A Robust Observer with Gyroscopic Bias Correction for Rotational Dynamics

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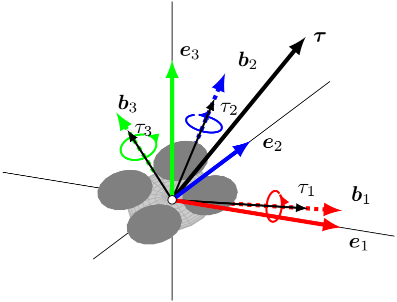
Introduction

Motivation

- Attitude estimation for aerospace applications
- Often derived using attitude *kinematics* [1]
- Design estimator targeted for UAV estimation
- Joint estimation of gyroscopic rates and biases

Contributions

- Result for attitude *dynamics* (using torques)
- Observer with *uniform* stability properties
- Produces filtered attitude rates for control



Takeaway

- *Extension of result in [1] to include torques and known inertia for UAV applications*
- *Proof of uniform LES (ULES) and uniform almost GAS (UaGAS) by Matrosov theory [2]*

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[1] - Mahony, Hamel, and Pflimlin, "Nonlinear complementary filters on the special orthogonal group" 2008

[2] - Loría, Panteley, Popovic, *et al.*, "A nested Matrosov theorem and persistency of excitation for uniform convergence in stable nonautonomous systems" 2005

Introduction

Presentation outline

- Setting and definitions
- Observer design
  - Step 1: Angular momentum estimator
  - Step 2: Gyroscopic bias estimator
  - Step 3: Convex combination of estimators
  - Step 4: Implementation aspects
- Simulations
  - Qualitative example
  - Quantitative comparison
  - Tuning the observer
- Conclusions

Setting and Definitions

- The motion of a rotating rigid body configured on $R \in \mathrm{SO}(3)$ is governed by the dynamics

$$\dot{R} = RS(\omega) \tag{1a}$$
$$J\dot{\omega} = S(J\omega)\omega + \tau, \tag{1b}$$

  where $J = J^\top > 0$ is the inertia matrix and $\tau \in R^3$ is the total moment vector in the body frame.
- Measured outputs

$$y_0 = \omega + b \qquad y_i = R^\top v_i \qquad i = 1, \dots, n, \tag{2}$$

  where b is an unknown constant, and v_i denote n known inertial directions (n=2, but wlog $n = 3$).

Problem

Consider the outputs (2). Design an observer/filter which produces estimates $\hat{R}$, $\hat{\omega}$, and $\hat{b}$ such that the point $(I, 0, 0)$ of the estimation error dynamics $(\tilde{R}, \tilde{\omega}, \tilde{b})$, given by

$$\tilde{R} = \hat{R}R^\top \qquad \tilde{\omega} = \hat{\omega} - \omega \qquad \tilde{b} = \hat{b} - b, \tag{3}$$

is almost globally and locally exponentially stable.

Observer Design - Step #1

Angular momentum estimator

- Estimation errors

$$\tilde{R} = \hat{R}R^\top \qquad \tilde{\ell} = \hat{\ell} - RJ\omega \tag{4}$$

- Observer update

$$\dot{\hat{R}} = \hat{R}S\left(J^{-1}R^\top\hat{\ell} - k_R\tilde{r}_k\right) \qquad \dot{\hat{\ell}} = R\tau - k_\ell RJ^{-1}\tilde{r}_k \tag{5a}$$

where $k_R > 0$, $k_\ell > 0$, and

$$\tilde{r}_k = \sum_{i=1}^{n} k_i S(\hat{R}^\top v_i)R^T v_i = \sum_{i=1}^{n} k_i S(\hat{R}^\top v_i)y_i. \tag{5b}$$

Proposition (1)

Consider the observer (5) in closed-loop with the attitude dynamics. If the weights k_i are chosen such that $\sum_{i=1}^{n} k_i v_i v_i^\top$ has distinct eigenvalues λ_i, i.e., $\lambda_3 > \lambda_2 > \lambda_1 > 0$, then the estimation errors (4) are UaGAS and ULES towards $(I, 0)$.

Observer Design - Step #2

Gyroscopic bias estimator [1]

- Estimation errors

$$\tilde{b} = \hat{b} - b \qquad\qquad \tilde{R} = \hat{R}R^\top \tag{6}$$

- Observer update

$$\dot{\hat{b}} = k_b \tilde{r}_k \qquad\qquad \dot{\hat{R}} = \hat{R}S(y_0 - \hat{b} - k_R \tilde{r}_k) \tag{7}$$

with $k_b > 0$, $k_R > 0$, $J = J^\top > 0$ and $\tilde{r}_k$ as in (5b) previously.

Proposition (2)

Consider the observer (7) in closed-loop with the kinematics. If ω and $\dot{\omega}$ are bounded and the weights k_i are chosen such that $\sum_{i=1}^{n} k_i v_i v_i^\top$ has distinct eigenvalues λ_i, i.e., $\lambda_3 > \lambda_2 > \lambda_1 > 0$, then the estimation errors (6) are UaGAS and ULES towards $(I, 0)$.

[1] - Mahony, Hamel, and Pflimlin, "Nonlinear complementary filters on the special orthogonal group" 2008

Observer Design - Step #3

Convex combination of the observers from Step #1 and Step #2 and some additional terms

- Estimation errors

$$\tilde{R} = \hat{R}R^\top \qquad \tilde{\ell} = \hat{\ell} - RJ\omega \qquad \tilde{b} = \hat{b} - b, \tag{8}$$

- Observer update

$$\dot{\hat{b}} = k_b \tilde{r}_k - \alpha k_b k_\alpha J R^\top [\hat{\ell} - RJ(y_0 - \hat{b})] \tag{9a}$$

$$\dot{\hat{R}} = \hat{R} S\left(\alpha J^{-1} R^\top \hat{\ell} - (1-\alpha)(y_0 - \hat{b}) - k_R \tilde{r}_k\right) \tag{9b}$$

$$\dot{\hat{\ell}} = R\tau - k_\ell R J^{-1} \tilde{r}_k - (1-\alpha) k_\ell k_\alpha [\hat{\ell} - RJ(y_0 - \hat{b})] \tag{9c}$$

$$\hat{\omega} = J^{-1} \hat{R}^\top \hat{\ell}, \tag{9d}$$

with $k_b > 0$, $k_\alpha > 0$, $k_b > 0$, $k_R > 0$, $k_\ell > 0$, $0 < \alpha < 1$, and $\tilde{r}_k$ as defined in (5b) previously.

Proposition (3)

Consider the observer (9) in closed-loop with the dynamics. If k_i are chosen such that $\sum_{i=1}^{n} k_i v_i v_i^\top$ has distinct eigenvalues λ_i, then the estimation errors are UaGAS and ULES towards $(I, 0, 0)$.

Observer Design - Step #3

Using the Lyapunov function from Step #1 and #2

$$\text{Step } \#1\text{:} \quad V_{(\#1)} = k_\ell \sum_{i=1}^{n} \frac{k_i}{2} \left\| \tilde{R} v_i - v_i \right\|_2^2 + \frac{1}{2} \tilde{\ell}^\top \tilde{\ell} \tag{10a}$$

$$\text{Step } \#2\text{:} \quad V_{(\#2)} = k_b \sum_{i=1}^{n} \frac{k_i}{2} \left\| \tilde{R} v_i - v_i \right\|_2^2 + \frac{1}{2} \tilde{b}^\top \tilde{b} \tag{10b}$$

$$\text{Step } \#3\text{:} \quad V_{(\#3)} = k_\ell k_b \sum_{i=1}^{n} \frac{k_i}{2} \left\| \tilde{R} v_i - v_i \right\|_2^2 + \frac{k_\ell}{2} (1-\alpha) \tilde{b}^\top \tilde{b} + \frac{k_b}{2} \alpha \tilde{\ell}^\top \tilde{\ell} \tag{10c}$$

Differentiation along the error dynamics yields

$$\dot{V}_{(\#3)} = -k_\ell k_b k_R \left\| \tilde{r}_k \right\|_2^2 - \alpha(1-\alpha) k_\ell k_b k_\alpha \|\tilde{\delta}_L\|_2^2 \leq 0, \tag{11}$$

yielding boundedness of both $\dot{\tilde{r}}_k$ and $\ddot{\tilde{r}}_k$, and the proof is completed with a Matrosov result [2].

[2] - Loría, Panteley, Popovic, *et al.*, "A nested Matrosov theorem and persistency of excitation for uniform convergence in stable nonautonomous systems" 2005

Observer Design - Step #3

Properties

- Approaches the observers in step #1 and step #2 if $\alpha \to \{1, 0\}$
- Potential to get the best of both (smooth rate estimates and bias estimates)
- All measurement noise passes through an integrator $\Rightarrow$ impacts noise-error gain
- Uniform stability properties $\Rightarrow$ robustness to bounded perturbations [3]

Implementation aspects

- Possible to get slow error decays (e.g., if $\alpha \to \{0, 1\}$), needs to be tuned with care
- Not strictly implementable as written (R is not known)

But there is a remedy!

[3] - Khalil, *Nonlinear Systems* 2002

Observer Design - Step #4

Making the estimator implementable

- In the estimate update (9) of Proposition 3, we require R which is not known
- Here, we can replace instances of R in the estimate update by

$$\bar{R} = \left(\sum_{i=1}^{n} k_i v_i v_i^\top \right)^{-1} \sum_{i=1}^{n} k_i v_i y_i^\top . \tag{12}$$

Proposition (4)

Consider the observer (9) in closed-loop with the dynamics. If the observer is defined as in Proposition 3 with R replaced by $\bar{R}$, then the estimation errors are UaGAS and ULES towards $(I, 0, 0)$.

Remark

This is only used on the right-hand-side of the ODE, and thus this (potentially noisy) reconstruction of R is solely used to define the innovations of the observer, and filtered through the integrators.

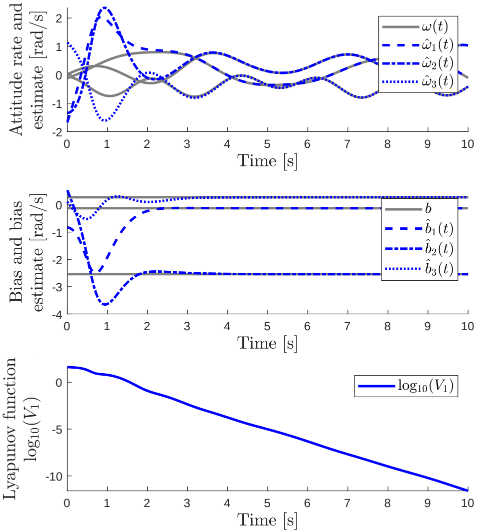
Simulations - Qualitative

Qualitative simulation result

- Observer in Proposition 4
- Discrete-time implementation (500Hz)
- This is an ideal setting, no noise
- Dense inertia matrix
- Large initial estimation errors

Takeaway

- *Proof of UaGAS/ULES, but initial transients decay with rates similar to the local errors*
- *The transient system response depends on the observer tuning, can be made slow if* $\alpha \in (0, 1)$ *is close to end-points*



Simulations - Quantitative

- Parameters: $\theta = \{R_0, \omega_0, b, \hat{R}_0, \hat{\omega}_0, \hat{b}_0\}$ sampled uniformly for each MC run
- Movement: significant, driven by $\tau(t) = (\sin(t+1), \sin(2t+2), \sin(3t+3))^\top \in \mathbb{R}^3$
- Noise: sampled at 500 Hz with $\delta_i \sim \mathcal{N}(0, 0.1^2 I)$, where

$$
\begin{aligned}
y_0(hk) &= \omega(hk) + b(hk) + \delta_0(hk) \\
\bar{y}_i(hk) &= R(hk)^\top v_i + \delta_i(hk) & i = 1, \ldots, 3, \\
y_i(hk) &= \bar{y}_i(hk) / \|\bar{y}_i(hk)\|_2 & i = 1, \ldots, 3,
\end{aligned}
$$

- Measure: Average $\mathcal{L}_2$-norms over different time intervals

$$
\mathrm{RMSE}_{\mathcal{L}_2([a,b])}(x) = \left(\frac{1}{N_{MC}} \sum_{i=1}^{N_{MC}} \int_a^b \|x^{(i)}(t)\|_2^2 \mathrm{d}t \right)^{1/2}.
$$

- Signals: attitude error $\Psi(\tilde{R}) = \frac{1}{2}\mathrm{Tr}(I - \tilde{R})$, attitude rate error $\tilde{\omega}$ and bias error $\tilde{b}$

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- Measure: Average $\mathcal{L}_2$-norms over different time intervals
- Signals: attitude error $\Psi(\tilde{R}) = \frac{1}{2}\mathrm{Tr}(I - \tilde{R})$, attitude rate error $\tilde{\omega}$ and bias error $\tilde{b}$

Table: RMSEs of transient and stationary errors categorized by signals and observers.

Measure	$\mathrm{RMSE}_{\mathcal{L}_2([0,T])}(x)$			$\mathrm{RMSE}_{\mathcal{L}_2([T-1,T])}(x)$		
Signal	$\Psi(\tilde{R})$	$\tilde{\omega}$	$\tilde{b}$	$\Psi(\tilde{R})$	$\tilde{\omega}$	$\tilde{b}$
Prop. 1	0.560	2.571	2.629	$3.043 \cdot 10^{-5}$	**0.022**	0.178
Prop. 2	0.577	2.463	2.401	$2.718 \cdot 10^{-5}$	0.177	**0.016**
Prop. 4	0.570	2.389	2.226	$2.809 \cdot 10^{-5}$	**0.021**	**0.016**

Simulations - Quantitative

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Takeaway

Relatively small differences in the transients, but significant differences in the stationary errors $\tilde{\omega}$ and $\tilde{b}$. The observer in Proposition 4 is less impacted by noise in these errors.

Simulations - Tuning

Observer tuning

- Express local perturbation about identity error

$$\tilde{R} \approx I + S(\tilde{\epsilon}) \tag{13}$$

- Consider noise $\delta_i \in \mathbb{R}^3$ on the measurements, as

$$y_0 = w + b + \delta_0, \qquad y_i = R^\top (I + S(\delta_i)) v_i. \tag{14}$$

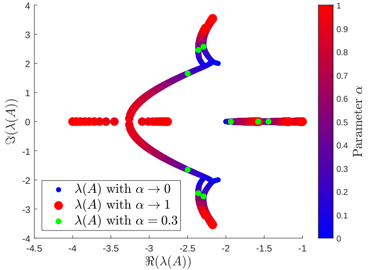
- Under a stationary rotation R, we get a linear system

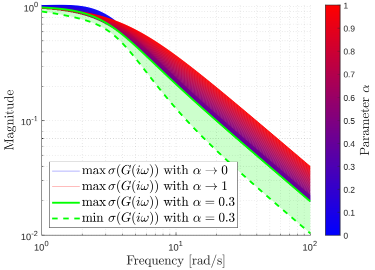
$$\dot{\tilde{x}} = A\tilde{x} + B\delta. \tag{15}$$

from measurement noises $\delta^\top = (\delta_0^\top, \delta_1^\top, \delta_2^\top, \delta_3^\top) \in \mathbb{R}^{12}$ to the local estimation errors $\tilde{x}^\top = (\tilde{\epsilon}^\top, \tilde{\omega}^\top, \tilde{b}^\top) \in \mathbb{R}^9$.

Takeaway

Tune based on the spectrum of A, and the properties of the system $G(s) = (sI - A)^{-1}B$. For example, we get a balanced spectrum and good attenuation of the measurement noise with $\alpha \approx 0.3$ when fixing the other controller parameters.





Conclusions

Summary

- Observer for angular momentum (step #1)
- Combined with result by Mahony [1] (step #3)
- Addressing implementation aspects (step #4)
- Robust attitude observer, demonstrated both in theory and simulations
- Code for discrete-time implementation with quaternions and RK4
- Additional mathematical details in extended arXiv paper [4]

Main insights

- We can take a "convex combination" of observers
- Proofs come with *uniform* stability properties

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[1] - Mahony, Hamel, and Pflimlin, "Nonlinear complementary filters on the special orthogonal group" 2008

[4] - Lefeber, Greiff, and Robertsson, "A Robust Observer with Gyroscopic Bias Correction for Rotational Dynamics" 2023

References

[1] R. Mahony, T. Hamel, and J.-M. Pflimlin, "Nonlinear complementary filters on the special orthogonal group," *IEEE Trans. on Aut. Cont.*, vol. 53, no. 5, pp. 1203–1218, 2008.

[2] A. Loría, E. Panteley, D. Popovic, and A. R. Teel, "A nested Matrosov theorem and persistency of excitation for uniform convergence in stable nonautonomous systems," *IEEE Trans. on Aut. Cont.*, vol. 50, no. 2, pp. 183–198, 2005.

[3] H. Khalil, *Nonlinear Systems*, 3rd. Upper Saddle River, NJ, USA: Prentice-Hall, 2002.

[4] E. Lefeber, M. Greiff, and A. Robertsson, "A robust observer with gyroscopic bias correction for rotational dynamics,", 2023. arXiv: `2304.02763 [eess.SY]`. [Online]. Available: `arxiv.org/abs/2304.02763v1`.