Lyapunov stability: Why uniform results are important, and how to obtain them

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Result derived last week during visit in Lund

Consider the following dynamics

$$\begin{aligned}\dot{\rho}_e &= -S(\omega_r)\rho_e + \nu_e \\ \dot{\nu}_e &= -S(\omega_r)\nu_e - (k_\rho \rho_e + k_\nu \nu_e)\end{aligned}$$

where $S(\omega_r) = -S(\omega_r)^T$, $k_\rho > 0$, $k_\nu > 0$, and $\omega_r(t)$, $\dot{\omega}_r(t)$ bounded. Differentiating

$$V = \frac{k_\rho}{2}\nu_e^T \nu_e + \frac{1}{2}(k_\rho \rho_e + k_\nu \nu_e)^T (k_\rho \rho_e + k_\nu \nu_e)$$

along solutions yields (using $x^T S(\omega_r) x = 0$):

$$\begin{aligned}\dot{V} &= k_\rho \nu_e^T \dot{\nu}_e + (k_\rho \dot{\rho}_e^T + k_\nu \dot{\nu}_e^T)(k_\rho \rho_e + k_\nu \nu_e) \\ &= -k_\rho \nu_e^T S(\omega_r)\nu_e - k_\rho \nu_e^T (k_\rho \rho_e + k_\nu \nu_e) + k_\rho \rho_e^T S(\omega_r)(k_\rho \rho_e + k_\nu \nu_e) + \\ &\quad + k_\rho \nu_e^T (k_\rho \rho_e + k_\nu \nu_e) + k_\nu \nu_e^T S(\omega_r)(k_\rho \rho_e + k_\nu \nu_e) - k_\nu (k_\rho \rho_e + k_\nu \nu_e)^T (k_\rho \rho_e + k_\nu \nu_e) \\ &= -k_\nu (k_\rho \rho_e + k_\nu \nu_e)^T (k_\rho \rho_e + k_\nu \nu_e) \leq 0.\end{aligned}$$

Result derived last week during visit in Lund

Consider the following dynamics

$$\dot{\rho}_e = -S(\omega_r)\rho_e + \nu_e$$
$$\dot{\nu}_e = -S(\omega_r)\nu_e - (k_\rho \rho_e + k_\nu \nu_e)$$

where $S(\omega_r) = -S(\omega_r)^T$, $k_\rho > 0$, $k_\nu > 0$, and $\omega_r(t)$, $\dot{\omega}_r(t)$ bounded. Differentiating

$$V = \frac{k_\rho}{2}\nu_e^T \nu_e + \frac{1}{2}(k_\rho \rho_e + k_\nu \nu_e)^T (k_\rho \rho_e + k_\nu \nu_e)$$

along solutions yields (using $x^T S(\omega_r) x = 0$):

$$\dot{V} = -k_\nu (k_\rho \rho_e + k_\nu \nu_e)^T (k_\rho \rho_e + k_\nu \nu_e) \leq 0.$$

Only negative semi-definite. LaSalle fails to conclude GAS. Nevertheless, consider

$$k_\rho \dot{\rho}_e + k_\nu \dot{\nu}_e = k_\rho \nu_e - S(\omega_r)(k_\rho \rho_e + k_\nu \nu_e) + k_\nu (k_\rho \rho_e + k_\nu \nu_e)$$

We can hope to be able to conclude that both $k_\rho \rho_e + k_\nu \nu_e$ and $k_\rho \nu_e$ converge to zero

Questions

Assume that $\lim_{t\to\infty} x(t) = 0$. Do we have $\lim_{t\to\infty} \dot{x}(t) = 0$?

No: Consider $x(t) = e^{-t} \sin e^{2t}$ for which $\dot{x}(t) = -e^{-t} \sin e^{2t} + 2e^{t} \cos e^{2t}$

Assume that $x(t)$ is bounded and $\lim_{t\to\infty} \dot{x}(t) = 0$. Do we have $\lim_{t\to\infty} x(t) = C$ for some constant C?

No: Consider $\dot{x}(t) = \frac{\cos(\ln(t+1))}{t+1}$ for which $x(t) = \sin(\ln(1+t))$

We need some results to complete the proof

Commonly used tools for completing the proof

Lemma (Barbălat, 1959)

Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ *be a uniformly continuous function (e.g.,* $\dot{\phi}$ *bounded). Suppose that* $\lim_{t\to\infty} \int_0^t \phi(\tau)\mathrm{d}\tau$ *exists and is finite. Then* $\lim\limits_{t\to\infty} \phi(t) = 0$.

Idea: For $\phi(t)$ use $\dot{V}(t)$.

Lemma (Micaelli, Samson, 1993)

Let $f : \mathbb{R}_+ \to \mathbb{R}$ *be any differentiable function. If* $\lim_{t\to\infty} f(t) = 0$ *and*

$$\dot{f}(t) = f_0(t) + \eta(t) \qquad t \geq 0$$

where f_0 *is a uniformly continuous function (e.g.,* $\dot{f}_0$ *is bounded) and* $\lim_{t\to\infty} \eta(t) = 0$, *then* $\lim_{t\to\infty} \dot{f}(t) = \lim_{t\to\infty} f_0(t) = 0$.

Idea: Signal chasing by (repeatedly) applying to signals that converge to zero

Completing the proof of last week

Consider the following dynamics

$$\dot{\rho}_e = -S(\omega_r)\rho_e + \nu_e$$
$$\dot{\nu}_e = -S(\omega_r)\nu_e - (k_\rho \rho_e + k_\nu \nu_e)$$

where $S(\omega_r) = -S(\omega_r)^T$, $k_\rho > 0$, $k_\nu > 0$, and $\omega_r(t)$, $\dot{\omega}_r(t)$ bounded. Differentiating

$$V = \frac{k_\rho}{2}\nu_e^T \nu_e + \frac{1}{2}(k_\rho \rho_e + k_\nu \nu_e)^T (k_\rho \rho_e + k_\nu \nu_e)$$

along solutions yields (using $x^T S(\omega_r) x = 0$):

$$\dot{V} = -k_\nu (k_\rho \rho_e + k_\nu \nu_e)^T (k_\rho \rho_e + k_\nu \nu_e) \leq 0.$$

Applying Barbălat to $\dot{V}$ results in: $k_\rho \rho_e + k_\nu \nu_e$ converges to 0. Consider

$$k_\rho \dot{\rho}_e + k_\nu \dot{\nu}_e = k_\rho \nu_e - S(\omega_r)(k_\rho \rho_e + k_\nu \nu_e) + k_\nu (k_\rho \rho_e + k_\nu \nu_e)$$

Applying Micaelli-Samson result in: $k_\rho \nu_e$ converges to 0.

Example (Panteley, Loría, Teel, 1999)

Consider the system

$$\dot{x} = \begin{cases} \frac{1}{1+t} & \text{if } x \leq -\frac{1}{1+t} \\ -x & \text{if } |x| \leq \frac{1}{1+t} \\ -\frac{1}{1+t} & \text{if } x \geq \frac{1}{1+t} \end{cases}$$

For each $r > 0$ and $t_0 \geq 0$ there exist $k > 0$ and $\gamma > 0$ such that for all $t \geq t_0$ and $|x(t_0)| \leq r$:

$$|x(t)| \leq k|x(t_0)|e^{-\gamma(t-t_0)} \qquad \forall t \geq t_0 \geq 0$$

However, always a bounded (arbitrarily small) additive perturbation $\delta(t,x)$ and a constant $t_0 \geq 0$ exist such that the trajectories of the perturbed system $\dot{x} = f(t,x) + \delta(t,x)$ are unbounded.

Main reason for this negative result: the constants k and γ are allowed to depend on t_0, i.e., for each value of t_0 different constants k and γ may be chosen.

Stability definitions

Consider the system

$$\dot{x} = f(t, x) \qquad \text{where} \qquad f(t, 0) = 0 \qquad \forall t \geq 0 \tag{1}$$

- The equilibrium point $x = 0$ of (1) is said to be globally asymptotically stable (GAS) if for all $t_0 \in \mathbb{R}_+$ a function $\beta \in \mathcal{KL}$ exists such that for all $x(t_0) \in \mathbb{R}^n$

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) \qquad \forall t \geq t_0 \geq 0$$

- The equilibrium point $x = 0$ of (1) is said to be uniformly globally asymptotically stable (UGAS) if a function $\beta \in \mathcal{KL}$ exists such that for all $(t_0, x(t_0)) \in \mathbb{R}_+ \times \mathbb{R}^n$

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) \qquad \forall t \geq t_0 \geq 0$$

Robustness to perturbations for UGAS

Lemma (Khalil 1996, Lemma 5.3)

Let $x = 0$ be a uniformly asymptotically stable equilibrium point of the nominal system $\dot{x} = f(t, x)$ where $f : \mathbb{R}_+ \times B_r \to \mathbb{R}^n$ is continuously differentiable, and the Jacobian $\left[\frac{\partial f}{\partial x}\right]$ is bounded on B_r, uniformly in t. Then one can determine constants $\Delta > 0$ and $R > 0$ such that for all perturbations $\delta(t, x)$ that satisfy the uniform bound $\|\delta(t, x)\| \le \delta < \Delta$ and all initial conditions $\|x(t_0)\| \le R$, the solution $x(t)$ of the perturbed system $\dot{x} = f(t, x) + \delta(t, x)$ satisfies

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0) \qquad \forall t_0 \le t \le t_1 \qquad \text{and} \qquad \|x(t)\| \le \rho(\delta) \qquad \forall t \ge t_1$$

for some $\beta \in \mathcal{KL}$ and some finite time t_1, where $\rho(\delta)$ is a class $\mathcal{K}$ function of δ. Furthermore, if $x = 0$ is a uniformly globally exponentially stable equilibrium point, we can allow for arbitrarily large δ by choosing $R > 0$ large enough.

Problem

Lesson learned from example

For robustness we need uniform global asymptotic stability.

Subject of this talk

How to show UGAS when we do not have a proper Lyapunov function, i.e, when $\dot{V}$ is negative semi-definite, but are able to complete the proof using Barbălat + signal chasing

After this talk, you (hopefully) know:

- How to complete a proof using Barbălat + signal chasing
- Using Barbălat + signal chasing shows only GAS, whereas we want UGAS.
- How to show UGAS using different tools

Standard approach (general case)

More general case: $\dot{x}_1 = f_1(x_1, x_2, t)$, $\dot{x}_2 = f_2(x_1, x_2, t)$

- Lyapunov function: $V(x_1, x_2, t)$ positive definite.
- Derivative along dynamics: $\dot{V}(x_1, t)$ negative semi-definite.
- Using Barbălat: $x_1 \to 0$.
- Using Micaelli, Samson: $f_1(0, x_2, t) \to 0$, which implies $x_2 \to 0$.

Or even more general: $\dot{x}_1 = f_1(x_1, x_2, x_3, t)$, $\dot{x}_2 = f_2(x_1, x_2, x_3, t)$, $\dot{x}_3 = f_3(x_1, x_2, x_3, t)$

- Lyapunov function: $V(x_1, x_2, x_3, t)$ positive definite.
- Derivative along dynamics: $\dot{V}(x_1, t)$ negative semi-definite.
- Using Barbălat: $x_1 \to 0$.
- Using Micaelli, Samson: $f_1(0, x_2, x_3, t) \to 0$, which implies $x_2 \to 0$.
- Using Micaelli, Samson: $f_2(0, 0, x_3, t) \to 0$, which implies $x_3 \to 0$.

Or even more general…

I this way we show global asymptotic stability. However, we look for uniform result!

Matrosov like theorem (Loría et.al., 2005)

Consider the dynamical system

$$\dot{x} = f(t, x) \qquad x(t_0) = x_0 \qquad f(t, 0) = 0 \tag{2}$$

$f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ locally bounded, continuous a.e., locally uniformly continuous in t. If there exist

- ∘ j differentiable functions $V_i : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$, bounded in t, and
- ∘ continuous functions $Y_i : \mathbb{R}^n \to \mathbb{R}$ for $i \in \{1, 2, \ldots j\}$ such that
- V_1 is positive definite and radially unbounded,
- $\dot{V}_i(t, x) \leq Y_i(x)$, for all $i \in \{1, 2, \ldots, j\}$,
- $Y_i(x) = 0$ for $i \in \{1, 2, \ldots, k-1\}$ implies $Y_k(x) \leq 0$, for all $k \in \{1, 2, \ldots, j\}$,
- $Y_i(x) = 0$ for all $i \in \{1, 2, \ldots, j\}$ implies $x = 0$,

then the origin $x = 0$ of (2) is uniformly globally asymptotically stable.

Question: how to determine suitable functions V_i and Y_i (for $i > 1$)?

Lund-result (revisited)

Dynamics:

$$\dot{\rho}_e = -S(\omega_r)\rho_e + \nu_e$$
$$\dot{\nu}_e = -S(\omega_r)\nu_e - (k_\rho \rho_e + k_\nu \nu_e)$$

Lypunov function candidate:

$$V_1 = \frac{k_\rho}{2}\nu_e^T \nu_e + \frac{1}{2}(k_\rho \rho_e + k_\nu \nu_e)^T (k_\rho \rho_e + k_\nu \nu_e)$$

Differentiating along solutions:

$$\dot{V}_1 = -k_\nu (k_\rho \rho_e + k_\nu \nu_e)^T (k_\rho \rho_e + k_\nu \nu_e) = Y_1 \leq 0$$

Furthermore:

$$k_\rho \dot{\rho}_e + k_\nu \dot{\nu}_e = k_\rho \nu_e + \underbrace{k_\nu (k_\rho \rho_e + k_\nu \nu_e) - S(\omega_r)(k_\rho \rho_e + k_\nu \nu_e)}_{\eta}$$

Consider $V_2 = -(k_\rho \rho_e + k_\nu \nu_e)^T (k_\rho \dot{\rho}_e + k_\nu \dot{\nu}_e)$. Then

$$\begin{aligned}
\dot{V}_2 &= -(k_\rho \dot{\rho}_e + k_\nu \dot{\nu}_e)^T (k_\rho \dot{\rho}_e + k_\nu \dot{\nu}_e) - (k_\rho \rho_e + k_\nu \nu_e)^T (k_\rho \ddot{\rho}_e + k_\nu \ddot{\nu}_e) \\
&= -(k_\rho \nu_e + \eta)^T (k_\rho \nu_e + \eta) - (k_\rho \rho_e + k_\nu \nu_e)^T (k_\rho \ddot{\rho}_e + k_\nu \ddot{\nu}_e) \\
&= -k_\rho^2 \nu_e^T \nu_e + \underbrace{2\eta^T k_\rho \nu_e + \eta^T \eta - (k_\rho \rho_e + k_\nu \nu_e)^T (k_\rho \ddot{\rho}_e + k_\nu \ddot{\nu}_e)}_{=0 \text{ for } Y_1 = 0} \leq Y_2(\rho_e, \nu_e)
\end{aligned}$$

Note that $Y_1 = 0$ implies $Y_2 \leq 0$. Furthermore, $Y_1 = Y_2 = 0$ implies $\rho_e = \nu_e = 0$.

Therefore: uniform global asymptotic stability.

New standard approach for uniform results

More general case: $\dot{x}_1 = f_1(x_1, x_2, t), \dot{x}_2 = f_2(x_1, x_2, t)$

- Lyapunov function: $V_1(x_1, x_2, t)$ positive definite.
- Derivative along dynamics: $\dot{V}_1(x_1, t) \leq Y_1(x_1)$ negative semi-definite.
- Use $V_2 = -x_1^T \dot{x}_1$. Then $\dot{V}_2 = Y_2$.
- Note that $Y_1 = 0$ implies $Y_2 = -f_1(0, x_2, t)^2 \leq 0$. Furthermore $Y_1 = Y_2 = 0$ implies $x_1 = x_2 = 0$.
- Conclusion: uniform global asymptotic stability.

Or even more general: $\dot{x}_1 = f_1(x_1, x_2, x_3, t), \dot{x}_2 = f_2(x_1, x_2, x_3, t), \dot{x}_3 = f_3(x_1, x_2, x_3, t)$

- Lyapunov function: $V_1(x_1, x_2, x_3, t)$ positive definite.
- Derivative along dynamics: $\dot{V}_1(x_1, t) \leq Y_1(x_1)$ negative semi-definite.
- Use $V_2 = -x_1^T \dot{x}_1$. Then $\dot{V}_2 = Y_2$.
- $Y_1 = 0$ implies $Y_2 = -f_1(0, x_2, x_3, t)^2 \leq 0$. Furthermore $Y_1 = Y_2 = 0$ implies $x_1 = x_2 = 0$.
- Use $V_3 = -x_2^T \dot{x}_2$. Then $\dot{V}_3 = Y_3$.

Conclusions

- We recalled the standard approach of using Barbălat + signal chasing
- We illustrated the need for uniform asymptotic stability
- We showed how to modify the standard approach for showing GAS to prove UGAS instead.