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Robust optimal control of material flows in demand-driven supply networks

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Abstract

We develop a model based on stochastic discrete-time controlled dynamical systems in order to derive optimal policies for controlling the material flow in supply networks. Each node in the network is described as a transducer such that the dynamics of the material and information flows within the entire network can be expressed by a system of first-order difference equations, where some inputs to the system act as external disturbances. We apply methods from constrained robust optimal control to compute the explicit control law as a function of the current state. For the numerical examples considered, these control laws correspond to certain classes of optimal ordering policies from inventory management while avoiding, however, any a priori assumptions about the general form of the policy.
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1. Introduction

A supply network is a set of facilities, connected by transportation links, whose function comprises the procurement of (raw-) materials, transformation of these materials into intermediate and finished products and distribution of the finished products to consumers. Generalizing the original concept of a supply chain, where mainly linear or tree-structured arrangements of facilities and corresponding material flows were considered, the term 'supply network' explicitly acknowledges the possibility of more complex structures, where in principle any of the involved components can supply each other. Clearly, this is a more realistic model in today's networked economy.

An important task in managing supply networks is to organize the flow of goods and information between the different components. This includes both the design of the network structure (building and configuring the links) as well as the control of the material flow through the network over time. A major challenge in this respect is the handling of uncertainty, such as in the customer demand, availability of materials, variability of production and transportation times.

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Considerable research effort has been invested into deriving optimization models for either network structure or control policies [1], with increasing focus on incorporating randomness [2]. A related aspect of investigation is the stability of the supply network with respect to uncertain inputs. Existing studies typically assume a given (parameterized) control strategy and analyze how the dynamics depends on the parameters [3–8] or, additionally, on the structure of the network [9].

The focus of this study is on finding optimal control strategies for material flows in demand-driven supply networks. More precisely, the individual nodes are controlling their inflows by placing orders to their immediate suppliers. These orders represent the information flow, which might be instantaneous (so that the supplier's outflow can directly be controlled by its customer) or involve a time delay. Thus, we have a local control, resulting in a local coordination of the supplier–customer pairs, which nevertheless might be conceived with complete information of the whole network's (global) state.

Our approach is to view supply networks as controlled dynamical systems. The goal is to derive an explicit state-feedback control, i.e., control laws as functions of the current state of the system. The major difference from previous approaches is that we do not assume a certain family of strategies a priori. In principle, we allow for any possible mapping of the system's state into the range of possible values for the control inputs. The specific form of this mapping will emerge as the result.

Our assumptions are that the system works in discrete time and that we have fixed time delays for both material and information flows between the nodes in the network. The demand (orders) of the end customers (sinks in the network) is regarded as an external disturbance. We assume that the range of these disturbances is bounded, but no further assumptions about the distribution are needed. The control law to be derived should be robust in the sense that the system remains in a pre-specified state region for a pre-specified range of disturbances and optimal with respect to the worst case realization of the uncertain external demand.

2. Supply networks as controlled dynamical systems

We consider supply networks as discrete-time controlled dynamical systems with uncertainty

$$x(t+1) = f(x(t), u(t), d(t)), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ denotes the state of the system, $u(t) \in \mathbb{R}^{n_u}$ the control input and $d(t) \in \mathbb{R}^{n_d}$ the (uncertain) disturbances, at time t. The map $f : \mathbb{R}^{n+n_u+n_d} \longrightarrow \mathbb{R}^n$ depends on the structure of the network.

As the basic network structure for the material flow we assume a directed connected graph $G = (V, S)$ whose vertex set $V := \{v_1, \ldots, v_{n_V}\}$ contains the facilities (nodes) of the supply network and arcs $\{s_1, \ldots, s_{n_S}\} =: S \subseteq V \times V$ represent the material flows (see Fig. 1, top). On the same vertex set V, a further relation $\{r_1, \ldots, r_{n_R}\} := R \subseteq V \times V$ is given as the communication channels carrying the information flow. The information flow is intended to carry the orders, i.e., the desired (future) material flow on the corresponding arc from R. We require any control of the material flow to be employed via the information flow channels,
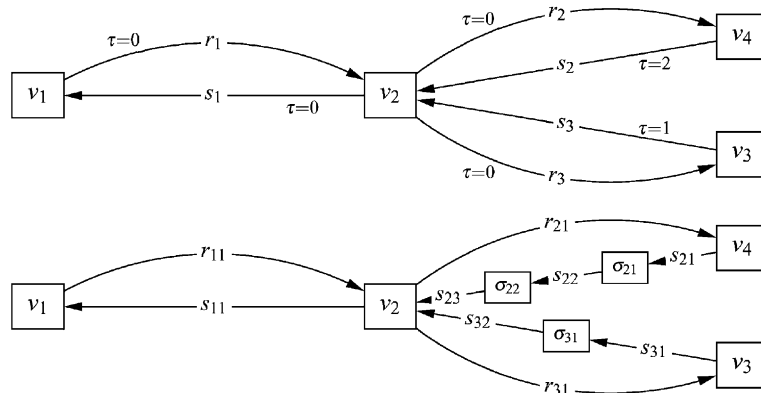

Fig. 1. Example of a basic network (top), where the arcs are annotated by their time delays τ, and the corresponding extended synchronous network (bottom).

hence the information flow 'coordinates' the material flow in the network. Let therefore $R := \{(v, w) \in V \times V : (w, v) \in S\}$ such that each material flow arc from w to v has a corresponding information flow arc, where v can place its orders from w.

Let $\tau_{s_i} \in \mathbb{N}_0$ be the time delay (transportation time) of an arc $s_i \in S$. As we want to derive a first-order difference equation model for the dynamics of the system, additional nodes are needed in our network model such that the transport over each arc takes exactly one time unit. The basic network is thus transformed into an extended network by replacing each arc $s_i \in S$ by a path (chain) of $\tau_{s_i} + 1$ arcs $S_i := \{s_{i1}, \ldots, s_{i\tau_{s_i}}\}$ through τ_{s_i} additionally defined auxiliary nodes $V_i = \{\sigma_{i1}, \ldots, \sigma_{i\tau_{s_i}}\}$. For any arc $s_i = (v, w) \in S$, the resulting path is $\{(v, \sigma_{i1}), (\sigma_{i1}, \sigma_{i2}), \ldots, (\sigma_{i\tau_{s_i}-1}, \sigma_{i\tau_{s_i}}), (\sigma_{i\tau_{s_i}}, w)\}$. Let $V_S := V_1 \cup \cdots \cup V_{n_S}$. Analogously, sets $R_i = \{r_{i1}, \ldots, r_{i\tau_{r_i}}\}$, $V_i' = \{\rho_{i1}, \ldots, \rho_{i\tau_{r_i}}\}$ and $V_R = V_1' \cup \cdots \cup V_{n_R}'$ are defined for the information flows. Let $\tilde{V} := V \cup V_S \cup V_R$, $\tilde{S} := S_1 \cup \cdots \cup S_{n_S}$ and $\tilde{R} := R_1 \cup \cdots \cup R_{n_R}$. Then the graph $\tilde{G}_S := (\tilde{V}, \tilde{S})$ represents the resulting synchronous material flow network and the graph $\tilde{G}_R := (\tilde{V}, \tilde{R})$ the synchronous information flow network. The pairs of corresponding information and material flow arcs have thus been extended to pairs of information and material flow paths between the nodes of the original basic network G (see Fig. 1, bottom). For all $v \in \tilde{V}$ define

$$\check{S}^{(v)} := \{(w, v) \in \tilde{S}\}, \ \check{\delta}^{(v)} := |\check{S}^{(v)}| \quad \text{for incoming material flow arcs,}$$

$$\hat{S}^{(v)} := \{(v, w) \in \tilde{S}\}, \ \hat{\delta}^{(v)} := |\hat{S}^{(v)}| \quad \text{for outgoing material flow arcs,}$$

$$\check{R}^{(v)} := \{(w, v) \in \tilde{R}\}, \ \check{\delta}'^{(v)} := |\check{R}^{(v)}| \quad \text{for incoming information flow arcs,}$$

$$\hat{R}^{(v)} := \{(v, w) \in \tilde{R}\}, \ \hat{\delta}'^{(v)} := |\hat{R}^{(v)}| \quad \text{for outgoing information flow arcs,}$$

whose elements are indexed as $\check{s}_i^{(v)}$, $\hat{s}_i^{(v)}$, $\check{r}_i^{(v)}$ and $\hat{r}_i^{(v)}$, respectively, where $i \in \mathbb{N}$. Each vertex $v \in \tilde{V}$ can now be considered as a transducer whose internal state $x^{(v)}(t) \in \mathbb{R}$ corresponds to its inventory of goods at time t. Its input $u^{(v)}(t) = [u_S^{(v)}(t), u_R^{(v)}(t)]^{\mathrm{T}} \in \mathbb{R}^{\check{\delta}^{(v)} + \check{\delta}'^{(v)}}$ corresponds to the flow on the incoming arcs and the output $y^{(v)}(t) = [y_S^{(v)}(t), y_R^{(v)}(t)]^{\mathrm{T}} \in \mathbb{R}^{\hat{\delta}^{(v)} + \hat{\delta}'^{(v)}}$ to the flow on the outgoing arcs at time t. The resulting dynamics of a node $v \in V$ is given by the map

$$x^{(v)}(t+1) = x^{(v)}(t) + e^{\check{\delta}^{(v)}} \cdot u_S^{(v)}(t) - e^{\hat{\delta}^{(v)}} \cdot z_S^{(v)}(t),$$

$$y_R^{(v)}(t) = z_R^{(v)}(t),$$

$$y_S^{(v)}(t) = z_S^{(v)}(t), \tag{2}$$

where $e^k \in \mathbb{R}^k$ denotes a row vector with all components equal to one. The values $z_S^{(v)}(t) \in \mathbb{R}^{\hat{\delta}^{(v)}}$ represent the goods that node v decides to ship towards each of its customers along arcs $\hat{S}^{(v)}$, and $z_R^{(v)}(t) \in \mathbb{R}^{\hat{\delta}'^{(v)}}$ represent the orders that node v decides to place to each of its suppliers via the arcs $\hat{R}^{(v)}$. These are additional control inputs to the nodes that determine the nodes' transfer behavior. They will remain the only variables that we can manipulate on the system level and hence become our control inputs to the system as a whole.

The dynamics for the auxiliary nodes $w \in V_S$ and $w' \in V_R$ simply reduces, as desired, to that of first-order delay elements

$$x^{(w)}(t+1) = u_S^{(w)}(t), \quad x^{(w')}(t+1) = u_R^{(w')}(t),$$

$$y_S^{(w)}(t) = x^{(w)}(t), \quad y_R^{(w')}(t) = x^{(w')}(t). \tag{3}$$

There are certain nodes $v_C \in V$ that act as sinks as they do not have any outgoing material flow arcs. These nodes are called 'consumers' and simply generate an uncertain demand $d^{(v_C)}$. Their dynamics reduces to

$$y_R^{(v_C)}(t) = d^{(v_C)}(t) \tag{4}$$

for all $v_C \in V_C := \{v \in V : \hat{\delta}^{(v_C)} = \check{\delta}'^{(v_C)} = 0\}$. Similarly, some $v_M \in V$ that have no incoming material flows act as sources of infinite supply capacity. They simply transform incoming orders (information flow) into outgoing goods (material flow), hence their dynamics is

$$y_S^{(v_M)}(t) = u_R^{(v_M)}(t) \tag{5}$$

for all $v_M \in V_M := \{v \in V : \check{\delta}^{(v_M)} = \hat{\delta}'^{(v_M)} = 0\}$. Source and sink nodes have no internal state variables.

The dynamics of the whole system can now easily be derived by connecting the corresponding inputs $u^{(v)}$ and outputs $y^{(v)}$ of all $v \in \tilde{V}$, thus eliminating all these variables. Let the nodes $v_i \in V_z := \tilde{V} \backslash (V_C \cup V_M)$, $w_j \in V_x := V \backslash (V_C \cup V_M)$ and $w'_k \in V_C$ be indexed such that $1 \leqslant i \leqslant |V_z|$, $1 \leqslant j \leqslant |V_x|$ and $1 \leqslant k \leqslant |V_C|$. Defining $x(t) := [x^{(w_1)}(t) \cdots x^{(w_{|V_x|})}(t)]^T$ as the state vector, $u(t) := [z_1(t) \cdots z_{|V_z|}(t)]^T$ with $z_i(t) := [[z_S^{(v_i)}(t)]^T [z_R^{(v_i)}(t)]^T]^T$ as the input vector, and $d(t) := [[d^{(w'_1)}(t)]^T \cdots [d^{(w'_{|V_C|})}(t)]^T]^T$ as the disturbance vector gives a first-order difference equation of the form (1) with $n = |\tilde{V} \backslash (V_C \cup V_M)|$ state variables, $n_u = \sum_{v \in V} (\hat{\delta}^{(v)} + \hat{\delta}'^{(v)})$ control variables and $n_d = \sum_{v \in V_C} \hat{\delta}'^{(v)}$ external disturbances. As the equations for the node dynamics (2–5) are all linear, the dynamics of the entire system can be written as

$$x(t+1) = Ax(t) + Bu(t) + Ed(t), \tag{6}$$

with matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_u}$, and $E \in \mathbb{R}^{n \times n_d}$.

2.1. Constrained robust optimal control

We consider the problem of finding optimal control inputs $u(t)$ to the system (6) with respect to costs on state and control variables, given by $\|Qx(t)\|_1$ and $\|Ru(t)\|_1$, $Q \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{n_u \times n_u}$, and constraints

$$Fx(t) + Gu(t) \leqslant g, \quad F \in \mathbb{R}^{n_g \times n}, \ G \in \mathbb{R}^{n_g \times n_u}, \ F \in \mathbb{R}^{n_g}, \ n_g \in \mathbb{N}_0.$$

Typically, the costs represent inventory holding costs on x and ordering costs on u. The constraints are used to specify upper and lower bounds on the state variables (e.g., to enforce non-negative inventories or limited storage capacity) and on the control variables (no negative orders, maximum allowed order per time unit, limits on transportation capacity). The constraints can also be used to make allowed control inputs state-dependent, e.g. to ensure that a node cannot send more material than it has available.

The goal in robust constrained optimal control is to find control inputs that guarantee satisfaction of the constraints for all possible disturbance realizations and that are favorable with respect to the resulting cost distribution due to the disturbances. The main approach, which we also follow here, is to optimize the worst-case cost via a min-max approach (minimum over u and maximum over d).

We assume that the disturbances $d(t) \in \mathcal{D} \subset \mathbb{R}^{n_d}$ are bounded such that $\mathcal{D} = \{d : Ld \leqslant l\}$, $L \in \mathbb{R}^{n_l \times n_d}, l \in \mathbb{R}^{n_l}, n_l \in \mathbb{N}_0$. Given that the system (6) is in state $x(t_0)$ at time $t_0 = 0$ and a horizon $K \in \mathbb{N}$, we are looking for an optimal control input sequence $(u^{(k)})_{k=0}^{K-1}$ such that

$$J^{*(k)}(x^{(k)}) = \min_{u^{(k)}} \quad J^{(k)}(x^{(k)}, u^{(k)})$$

$$\text{s.t.} \quad \left. \begin{cases} Fx^{(k)} + Gu^{(k)} \leqslant g \\ Ax^{(k)} + Bu^{(k)} \in \mathcal{X}^{(k)} \end{cases} \right\} \quad \forall d^{(k)} \in \mathcal{D},$$

$$J^{(k)}(x^{(k)}, u^{(k)}) = \max_{d^{(k)} \in \mathcal{D}} \|Qx^{(k)}\|_1 + \|Ru^{(k)}\|_1 + J^{*(k+1)}(Ax^{(k)} + Bu^{(k)} + Ed^{(k)}) \tag{7}$$

for $k = K - 1, \ldots, 0$, where $x^{(k)}$ is the state vector at time $t_0 + k$ given the system started in $x(t_0) = x^{(0)}$ and was exposed to input sequence $(u^{(j)})_{j=0}^k$ and disturbance sequence $(d^{(j)})_{j=0}^k$, and where the set of feasible states is

$$\mathcal{X}^{(k)} = \{x \in \mathbb{R}^n : \forall d \in \mathcal{D} \ \exists u \in \mathbb{R}^{n_u} \text{ with } Fx + Gu \leqslant g \text{ and } Ax + Bu + Ev \in \mathcal{X}^{(k+1)}\}.$$

As boundary conditions we choose $J^{*K}(x^{(K)}) = 0$ and $\mathcal{X}^{(K)} = \{x \in \mathbb{R}^n : Fx \leqslant g\}$. The finite time K-step optimal control $u^* : \mathbb{R}^n \longrightarrow \mathbb{R}^{n_u}$ is given by the first component $u^{(0)}$ of the above problem for a fixed horizon K, and the infinite-time optimal control law is its limit for $K \to \infty$.

We can compute the optimal control law u^* by dynamic programming over k. It was shown in Ref. [10] that u^* is a piecewise affine and continuous function of the current state $x(t_0)$ and that each dynamic programming step, i.e., the solution of (7), can be obtained by solving a multi-parametric linear program.

3. Examples

3.1. First example: beergame, single node

We give some numerical examples to illustrate the proposed model and solution procedure to arrive at a robust optimal control strategy. The examples are inspired by (and variations of) the Beer Distribution Game, which is a supply chain simulation for investigating dynamic behavior and potential sources of instability with respect to the formation of control rules applied by the individual actors in the chain [11]. In particular, the family of rules that were empirically derived from the behavioral experiment were shown to cause various kinds of chaotic and hyperchaotic behavior of the system [3–5].

The basic network structure of the game is given in Fig. 2. We assume in the following that $d(t) \in [0, 8]$, i.e., that the end customers can order any quantity between zero and eight. Throughout the examples the inventory holding costs in all four stages B, W, D, and F are 0.5 per unit per time step. Furthermore, the infinite-time optimal control problem has been considered, which guarantees asymptotic stability of the derived controller.

As a first example, we consider only one stage of the game, the retailer branch B, which supplies its customers as shown in the Fig. 2. B receives its supply from W with a shipping delay of $\tau_{s_1} = 2$ and ordering delay of $\tau(r_1) = 1$, where $s_1 = (W, B)$ and $r_1 = (B, W)$. W acts as a source with infinite inventory and dynamics given by (5). The four necessary state variables are $x = [x^{(B)}, x^{(\sigma_{12})}, x^{(\sigma_{11})}, x^{(\rho_{11})}]$. A constraint is set to enforce $x^{(B)}(t) \geq 0$, and $z_S^{(B)}(t)$ will be set equal to $u_R^{(B)}(t) = d(t)$ so that B is required to always deliver what is ordered. The remaining control variable is $u(t) = z_R^{(B)}(t)$, the order rate of B, which we require to be bounded as $0 \leq u(t) \leq 8$. The dynamics according to (6) is

$$x(t+1) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} d(t).$$

By solving the corresponding optimization problem, it follows that the feasible region is given by $x_i \geq 0$ for all $1 \leq i \leq 4$ and

$$x_1 + x_2 \geq 8,$$

$$x_1 + x_2 + x_3 \geq 16,$$

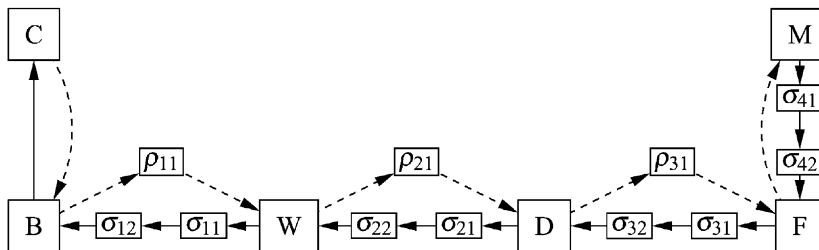$$x_1 + x_2 + x_3 + x_4 \geq 24 \tag{8}$$

Fig. 2. Synchronous supply network of the Beer Distribution Game. The large squares denote the nodes from V which represent the different stages: end customers (C), retailer branch (B), wholesaler (W), distributor (D), factory (F) and raw materials source (M). The solid arcs denote material flows (deliveries) $\tilde{S}$ and the dashed arcs information flows (orders) $\tilde{R}$. The small squares represent the auxiliary delay nodes, $\sigma_{ij} \in V_S$ and $\rho_{ij} \in V_R$.

and the resulting optimal control law is

$$z_R^{(B)}(x) = u^*(x) = \max\{32 - x_1 - x_2 - x_3 - x_4, 0\}. \tag{9}$$

This essentially represents the common order-up-to policy, sometimes also called hedging point policy [12], which is plausible in this case. The order-up-to level, or hedging point, is the maximum demand per period multiplied with the time it takes from placing the order until the goods arrive in the inventory (lead time). Naturally, the unfilled orders (state variables x_2, x_3 and x_4) have to be accounted for when determining the next order. The resulting level of work in progress (WIP) of 32 is the minimum WIP level that guarantees a service level of 100% for any sequence of demand realizations from the given interval. Furthermore, the resulting policy prevents a bullwhip effect in the sense that the variations in orders are not higher than variations in consumer demand.

3.2. Second example: beergame, all four stages

Next we consider all four stages as displayed in Fig. 2. As above, $x^{(v)}(t) \geqslant 0$ is enforced and $z_S^{(v)}(t) = u_R^{(v)}(t)$ is set for all $v \in \{B, W, D, F\}$. The remaining four control variables are $u = [z_R^{(B)} z_R^{(W)} z_R^{(D)} z_R^{(F)}]$. By solving the optimization problem, it follows that the feasible region in the state space is defined by inequalities (8) plus

$$32 \leqslant x^{(B)} + x^{(W)} + \sum_{i=1}^{2}(x^{(\sigma_{i2})} + x^{(\sigma_{i1})}), \quad x^{(W)} + x^{(\sigma_{22})} \geqslant x^{(\rho_{11})},$$

$$40 \leqslant x^{(B)} + x^{(W)} + \sum_{i=1}^{2}(x^{(\sigma_{i2})} + x^{(\sigma_{i1})}) + x^{(\rho_{21})}, \quad x^{(D)} + x^{(\sigma_{32})} \geqslant x^{(\rho_{21})},$$

$$48 \leqslant x^{(B)} + x^{(W)} + x^{(D)} + \sum_{i=1}^{3}(x^{(\sigma_{i2})} + x^{(\sigma_{i1})}), \quad x^{(F)} + x^{(\sigma_{42})} \geqslant x^{(\rho_{31})},$$

$$56 \leqslant x^{(B)} + x^{(W)} + x^{(D)} + \sum_{i=1}^{3}(x^{(\sigma_{i2})} + x^{(\sigma_{i1})}) + x^{(\rho_{31})},$$

$$64 \leqslant x^{(B)} + x^{(W)} + x^{(D)} + x^{(F)} + \sum_{i=1}^{4}(x^{(\sigma_{i2})} + x^{(\sigma_{i1})}),$$

and a resulting optimal control law is (9) for u_1^* and

$$u_2^*(x) = \max\{48 - x^{(B)} - x^{(W)} - \sum_{i=1}^{2}(x^{(\sigma_{i2})} + x^{(\sigma_{i1})}) - x^{(\rho_{21})}, 0\},$$

$$u_3^*(x) = \max\{64 - x^{(B)} - x^{(W)} - x^{(D)} - \sum_{i=1}^{3}(x^{(\sigma_{i2})} + x^{(\sigma_{i1})}) - x^{(\rho_{31})}, 0\},$$

$$u_4^*(x) = \max\{72 - x^{(B)} - x^{(W)} - x^{(D)} - x^{(F)} - \sum_{i=1}^{4}(x^{(\sigma_{i2})} + x^{(\sigma_{i1})}), 0\}, \tag{10}$$

which is again a hedging point policy for all stages, where each stage has to make sure that downstream inventories (including themselves) contain enough goods for any possible demand sequence. This is the solution for the global information case, where all stages know all state variables of the entire system. In contrast, if all stages only knew their state variables (inventory and past orders), they would have to treat incoming orders as unknown disturbances, which would correspond to a sequence of four single stage models as treated in the previous example. Clearly, the solution of the independent case would be a hedging point of

32 for each stage, which would lead to higher inventories and hence higher cost than in the global information case.

The optimal control strategy (9,10) is not unique though. The same cost can be achieved by

$$u_1^*(x) = x^{(W)} + x^{(\sigma_{22})} - x^{(\rho_{11})},$$

$$u_2^*(x) = x^{(D)} + x^{(\sigma_{32})} - x^{(\rho_{21})},$$

$$u_3^*(x) = x^{(F)} + x^{(\sigma_{42})} - x^{(\rho_{31})}$$

and u_4^* as above, or any linear combination of these. This 'order' policy simply states that all incoming goods at W, D and F should be shipped out as soon as they arrive. This results in a simple 'push' chain, where only F makes sure that the whole amount of goods in the chain is sufficient. No inventories are kept in W, D or F but accumulate only at B. The existence of different optimal control strategies in this case is related to the degeneracy of the dual of the linear programs to solve which is caused by the cost structure of this example. In general, excess inventories would be held only at stages whose downstream successor has higher inventory cost. This is ambiguous in our case as the inventory holding costs are all identical for the different stages.

3.3. Third example: one retailer with two suppliers

Finally, we consider the network of Fig. 1, where v_2 is a retailer facing uncertain demand $d(t) \in [0, 8]$ by its customer v_1. The retailer can choose to order $u_1(t) \in [0, 6]$ from supplier v_4 for a cost of 1 per unit or $u_2(t) \in [0, 6]$ from v_3 for a cost of 4 per unit. Inventory holding costs are 1 per unit per time step. The dynamics is given by

$$x(t+1) = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(t) + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} d(t).$$

By solving the corresponding optimization problem, it follows that the resulting feasible region is given by $x_1 + x_2 + x_4 \geqslant 8$ and $x_1 + x_2 + x_3 + x_4 \geqslant 10$, and the resulting robust optimal control is

$$u_1^*(x) = \min\{\max\{20 - x_1 - x_2 - x_3 - x_4, 0\}, 4\},$$

$$u_2^*(x) = \max\{16 - x_1 - x_2 - x_3 - x_4, 0\}.$$

This is a so-called dual base-stock policy, which has been proved to be optimal for certain discrete-time inventory models with two sources [12]. Of course, the optimal strategy depends on the different order costs and is constrained by the maximum allowed order quantities that have been agreed upon by the retailer and each of its suppliers.

4. Conclusions

We have presented a very general supply network model that allows to describe each component of the network as a transducer and to state the entire dynamics of the material and information flow within the network by a system of first-order difference equations. Various sources of uncertainty (e.g., customer demand, transportation times, perishing of goods) can enter the model via disturbances. The model allows recent techniques from constrained robust optimal control to be used for deriving optimal control policies.

For the examples of manufacturing supply chains considered in this study our approach led to the traditional, optimal ordering policies without any prior assumption about the structure of the policy. As the approach is independent of any structural assumptions about the network, it should be possible to derive optimal control also for more complex network types, e.g., cyclic or recurrent flows. The approach also opens the possibility to study the effects of contracts limiting the allowed order range between pairs of nodes or the choice from a set of different suppliers, as alluded to in our last example.

A further interesting issue is to investigate the relationship of this discrete-time approach to the continuous-time supply network models as both appear structurally very similar. In particular, the continuous models mainly focus on stability questions with respect to given control strategy and the question of finding optimal strategies has not yet been followed.

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