e \\ \theta_e \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_r - x_c \\ y_r - y_c \\ \theta_r - \theta \end{bmatrix} \tag{24}$$

Obviously, for any value of $\theta$, $(x_e, y_e, \theta_e) = 0$ if and only if $(x_c, y_c, \theta) = (x_r, y_r, \theta_r)$.

As it can be directly checked, the tracking error dynamics of the robot satisfy

$$\begin{aligned} \dot{x}_e &= \omega y_e \,-\, \nu \,+\, \nu_r \cos\theta_e \\ \dot{y}_e &= -\omega x_e \,+\, \nu_r \sin\theta_e \\ \dot{\theta}_e &= \omega_r \,-\, \omega\;. \end{aligned} \tag{25}$$

We show next that the following control laws solve our tracking problem:

$$\omega \;=\; \omega_r + \frac{\lambda_1 \nu_r y_e}{1 + x_e^2 + y_e^2} \int_0^1 \cos(s\theta_e(t))ds + h_{\lambda_2}(\theta_e) \;:=\; \beta_1(t, \theta_e, x_e, y_e) \tag{26}$$

$$\nu \;=\; \nu_r \cos\theta_e \;+\; h_{\lambda_3}(x_e) \;:=\; \beta_2(t, \theta_e, x_e) \tag{27}$$

where $\lambda_1$, $\lambda_2$, $\lambda_3$ are positive design parameters and $h_{\lambda_2} \in \mathcal{S}_{\lambda_1}$, $h_{\lambda_3} \in \mathcal{S}_{\lambda_2}$.

**Proposition 3** *Assume that $\omega_r$ and $\nu_r$ are bounded and uniformly continuous over* $[0, \infty)$. *If either $\omega_r(t)$ or $\nu_r(t)$ does not converge to zero, then the zero equilibrium of the closed-loop system (25), (26) and (27) is globally asymptotically stable. In particular, given any $\omega_{\max} > \sup_{t\geq 0} |\omega_r(t)|$ and $\nu_{\max} > \sup_{t\geq 0} |\nu_r(t)|$, we can always tune our design parameters $\lambda_1$, $\lambda_2$ and $\lambda_3$ so that the condition (23) is met for the controllers (26,27).*

**Proof.** Consider the positive definite and proper Lyapunov function candidate

$$W_1(x_e, y_e, \theta_e) \;=\; \frac{\lambda_1}{2} \log(1 + x_e^2 + y_e^2) \;+\; \frac{1}{2}\theta_e^2 \tag{28}$$

Differentiating $W_1$ along the solutions of the closed-loop system (25), (26) and (27) yields:

$$\dot{W}_1(x_e, y_e, \theta_e) = -\frac{\lambda_1 x_e h_{\lambda_3}(x_e)}{1 + x_e^2 + y_e^2} - \theta_e h_{\lambda_2}(\theta_e) \leq 0 \tag{29}$$

Therefore, the trajectories $(x_e(t), y_e(t), \theta_e(t))$ are uniformly bounded on $[0, \infty)$. It follows, as in [8], by direct application of Barbălat's lemma [13] that

$$\lim_{t\to\infty} [x_e(t) h_{\lambda_3}(x_e(t)) + \theta_e(t) h_{\lambda_2}(\theta_e(t)] \;=\; 0 \tag{30}$$

which, in turn, gives

$$\lim_{t\to\infty} (|x_e(t)| \;+\; |\theta_e(t)|) \;=\; 0 \tag{31}$$

It remains to prove that $y_e(t)$ goes to zero as $t \to \infty$. Indeed, this fact can be established by mimicking the arguments used in the proof of [8, Proposition 2].

The last statement of Proposition 3 is more or less direct. $\square$

### 3.2 Dynamic model

We extend the tracking result from subsection 3.1 to the simplified dynamic model (13) of the robot. In this case, the tracking error dynamics are described by

$$\begin{aligned} \dot{x}_e &= \omega y_e \;-\; \nu \;+\; \nu_r \cos\theta_e \\ \dot{y}_e &= -\omega x_e \;+\; \nu_r \sin\theta_e \\ \dot{\theta}_e &= \omega_r \;-\; \omega \\ \dot{\omega} &= u_1 \\ \dot{\nu} &= u_2 \end{aligned} \tag{32}$$

where $u_1$ and $u_2$ are torque-inputs subject to the contraint:

$$|u_1(t)| \;\leq\; u_{1,\max}\,, \quad |u_2(t)| \;\leq\; u_{2,\max} \quad \forall\, t \geq 0 \tag{33}$$

where $u_{1,\max}$ and $u_{2,\max}$ are two arbitrary saturation levels satisfying the property

$$u_{1,\max} \;>\; \sup_{t\geq 0} |\dot{\omega}_r(t)|\,, \quad u_{2,\max} \;>\; \sup_{t\geq 0} |\dot{\nu}_r(t)|\,. \tag{34}$$

Contrary to the kinematic model (25) considered in subsection 3.1, $\omega$ and $\nu$ are not the actual control inputs to the dynamic model (32) of the robot. Consequently, the tracking control laws obtained in (26) and (27) cannot be implemented in the present situation. To invoke integrator backstepping (see [15]) for the purpose of designing our true tracking controllers subject to (33), we introduce two new variables

$$\omega_e = \omega - \beta_1(t, \theta_e, x_e, y_e), \;\; \nu_e = \nu - \beta_2(t, \theta_e, x_e) \tag{35}$$

where $\beta_1$ and $\beta_2$ are defined as in (26) and (27), respectively.

Consider the positive definite and proper Lyapunov function candidate for system (32)

$$W_2(t, X_e) \;=\; \lambda_4 \log(1 + W_1(t, x_e, y_e, \theta_e)) + \frac{1}{2}\omega_e^2 + \frac{\lambda_5}{2}\log(1 + \nu_e^2) \tag{36}$$

where $X_e := (x_e, y_e, \theta_e, \omega_e, \nu_e)$ and $\lambda_4,\; \lambda_5 > 0$ are two design parameters to be chosen later.

Using (29), the time derivative of $W_2$ along the solutions of (32) satisfies

$$\begin{aligned}\dot{W}_2(t, X_e) &= -\left(\frac{\lambda_1 x_e h_{\lambda_3}(x_e)}{1+x_e^2+y_e^2} + \theta_e h_{\lambda_2}(\theta_e)\right)\frac{\lambda_4}{1+W_1} + \left(\frac{-\lambda_1 x_e}{1+x_e^2+y_e^2}\nu_e - \theta_e\omega_e\right)\frac{\lambda_4}{1+W_1} \\ &\quad + \omega_e(u_1 - \dot{\beta}_1) + \frac{\lambda_5\nu_e}{1+\nu_e^2}(u_2 - \dot{\beta}_2) \end{aligned} \tag{37}$$

where

$$\begin{aligned}\dot{\beta}_1 &= \frac{\partial\beta_1}{\partial t} + \frac{\partial\beta_1}{\partial x_e}(\omega y_e - \nu + \nu_r\cos\theta_e) + \frac{\partial\beta_1}{\partial y_e}(-\omega x_e + \nu_r\sin\theta_e) + \frac{\partial\beta_1}{\partial\theta_e}(\omega_r - \omega)\ , \\ \dot{\beta}_2 &= \frac{\partial\beta_2}{\partial t} + \frac{\partial\beta_2}{\partial x_e}(\omega y_e - \nu + \nu_r\cos\theta_e) + \frac{\partial\beta_2}{\partial\theta_e}(\omega_r - \omega) \\ &= \nu_r\omega_e\sin\theta_e + \dot{\nu}_r\cos\theta_e + h'_{\lambda_3}(x_e)\,(\omega y_e - \nu + \nu_r\cos\theta_e) \\ &\quad + \left(\frac{\lambda_1\nu_r y_e}{1+x_e^2+y_e^2}\int_0^1\cos(s\theta_e(t))ds + h_{\lambda_2}(\theta_e)\right)\nu_r\sin\theta_e\end{aligned}$$

Let $\lambda_6 > 0$, $\lambda_7 > 0$ be design parameters. By making the following choice of tracking control laws for the torques $u_1$ and $u_2$

$$u_1 = -h_{\lambda_6}(\omega_e) + \dot{\beta}_1 + \frac{\lambda_4\theta_e}{1+W_1} + \frac{\lambda_5\nu_e}{1+\nu_e^2}\nu_r\sin\theta_e \tag{38}$$

$$\begin{aligned}u_2 &= -h_{\lambda_7}(\nu_e) + \tfrac{\lambda_1\lambda_4 x_e(1+\nu_e^2)}{\lambda_5(1+W_1)(1+x_e^2+y_e^2)} + \dot{\nu}_r\cos\theta_e + h'_{\lambda_3}(x_e)\,(\omega y_e - \nu + \nu_r\cos\theta_e) \\ &\quad + \left(\frac{\lambda_1\nu_r y_e}{1+x_e^2+y_e^2}\int_0^1\cos(s\theta_e(t))ds + h_{\lambda_2}(\theta_e)\right)\nu_r\sin\theta_e\end{aligned} \tag{39}$$

with $h_{\lambda_6} \in \mathcal{S}_{\lambda_6}$, $h_{\lambda_7} \in \mathcal{S}_{\lambda_7}$, it follows from (37) that

$$\dot{W}_2(t, X_e) = -\left(\frac{\lambda_1 x_e h_{\lambda_3}(x_e)}{1+x_e^2+y_e^2} + \theta_e h_{\lambda_2}(\theta_e)\right)\frac{\lambda_4}{1+W_1} - \omega_e h_{\lambda_6}(\omega_e) - \nu_e h_{\lambda_7}(\nu_e) \tag{40}$$

We are now in the position to state our tracking result for the dynamic model (32).

**Proposition 4** *Assume that $\omega_r$, $\dot{\omega}_r$, $\nu_r$ and $\dot{\nu}_r$ are bounded over $[0,\infty)$. If either $\omega_r(t)$ or $\nu_r(t)$ does not converge to zero, then the zero equilibrium $X_e = 0$ of the closed-loop system (32), (38) and (39) is globally asymptotically stable. In particular, given any $u_{1,\max} > \sup_{t\geq 0}|\dot{\omega}_r(t)|$ and $u_{2,\max} > \sup_{t\geq 0}|\dot{\nu}_r(t)|$ and any compact set $\Omega_2$ in $\mathbb{R}^5$, we can always tune our design parameters $\lambda_1$ to $\lambda_7$ so that the condition (33) is also met for all trajectories starting from $\Omega_2$.*

**Proof.** As in the proof of Proposition 3, the first part of Proposition 4 follows from (40) together with a straightforward application of Barbălat's lemma [13].

The second part of Proposition 4 is more or less direct from the expressions of the time-varying feedbacks (38) and (39). □

### 3.3 Simulations

To support our results, we simulated the closed-loop system (25, 26, 27). The desired trajectory has been given to be $\omega_r(t) = 1$, $\nu_r(t) = 1$, i.e. a circle. Using $\lambda_1 = 1$ and $h_{\lambda_2}(s) = h_{\lambda_3} = \tanh(s)$, which guarantees us that $|\omega(t)| \leq \omega_{\max} = 3$ and $|\nu(t)| \leq \nu_{\max} = 2$ for all $t \geq 0$, we obtained starting from the initial condition $[x_e(0), y_e(0), \theta_e(0)]^T = [-0.5, 0.5, 1]^T$ the performance as depicted in Figure 3.

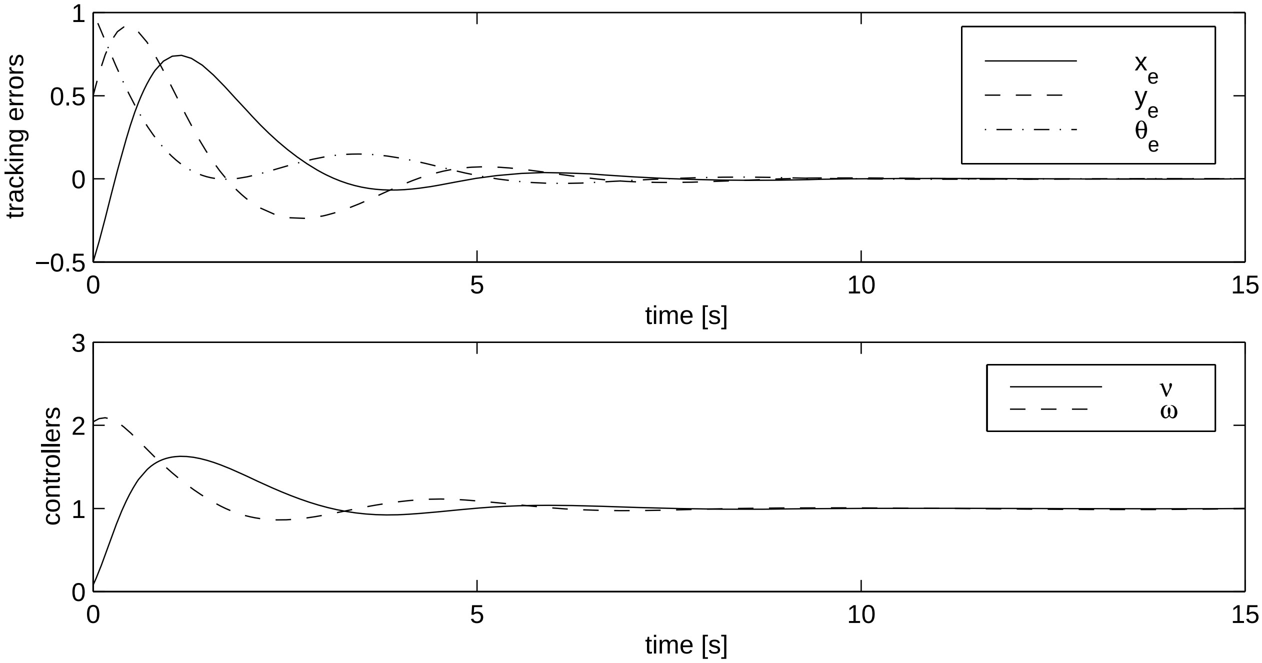


Figure 3: Tracking of the kinematic model with initial errors $[x_e(0), y_e(0), \theta_e(0)]^T = [-0.5, 0.5, 1]^T$.

We see that the control inputs obviously remain within their bounds and yield a quick convergence to the desired trajectory.

Next, we simulated the closed-loop system (32, 38, 39) where $\lambda_4 = \lambda_5 = 1$ and $h_{\lambda_6}(s) = h_{\lambda_7} = \tanh(s)$, where we want to track the same desired trajectory again. The resulting performance if we start from the initial condition $[x_e(0), y_e(0), \theta_e(0), \omega_e(0), \nu_e(0)]^T = [-0.5, 0.5, 1, 1, 1]^T$ is depicted in Figure 4.

We see an even quicker convergence of the tracking errors than in the previous case for the kinematic model.

# 4 Conclusions

Semiglobal and global solutions for the stabilization and tracking problem for the kinematic and simplified dynamic model of a wheeled mobile robot with input saturations are derived. On the basis of these results it becomes plausible that the same problems admit similar solutions if a complete dynamic model for the mobile robot is considered. Further research in this direction is however, still needed.

# References

[1] A.M. Bloch and S. Drakunov, Stabilization and tracking in the nonholonomic integrator via sliding modes, *Systems & Control Letters*, **29** (1996) 91–99.

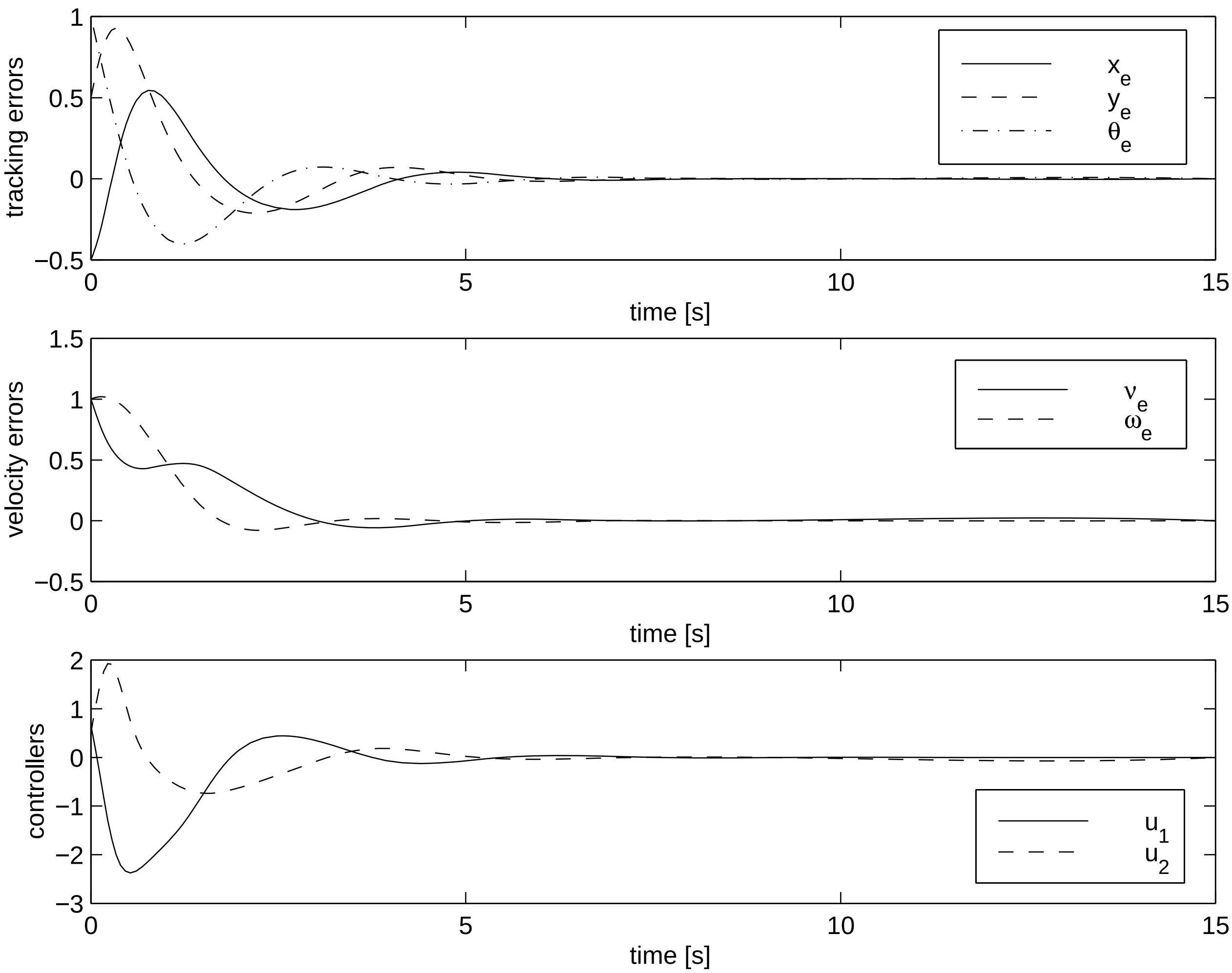


Figure 4: Tracking of the dynamic model with initial errors $[x_e(0), y_e(0), \theta_e(0), \omega_e(0), \nu_e(0)]^T = [-0.5, 0.5, 1, 1, 1]^T$.

[2] R. W. Brockett, Asymptotic stability and feedback stabilization, in: R. W. Brockett, R. S. Milman and H.J. Sussmann (Eds.), *Differential Geometric Control Theory*, Birkhauser, Boston, pp. 181-191, 1983.

[3] C. Canudas de Wit and B. Siciliano and G. Bastin (Eds.), *Theory of Robot Control*, London: Springer-Verlag, 1996,

[4] J.-M. Coron, Global asymptotic stabilization for controllable nonlinear systems without drift, *Math. Control, Systems Signals*, **5** (1992) 295–312.

[5] G. Escobar and R. Ortega and M. Reyhanoglu, Regulation and tracking of the nonholonomic double integrator: A field-oriented control approach, *Automatica*, **34** (1998) 125–132

[6] M. Fliess, J. Levine, P. Martin and P. Rouchon, Design of trajectory stabilizing feedback for driftless flat systems, *Proc. 3rd ECC*, pp. 1882–1887, Rome, 1995.

[7] Z. P. Jiang, Iterative design of time-varying stabilizers for multi-input systems in chained form, *Systems and Control Letters*, **28** (1996) 255–262.

[8] Z. P. Jiang and H. Nijmeijer, Tracking control of mobile robots: a case study in backstepping, *Automatica*, **33** (1997) 1393–1399.

[9] Z. P. Jiang and H. Nijmeijer, Backstepping-based tracking control of nonholonomic chained systems, *Proc. 4th European Control Conference*, Brussels, 1-4 July 1997.

[10] Z. P. Jiang and J.-B. Pomet, Global stabilization of parametric chained-form systems by time-varying dynamic feedback, *Int. J. Adaptive Contr. Signal Processing*, **10** (1996) 47–59.

[11] Z. P. Jiang and L. Praly, Preliminary results about robust Lagrange stability in adaptive regulation, *Int. J. of Adaptive Control and Signal Processing*, **6** (1992) 285–307.

[12] Y. Kanayama, Y. Kimura, F. Miyazaki and T. Noguchi, A stable tracking control scheme for an autonomous mobile robot, *Proc. IEEE Int. Conf. Robotics Automation*, pp. 384–389, 1990.

[13] H.K. Khalil, *Nonlinear Systems*, 2nd Edition, Upper Saddle River, NJ, 1996.

[14] I. Kolmanovsky and N. H. McClamroch, Developments in nonholonomic control systems, *IEEE Control Systems Magazine*, **15** (1995) 20–36.

[15] M. Krstić, I. Kanellakopoulos and P. V. Kokotović, *Nonlinear and Adaptive Control Design*, John Wiley & Sons, New York, 1995.

[16] W. Lin, Time-varying feedback control of nonaffine nonlinear systems without drift, *Systems & Control Letters*, **29** (1996) 101–110.

[17] R.T. M'Closkey and R.M. Murray, Exponential stabilization of driftless nonlinear control systems using homogeneous feedback, *IEEE Trans. Automat. Control.*, **42** (1997) 614–628.

[18] E. Panteley, E. Lefeber, A. Loria and H. Nijmeijer, Exponential tracking control of a mobile car using a cascaded approach, *IFAC Workshop on Motion Control*, Grenoble, 1998, to appear.

[19] J.-B. Pomet, Explicit design of time-varying stabilizing control laws for a class of controllable systems without drift, *Systems & Control Letters*, **18** (1992) 147–158.

[20] C. Samson and K. Ait-Abderrahim, Feedback control of a nonholonomic wheeled cart in Cartesian space, *Proc. IEEE Int. Conf. Robotics Automation*, pp. 1136–1141, Sacramento, 1991.

[21] G. Walsh, D. Tilbury, S. Sastry, R. Murray and J. P. Laumond, Stabilization of trajectories for systems with nonholonomic constraints, *IEEE Trans. Automat. Contr.*, **39** (1994) 216–222.