TWO OBSERVER-BASED TRACKING ALGORITHMS
FOR A UNICYCLE MOBILE ROBOT

JANUSZ JAKUBIAK*, ERIJEN LEFEBER**
KRZYSZTOF TCHOŃ*, HENK NIJMEIJER**

* Institute of Engineering Cybernetics, Wrocław University of Technology, ul. Janiszewskiego 11/17, 50–372 Wrocław, Poland e-mail: {jjakubia,tchon}@ict.pwr.wroc.pl

** Department of Mechanical Engineering, Eindhoven University of Technology PO Box 513, 5600 MB Eindhoven, The Netherlands e-mail: {A.A.J.Lefeber, H.Nijmeijer}@tue.nl

A trajectory tracking problem for the three-dimensional kinematic model of a unicycle-type mobile robot is considered. It is assumed that only two of the tracking error coordinates are measurable. By means of cascaded systems theory we develop observers for each of the error coordinates and show the $K$-exponential convergence of the tracking error in combined closed-loop observer-controller systems. The results are illustrated with computer simulations.

Keywords: observer, trajectory tracking, mobile robot

1. Introduction

In recent years the stabilization problem of non-holonomic systems has received considerable attention. One of the reasons is that for these systems Brockett’s necessary condition for smooth stabilization is not met (Brockett, 1983) and no smooth time-invariant stabilizing control law exists. For an overview, we refer the reader to the paper (Kolmanovsky and McClamroch, 1995) and references cited therein. The tracking problem has received less attention. In (Fierro and Lewis, 1995; Kanayama et al., 1990; Micaelli and Samson, 1993; Murray et al., 1992; Walsh et al., 1994) a linearization-based tracking control scheme was derived. The idea of input-output linearization was used in (Oelen and van Amerongen, 1994). In (Fliess et al., 1995) the trajectory stabilization problem was dealt with by means of a differentially flat system approach. A dynamic feedback linearization technique for a wheeled mobile robot was presented in (Canudas de Wit et al., 1996). All these publications solve the local tracking problem. The first global tracking control law that we are aware of was proposed in (Samson and Ait-Abderrahim, 1991). Another global tracking result was derived in (Jiang and Nijmeijer, 1997) using integrator backstepping. Global tracking results yielding exponential convergence were presented in (Dixon et al., 1999; Panteley et al., 1998) under a persistence-of-excitation assumption on the reference trajectory. A fuzzy PD controller using look-up tables for the unicycle robot is given in (Ulyanov et al., 1998).

In the paper (Panteley et al., 1998) a state feedback controller for the unicycle-type mobile robot was proposed. Here we adapt this result to develop an output-feedback trajectory tracking controller under the assumption that one of the tracking error coordinates is unknown. Our solution to this problem employs tools of cascaded systems and linear systems theory. By constructing reduced-order observers we have achieved global $K$-exponential stability in the case of uncertain position error, and local exponential stability in the case of unmeasurable orientation. Our stability analysis is based on the results of cascaded systems. A similar problem of motion planning with measurements of the position coordinates was solved in (Guillaume and Rouchon, 1998; Jiang and Nijmeijer, 1999). A part of the results included in this paper, concerning the position error observer, was presented in (Lefeber, 2000; Lefeber et al., 2001).

The organization of the paper is as follows. In Section 2 we recall definitions and theorems from stability theory and formulate the tracking problem. In Section 3 we present an observer for one of the position-error coordinates and the observer-based controller. In Section 4 the case of an unmeasured orientation angle is considered and an appropriate controller is proposed. Computer simulations illustrating the behaviour of both controllers are presented in Section 5. Section 6 concludes the paper.
2. Preliminaries and Problem Formulation

Below we recall some standard concepts of stability theory (Krstić et al., 1995).

2.1. Preliminaries

Definition 1. A continuous function \( \alpha : [0, a) \rightarrow [0, \infty) \) is said to belong to class \( K (\alpha \in K) \) if it is strictly increasing and \( \alpha(0) = 0 \). It is said to belong to class \( K_{\infty} \) if \( a = \infty \) and \( \alpha(r) \rightarrow \infty \) as \( r \rightarrow \infty \).

Definition 2. A continuous function \( \beta : [0, a) \times [0, \infty) \rightarrow [0, \infty) \) is said to belong to class \( KL (\beta \in KL) \) if for each fixed \( s \) the mapping \( \beta(r, s) \) belongs to class \( K \) with respect to \( r \), and if for each fixed \( r \) the mapping \( \beta(r, s) \) is decreasing with respect to \( s \) and \( \beta(r, s) \rightarrow 0 \) as \( s \rightarrow \infty \). It is said to belong to class \( KL_{\infty} \) if, in addition, for each fixed \( s \) the mapping \( \beta(r, s) \) belongs to class \( K_{\infty} \) with respect to \( r \).

Definition 3. The equilibrium point \( x = 0 \) of a non-autonomous system \( \dot{x} = f(t, x) \) is

- locally uniformly asymptotically stable (LUAS) if there exist a function \( \beta \in KL \) and a positive constant \( c \) such that for all \( t > t_0 > 0 \) and for all initial states \( \|x(t_0)\| < c \)
  \[ \|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0); \] (1)

- globally uniformly asymptotically stable (GUAS) if (1) is satisfied with \( \beta \in KL_{\infty} \) for any initial state \( x(t_0) \);

- locally exponentially stable (LES) if (1) is satisfied with \( \beta(r, s) = k \rho^{-\gamma} s, k > 0, \gamma > 0 \) for \( \|x(t_0)\| < c \);

- globally exponentially stable (GES) if (1) is satisfied with \( \beta(r, s) = k \rho^{-\gamma} s, k > 0, \gamma > 0 \) for any initial state \( x(t_0) \).

Definition 4. (Sordalen and Egeland, 1995, Def. 2) The equilibrium point \( x = 0 \) of a non-autonomous system \( \dot{x} = f(t, x) \) is said to be globally \( K \)-exponentially stable if there exist a function \( \kappa \in K \) and a constant \( \gamma > 0 \) such that for all \( (t_0, x(t_0)) \in \mathbb{R}^+ \times \mathbb{R}^n \) we have
\[ \|x(t)\| \leq \kappa(\|x(t_0)\|) e^{-\gamma(t-t_0)}, \quad \forall t \geq t_0 \geq 0. \]

Definition 5. A continuous function \( \phi : \mathbb{R}^+ \rightarrow \mathbb{R} \) is said to be persistently exciting (PE) if there exist constants \( \epsilon_1, \epsilon_2, \delta > 0 \) such that for all \( t \geq 0 \) we have
\[ \epsilon_1 \leq \int_t^{t+\delta} \phi^2(\tau) d\tau \leq \epsilon_2. \]

Lemma 1. (Khalil, 1996) Consider the system, \( x \in \mathbb{R}^2 \),
\[ \dot{x} = \begin{bmatrix} -c_1 & -c_2 \phi(t) \\ c_3 \phi(t) & 0 \end{bmatrix} x. \] (2)
If \( c_1 > 0, c_2 c_3 > 0 \) and \( \phi(t) \) is PE, then the system (2) is GES.

Theorem 1. (Lefeber et al., 2000) Consider the system, \( x \in \mathbb{R}^4 \),
\[ \dot{x} = \begin{bmatrix} -c_1 & -c_2 \phi(t) & d_1 & d_2 \phi(t) \\ \phi(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & -l_2 \phi(t) \\ 0 & 0 & \phi(t) & -l_1 \end{bmatrix} x. \] (3)
When \( \phi(t) \) is PE, \( c_1 > 0, c_2 > 0, l_1 > 0, l_2 > 0 \), then the system (3) is GES.

Theorem 2. (Ioannou and Sun, 1996, Thm. 3.4.6 (v)) The system \( \dot{x} = A(t)x \) is GES if and only if it is GUAS.

Theorem 3. (Krstić et al., 1995, Thm. A.5) Let \( x = 0 \) be an equilibrium point of a non-autonomous system \( \dot{x} = f(t, x) \) and \( D = \{ x \in \mathbb{R}^n : \|x\| < c \} \). Let \( V : D \times \mathbb{R}^n \rightarrow \mathbb{R}^+ \) be a continuously differentiable function such that \( \forall t \geq 0, \forall x \in D \)
\[ \alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|), \] (4)
\[ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -\alpha_3(\|x\|). \] (5)

Then the equilibrium point \( x = 0 \) is

- locally uniformly asymptotically stable if \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) are \( K \) functions on \( [0, c) \);

- globally uniformly asymptotically stable if \( D = \mathbb{R}^n, \alpha_1, \alpha_2 \) are \( K \) functions, and \( \alpha_3 \) is a \( K \) function on \( \mathbb{R}^+ \);

- locally exponentially stable if \( \alpha_i(\rho) = k_i \rho^\gamma \) on \( [0, c), \gamma > 0, k_i > 0, i = 1, 2, 3 \);

- globally exponentially stable if \( D = \mathbb{R}^n, \) and \( \alpha_i(\rho) = k_i \rho^\gamma \) on \( \mathbb{R}^+, \gamma > 0, k_i > 0, i = 1, 2, 3 \).

2.2. Cascaded Systems

Consider a system \( \dot{z} = f(t, z) \) that can be written as
\[ \dot{z}_1 = f_1(t, z_1) + g(t, z_1, z_2) z_2, \]
\[ \dot{z}_2 = f_2(t, z_2), \] (6)
where $z_1 \in \mathbb{R}^n$, $z_2 \in \mathbb{R}^m$, $(z_1, z_2) = (0, 0)$ is an equilibrium point of (6), $f_1(t, z_1)$ is continuously differentiable in $(t, z_1)$ and $f_2(t, z_2)$, $g(t, z_1, z_2)$ are continuous in their arguments, as well as locally Lipschitz in $z_2$ and $(z_1, z_2)$, respectively.

**Assumption 1.** Assume that there exist continuous functions $k_1: \mathbb{R}^+ \to \mathbb{R}$ and $k_2: \mathbb{R}^+ \to \mathbb{R}$ such that

$$\|g(t, z_1, z_2)\| \leq k_1(\|z_2\|) + k_2(\|z_2\|) \|z_1\|, \quad (7)$$

**Corollary 1.** Assume that the subsystem $\dot{z}_1 = f_1(t, z_1)$ of (6) is GES, the subsystem $\dot{z}_2 = f_2(t, z_2)$ is globally $K$-exponentially stable and $g(t, z_1, z_2)$ satisfies (7). Then the cascaded system (6) is globally $K$-exponentially stable.

### 2.3. Problem Formulation

A kinematic model of the unicycle-type mobile robot is given by the following equations:

$$\begin{align*}
\dot{x} &= v \cos \theta, \\
\dot{y} &= v \sin \theta, \\
\dot{\theta} &= \omega.
\end{align*}$$

The geometric interpretation of coordinates $x = (x, y, \theta)$ is shown in Fig. 1. The forward velocity $v$ and the angular velocity $\omega$ serve as the system controls.

![Fig. 1. The unicycle coordinates $(x, y, \theta)$, reference coordinates $(x_r, y_r, \theta_r)$ and moving frame coordinates $(x_e, y_e, \theta_e)$.](image)

We consider the problem of tracking a reference trajectory $x_r = (x_r, y_r, \theta_r)$ generated by the reference system

$$\begin{align*}
\dot{x}_r &= v_r \cos \theta_r, \\
\dot{y}_r &= v_r \sin \theta_r, \\
\dot{\theta}_r &= \omega_r,
\end{align*}$$

where $v_r$ and $\omega_r$ are continuous functions of time.

Following (Kanayama et al., 1990), we express the error coordinates in the moving frame in the form

$$\begin{bmatrix}
x_e \\
y_e \\
\theta_e
\end{bmatrix} =
\begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_r - x \\
y_r - y \\
\theta_r - \theta
\end{bmatrix}, \quad (8)$$

and compute the error dynamics as

$$\begin{bmatrix}
\dot{x}_e \\
\dot{y}_e \\
\dot{\theta}_e
\end{bmatrix} =
\begin{bmatrix}
\omega y_e - v + v_r \cos \theta_r \\
-\omega x_e + v_r \sin \theta_r \\
\omega_r - \omega
\end{bmatrix}. \quad (9)$$

We shall assume that in the dynamic system (8) only two error coordinates are measured while the remaining one is unknown. To this end, we define the output function

$$y = f(x_e), \quad \dim y = 2. \quad (10)$$

Upon defining the output $y$, the dynamic output-feedback state-tracking control problem can be formulated as follows:

*Find velocity control laws $v$ and $\omega$ of the form

$$v = v(t, y, z), \quad \omega = \omega(t, y, z), \quad (10)$$

where $z$ is generated by the observer

$$\dot{z} = g(t, y, z), \quad (11)$$

such that the closed-loop error system of (8), (10) and (11) is globally $K$-exponentially stable.*

The scheme of the closed-loop robot-observer-controller system is depicted in Fig. 2.

### 3. Position-Error Observer

In this section we address the problem of unmeasurable one of position error coordinates $x_e$ or $y_e$. For the purpose of designing an observer-based controller we choose a control law proposed in (Panteley et al., 1998):

$$\omega = \omega_r + c_1 \theta_e, \quad c_1 > 0, \quad (12a)$$

$$v = v_r + c_2 x_e - c_3 \omega_r y_e, \quad c_2 > 0, \quad c_3 > -1. \quad (12b)$$
In this case we obtain, in combination with the error dynamics (8), the cascaded structure
\[
\begin{bmatrix}
\dot{x}_e \\
\dot{y}_e
\end{bmatrix} =
\begin{bmatrix}
-c_2 & (1 + c_3)\omega_r \\
-\omega_r & 0
\end{bmatrix}
\begin{bmatrix}
x_e \\
y_e
\end{bmatrix}
\]
\[\hat{z}_1 = f_1(t, z_1)\]
\[
+ \begin{bmatrix}
c_1 y_e + v_r \cos \theta_e - 1 \\
-c_1 x_e + v_r \sin \theta_e
\end{bmatrix}
\begin{bmatrix}
\theta_e \\
g(t, z_1, z_2) z_2
\end{bmatrix},
\]
\[\hat{\theta}_e = -c_1 \dot{\theta}_e,\]
\[\hat{z}_2 = f_2(t, z_2)\]
The subsystem \(\hat{z}_2 = f_2(t, z_2)\) of (13b) is GES. Assume that \(v_r\) is bounded and \(\omega_r\) is persistently exciting. This being so, from Lemma 1 we obtain that the subsystem \(\hat{z}_1 = f_1(t, z_1)\) is also GES and the interconnection term \(g(t, z_1, z_2)\) satisfies Assumption 1. Hence, by means of Corollary 1, we conclude that the overall closed-loop system (13) is globally \(\mathcal{K}\)-exponentially stable.

Now we assume that we are unable to measure the forward-error \(x_e\), so only the values of \(y_e\) and \(\theta_e\) are available, i.e.
\[Y = [y_e \; y_e]^T = [y_e \; \theta_e].\]

The case of unmeasured \(y_e\) can be addressed analogously.

We notice that the control \(\omega\) in (12a) depends only on the available output \(y_e(\theta_e)\) and therefore it can be directly used in the observer-based controller; in the control \(v\) the unmeasurable state \(x_e\) must be replaced by its estimate. To find an estimate of \(x_e\), we first consider the subsystem \(\hat{z}_1 = f_1(t, z_1)\) of (13a) without the substitution of the control \(v\) (12b), which corresponds to the case of \(\theta_e = 0\). Further, in Proposition 1, we shall show that the same observer can be used for an arbitrary \(\theta_e\). We have
\[
\begin{bmatrix}
\dot{x}_e \\
\dot{y}_e
\end{bmatrix} =
\begin{bmatrix}
0 & \omega_r \\
-\omega_r & 0
\end{bmatrix}
\begin{bmatrix}
x_e \\
y_e
\end{bmatrix}
+ \begin{bmatrix}
v_r - v \\
0
\end{bmatrix}.
\]

We define a new variable \(z\) as a linear combination of the measured and unknown states
\[z = x_e - by_e,\]
where \(b\) is a function of time, still to be determined in order to guarantee the asymptotic stability of the reduced-order observer. Differentiating \(z\) with respect to time along the dynamics (15) yields
\[
\dot{z} = \omega_r y_e + (v_r - v) - by_e + b\omega_r x_e
\]
\[= b\omega_r (x_e - by_e) + b^2 \omega_r y_e + \omega_r y_e + (v_r - v) - by_e
\]
\[= b\omega_r z + \left(b^2 \omega_r + \omega_r - \dot{b}\right) y_e + (v_r - v).
\]

Defining the reduced-order observer dynamics as
\[
\dot{\hat{z}} = b\omega_r \dot{z} + \left(b^2 \omega_r + \omega_r - \dot{b}\right) y_e + (v_r - v),
\]
we obtain for the observation-error \(\hat{z} = z - \dot{z}\)
\[
\dot{\hat{z}} = b\omega_r \hat{z}.
\]

Solutions of (16) satisfy
\[
\hat{z}(t) = \hat{z}(t_0) e^{\int_{t_0}^t b(\tau) \omega_r(\tau) d\tau}.
\]

If we now take \(b = \frac{-l}{2}\) with \(l\) as a positive constant and assume furthermore that \(\omega_r\) is PE, we have the existence of \(\epsilon_1 > 0, \epsilon_2 > 0,\) and \(\delta > 0\) such that
\[
\frac{\epsilon_1}{\delta} (t - t_0) - \int_{t_0}^t \omega^2_r(\tau) d\tau < \frac{\epsilon_2}{\delta} (t - t_0),
\]
which enables us to conclude that (16) is GES and the estimate \(\hat{z}\) tends to \(z\). The estimate of \(x_e\) for the subsystem \(\hat{z}_1 = f_1(t, z_1)\), defined as
\[\hat{x}_e = \hat{z} - \dot{\omega}_r y_e,
\]
converges exponentially to the original state \(x_e\).

Now we plug the observer into the complete closed-loop system:

**Proposition 1.** Consider the tracking error dynamics (8) with output (14) in the closed loop with the control law
\[
\begin{align}
\omega &= \omega_r + c_1 \theta_e, \quad c_1 > 0, \quad (17a) \\
v &= v_r + c_2 \dot{x}_e - c_3 \omega_r y_e, \quad c_2 > 0, \quad c_3 > -1. \quad (17b)
\end{align}
\]
where \( \tilde{x}_e \) is generated by the reduced-order observer
\[
\dot{\hat{x}} = -l_\omega^2 \dot{\hat{x}} + (l^2 \omega_r^3 + \omega_r + l_\omega) y_e + (v_r - v), \tag{18a}
\]
\[
\ddot{x}_e = \ddot{x} - l_\omega y_e, \quad l > 0. \tag{18b}
\]
If \( v_r \) is bounded and \( \omega_r \) is persistently exciting (PE), then the closed-loop system (8), (17) and (18) is globally \( K \)-exponentially stable.

Proof. We can view the closed-loop system (8), (17) and (18) as a cascaded system, i.e. the system of the form (6), where
\[
z_1 = \begin{bmatrix} x_e & y_e & x_e - \hat{x}_e \end{bmatrix}^T, \quad z_2 = \theta_e.
\]
\[
f_1(t, z_1) = \begin{bmatrix} -c_2 & (c_3 + 1) \omega_r & c_2 \\ -\omega_r & 0 & 0 \\ 0 & 0 & -l_\omega^2 \end{bmatrix} z_1,
\]
\[
f_2(t, z_2) = -c_1 z_2,
\]
\[
g(t, z_1, z_2) = \begin{bmatrix} c_1 y_e + v_r \cos \theta_e - \frac{1}{\theta_e} \\ -c_1 x_e + v_r \sin \theta_e - \frac{1}{\theta_e} \\ c_1 y_e + v_r \cos \theta_e - \frac{1}{\theta_e} + l_\omega r \left( -c_1 x_e + v_r \sin \theta_e - \frac{1}{\theta_e} \right) \end{bmatrix}.
\]
To be able to apply Corollary 1, we need to verify the global exponential stability (GES) of the subsystem \( \ddot{z}_1 = f_1(t, z_1) \). To do so, we rewrite it in the cascaded form as
\[
\ddot{\hat{x}}_e = \begin{bmatrix} -c_2 & (c_3 + 1) \omega_r \\ -\omega_r & 0 \\ f_1(t, \hat{x}, \hat{y}) \end{bmatrix} \begin{bmatrix} x_e \\ y_e \\ \hat{x}_e \end{bmatrix} + \begin{bmatrix} c_2 \\ 0 \end{bmatrix} \ddot{z}_2. \tag{19a}
\]
\[
\ddot{z}_2 = -l_\omega^2 \ddot{z}_2. \tag{19b}
\]
Solutions of the subsystem (19b) are given by
\[
\ddot{z}_2(t) = \ddot{z}_2(t_0) e^{-t \int_0^t \omega^2(\tau) d\tau}.
\]
Since \( \omega_r \) is PE, the subsystem (19b) is GES. Furthermore, the term \( \hat{g}(t, z_1, z_2) \) is bounded and the system \( \dot{\hat{x}}_1 = f_1(t, z_1) \) is GES. From Corollary 1 we can conclude that the system \( \dot{z}_1 = f_1(t, z_1) \) is GUAS. Since it is a linear time-varying system, Theorem 2 allows us to conclude that \( \dot{z}_1 = f_1(t, z_1) \) is GES. Since also the system \( \dot{z}_2 = f_2(t, z_2) \) is GES and the boundedness of both \( v_r \) and \( \omega_r \) (cf. Definition 5) guarantees that the condition on \( g(t, z_1, z_2) \) is met, Corollary 1 yields the desired result.

\[\Box\]

4. Orientation-Error Observer

In this section we assume that the available output is
\[
Y = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T = \begin{bmatrix} x_e & y_e \end{bmatrix}^T. \tag{20}
\]
We notice that the unknown orientation error \( \theta_e \) appears in the system equation (8) only as an argument of the sine and cosine. Hence we expect that it is not possible to retrieve the exact value of \( \theta_e \) from the available output, but only \( \sin \theta_e \) and \( \cos \theta_e \), i.e. the value of \( \theta_e \) limited to one full period \( -\pi < \theta_e \leq \pi \).

Therefore we modify the controller (12) to include \( \sin \theta_e \) instead of \( \theta_e \):
\[
\omega = \omega_r + c_1 \sin \theta_e, \quad v = v_r + c_2 x_e - c_3 \omega_r y_e. \tag{21}
\]
The controller (21) ensures the local exponential stability of the closed-loop control system (8) and (21) if \( |\theta_e| \leq \theta_0 < \pi \).

The \( \theta_e \) dynamics in the closed loop are given by
\[
\dot{\theta}_e = -c_1 \sin \theta_e. \tag{22}
\]
Define the Lyapunov function
\[
V(\theta_e) = 1 - \cos \theta_e, \tag{23}
\]
and differentiate it along the dynamics (22):
\[
\dot{V} = -c_1 \sin^2 \theta_e \leq 0. \tag{24}
\]
If \( c_1 \) is a positive constant and \( |\theta_e| \leq \theta_0 < \pi \), the system (22) is asymptotically stable.

We can also find \( \delta(\theta_0) > 0 \) such that
\[
\sin^2 \theta_e \geq \delta(\theta_0)(1 - \cos \theta_e).
\]
Then (24) satisfies
\[
\dot{V} \leq -c_1 \delta(\theta_0)(1 - \cos \theta_e) = -c_1 \delta(\theta_0)V,
\]
and the system (22) is LES.

In order to modify the state-feedback controller (21) to an output-feedback controller for the system (8), (20), we shall apply an observer estimating \( \sin \theta_e \).
To this end, we define the new variable \( z \)

\[
z = \sin \theta_e - av_r y_e.
\]

Its derivative along the dynamics (8) is given by

\[
\dot{z} = (\omega_r - \omega) \cos \theta_e - av_r \omega x_e + av_r \omega x_e - av_r^2 \sin \theta_e.
\]

Set \( \psi = \sin \theta_e \) and its estimate \( \hat{\psi} = \hat{\sin} \theta_e \). Hence we define the observer

\[
\dot{\hat{z}} = -av_r y_e + av_r \omega x_e - av_r^2 \tilde{z} - a^2 v_r^3 y_e, \tag{25a}
\]

\[
\dot{\hat{\psi}} = \dot{\hat{z}} + av_r y_e. \tag{25b}
\]

With the observer error \( \hat{\psi} = \psi - \hat{\psi} \), we obtain the observer error dynamics

\[
\dot{\hat{\psi}} = (\omega_r - \omega) \cos \theta_e - av_r^2 \tilde{\psi}. \tag{26}
\]

Before we define the complete control law for the system (8), we examine the stability of the combined observer (25) with the control of angular velocity \( \dot{\omega} \)

\[
\omega = \omega_r + c_1(t) \hat{\psi}, \tag{27}
\]

where \( c_1(t) \) is a non-negative function of time. The system consisting of \( \theta_e \) and the observer error \( \hat{\psi} \) with the control (27) yields

\[
\dot{\hat{\psi}} = -c_1(t) \hat{\psi},
\]

\[
\dot{\hat{\psi}} = -c_1(t) \hat{\psi} \cos \theta_e - av_r^2 \tilde{\psi}.
\]

Then, for \( \hat{\psi} = \sin \theta_e - \hat{\psi} \), we obtain

\[
\dot{\hat{\theta}} = -c_1(t) (\sin \theta_e - \hat{\psi}),
\]

\[
\dot{\hat{\psi}} = -c_1(t) \frac{1}{2} \sin 2\theta_e + c_1(t) \hat{\psi} \cos \theta_e - av_r^2 \tilde{\psi}. \tag{28}
\]

Define a Lyapunov function \( V \) for the system (28):

\[
V = (1 - \cos \theta_e) + \frac{1}{2} \tilde{\psi}^2. \tag{29}
\]

The derivative of \( V \) along trajectories (28) is equal to

\[
\dot{V} = -c_1(t) \sin^2 \theta_e + c_1(t) \hat{\psi} \sin \theta_e - c_1(t) \sin 2\theta_e \hat{\psi}
\]

\[
- - (-c_1(t) \cos \theta_e + av_r^2) \tilde{\psi}^2 - c_1(t) \frac{1}{2} \sin 2\theta_e \hat{\psi}
\]

\[
\leq -c_1(t) \sin^2 \theta_e - (c_1(t) \cos \theta_e + av_r^2) \tilde{\psi}^2
\]

\[
+ c_1(t) \left( |\sin \theta_e| + \frac{1}{2} |\sin 2\theta_e| \right) |\hat{\psi}|.
\]

Since \( \frac{1}{2} |\sin 2\theta_e| \leq |\sin \theta_e| \) and \( c_1(t) \cos \theta_e \leq c_1(t) \),

\[
\dot{V} \leq -c_1(t) \sin^2 \theta_e - (c_1(t) + av_r^2) \tilde{\psi}^2 + 2c_1(t) |\sin \theta_e| |\hat{\psi}|.
\]

Assume that \( c_1(t) = \frac{1}{2} \gamma \alpha v_r^2 \), where \( 0 < \gamma < 1 \). Then

\[
\dot{V} \leq -av_r^2 \left( \frac{\gamma}{2} (|\sin \theta_e| - |\hat{\psi}|)^2 + (1 - \gamma) \tilde{\psi}^2 \right) \leq 0. \tag{30}
\]

We also assume that \( \theta_e \) is inside the interval \((-\pi, \pi)\), and we choose a very small constant \( \delta \) such that \( \cos \theta_e > -1 + \delta \) and \( \sin^2 \theta_e \geq \delta (1 - \cos \theta_e) \) hold and (30) can be transformed into the following form:

\[
\dot{V} \leq -av_r^2 \left( \alpha^2 \sin^2 \theta_e + \beta^2 \tilde{\psi}^2 - 2\alpha \beta |\sin \theta_e| |\hat{\psi}| \right.
\]

\[
+ \frac{\eta}{\delta} \sin^2 \theta_e + \left( \frac{\eta}{\delta} + \kappa \right) \tilde{\psi}^2 \right)
\]

\[
\leq -av_r^2 \left( \alpha |\sin \theta_e| - \beta |\hat{\psi}| \right)^2
\]

\[
- \eta av_r^2 \left( 1 - \cos \theta_e + \frac{1}{2} \tilde{\psi}^2 \right)
\]

\[
\leq - \eta av_r^2 \tilde{\psi},
\]

so the system (28) is locally exponentially stable. For given \( 0 < \gamma < 1 \) and small \( \delta > 0 \) we find constants \( \alpha, \beta, \eta \), and \( \kappa \) by solving the set of equations

\[
\alpha \beta = \gamma \frac{\gamma}{2},
\]

\[
\alpha^2 + \eta \frac{\eta}{\delta} = \gamma \frac{\gamma}{2},
\]

\[
\beta^2 + \frac{\eta}{\delta} + \kappa = 1 - \frac{\gamma}{2}.
\]

Finally, we shall extend our deliberations to the entire closed-loop controller.

**Proposition 2.** Consider the system (8) with the control law

\[
v = v_r + c_2 x_e - c_3 \omega_r y_e, \tag{31}
\]

\[
\omega = \omega_r + \frac{1}{2} \gamma \alpha v_r^2 \hat{\psi},
\]

and the observer given by

\[
\dot{\tilde{z}} = -av_r y_e + av_r \omega x_e - av_r^2 \tilde{z} - a^2 v_r^3 y_e,
\]

\[
\dot{\hat{\psi}} = \tilde{z} + av_r y_e,
\]

where \( c_3 > -1, c_2, \) and \( a \) are positive constants, \( 0 < \gamma < 1 \). If \( v_r, \omega_r \) are bounded and persistently exciting and \( \dot{v}_r, \dot{\omega}_r \) are bounded, the closed-loop system (8), (31) and (32) is locally exponentially stable.
Proof. The closed-loop dynamics, defined by (8), (31) and (32):
\[ \dot{x}_e = \left(1 + c_3\right)\omega_r + \frac{\gamma}{2}a \psi^2 \dot{\psi} \right) y_e
+ v_r (\cos \theta_e - 1) - c_2 x_e, \]
\[ \dot{y}_e = - \left(\omega_r + \frac{\gamma}{2}a \psi^2 \dot{\psi}\right) x_e + v_r \sin \theta_e, \]
\[ \dot{\theta}_e = - \frac{\gamma}{2}av^2_e \dot{\psi}. \]
\[ \dot{\psi} = - \frac{\gamma}{2}av^2_e \left(\frac{1}{2} \sin 2\theta_e - \dot{\psi} \cos \theta_e\right) - av^2_e \dot{\psi}, \]
can be transformed to the cascaded form
\[
\begin{bmatrix}
\dot{x}_e \\
\dot{y}_e \\
\dot{\theta}_e \\
\dot{\psi}
\end{bmatrix}
= 
\begin{bmatrix}
eg_2 & (1 + c_3)\omega_r & 0 \\
-\omega_r & \omega_r & 0 \\
-\frac{\gamma}{2}av^2_e & \frac{\gamma}{2}av^2_e & 0 \\
0 & -\frac{\gamma}{2}av^2_e & -\frac{\gamma}{2}av^2_e \\
\end{bmatrix}
\begin{bmatrix}
x_e \\
y_e \\
\theta_e \\
\psi
\end{bmatrix}
+ 
g(t, [x_e, y_e]^T, [\theta_e, \dot{\psi}]^T, \frac{\theta_e}{\dot{\psi}}). \tag{34a}
\]
\[ \dot{z}_1 = f_1(t, z_1), \]
\[ \dot{z}_2 = f_2(t, z_2), \]
\[ g(t, [x_e, y_e]^T, [\theta_e, \dot{\psi}]^T) \]
\[ = \begin{bmatrix}
\frac{\gamma}{2}av^2_e y_e & \frac{1}{2} \cos \theta_e \dot{s} + v_r & \frac{1}{2} \sin \theta_e \dot{s} \\
0 & 0 & 0 \\
\frac{\gamma}{2}av^2_e x_e & \frac{1}{2} \cos \theta_e \dot{s} + v_r & \frac{1}{2} \sin \theta_e \dot{s} \\
0 & 0 & 0
\end{bmatrix}. \]
If \( v_r \) is persistently exciting and bounded, the subsystem (34b) is locally exponentially stable and the interconnection term satisfies Assumption 1. Furthermore, if \( \omega_r \) is PE, we obtain that the subsystem \( \dot{z}_1 = f_1(t, z_1) \) is GES. From Corollary 1 we conclude that the system (33) is locally exponentially stable.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig3.png}
\caption{Tracking errors and inputs for the state-feedback controller (12) with the controller gains (35).}
\end{figure}

Remark 1. We notice that both forward and angular velocities need to be persistently exciting. The assumption on \( v_r \) is needed to ensure the convergence of the observer, while the condition on \( \omega_r \) results from the controller used.

5. Simulations

In order to illustrate the behaviour of the output-feedback state-tracking controllers derived in this paper, a number of simulations have been done. The simulations were carried out using MATHEMATICA. We considered the problem of tracking a circle with a constant velocity, i.e. a reference trajectory that is given by \( v_r = 1, \omega_r = 1 \), where, as in (Jiang and Nijmeijer, 1997), we took for the initial error \( (x_r(0), y_r(0), \theta_r(0)) = (-0.5, 0.5, 1) \).

For comparison, we first simulated the state-feedback controller (12) using the gains
\[ c_1 = 5.9460, \quad c_2 = 1.3522, \quad c_3 = -0.4142, \tag{35} \]
which arise by minimizing the cost
\[ \int_0^\infty x^2_e(\tau) + y^2_e(\tau) + (v_r(\tau) - v(\tau))^2 \, d\tau \]
for the system (15) with an arbitrarily chosen convergence of \( \theta_r \). The resulting performance is depicted in Fig. 3.

For comparison, a simulation was performed for the controller (21) with the use of the same initial values and gains. The results are shown in Fig. 4.

For studying the behaviour of the position-error observer, we simulated the output-feedback controller (17) and (18) with the controller gains (35) and the observer gain
\[ l = 24.7461, \tag{36} \]
which guarantees that the error dynamics for the convergence of the controller (17) and (18) are comparable to the state-feedback controller (12). The results are depicted in Fig. 5.
Fig. 4. Tracking errors and inputs for the state-feedback controller (21).

Fig. 5. Tracking errors and inputs for the output-feedback controller (17) and (18) with the controller and observer gains (35) and (36).

Fig. 6. Tracking errors and inputs for the output-feedback controller (31) and (32).

6. Concluding Remarks

In this paper we have designed two output-feedback tracking controllers for the unicycle-type mobile robot assuming that only the measurements of two out of three state variables are available. It corresponds to two situations encountered in some pursuit navigation problems: the position-error observer can be used when one of the distances between escaper and pursuer robots is outside the range of pursuer robot sensors, or measured with high disturbance error, or for any other reason unreliable. The second observer, estimating the orientation error, replaces the requirement of practically difficult measurements of the orientation angle with much simpler measurements of distances. Both observers can be used either to replace the real sensors or to stand as a parallel system to provide data for the controller in the case when the measurements are temporarily unavailable. In our solution we took advantage of the fact that modified observers for linear systems might be in some cases applied to nonlinear systems. We considered the tracking problem when one of the trajectory tracking error coordinates was unmeasurable. When the position error coordinate is unavailable, we are able to achieve global $\mathcal{K}$-exponential stability. In the case of the unmeasured orientation angle, only local exponential stability was shown.

In Fig. 6 the results for the orientation-angle observer (32), combined with the controller (31), are presented. To draw a comparison of this controller with the previous ones, we used the same controller gains (35) and the observer gains

$$a = 10, \quad \gamma = 0.5,$$

which ensure the fast convergence of the observer error.
It is worth noticing that the stability of both controllers assumed persistent excitation of the angular reference velocity. As a result, the output-feedback controllers are not capable of tracking, e.g., straight line trajectories. The additional requirement of the persistent excitation of $v_r$, appearing in the case of an unmeasured orientation angle error, means that the turning of the steering wheel is not a sufficient movement to estimate the orientation angle of the vehicle. A way of overcoming the PE-problem with the use of the concept of $\omega$–PE was presented in (Loria et al., 1999). We believe that it is worth investigating if it also applies to the output-feedback case.

Acknowledgement

The work of the first and the third author was supported by the State Committee for Scientific Research within a statutory research project.

References


Received: 8 September 2001
Revised: 24 April 2002
Re-revised: 8 October 2002