Lyapunov stability: Why uniform results are important, and how to obtain them

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1 Why are uniform results important?

Consider the dynamical system

$$\dot{x}(t) = \begin{cases} -\frac{1}{t+1}\operatorname{sgn}(x) & \text{if } |x| \geq \frac{1}{t+1} \\ -x & \text{if } |x| \geq \frac{1}{t+1} \end{cases} \tag{1}$$

where

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}.$$

Then for each $r > 0$ and $t_0 \geq 0$ there exist constants $k > 0$ and $\gamma > 0$ such that for all $t \geq t_0$ and $|x(t_0)| < r$:

$$|x(t)| \leq k|x(t_0)|e^{-\gamma(t-t_0)} \qquad \forall t \geq t_0 \geq 0. \tag{2}$$

However, always a bounded (arbitrarily small) additive perturbation $\delta(t,x)$ and a constant $t_0 \geq 0$ exist such that the trajectories of the perturbed system

$$\dot{x}(t) = \delta(t,x) + \begin{cases} -\frac{1}{t+1}\operatorname{sgn}(x) & \text{if } |x| \geq \frac{1}{t+1} \\ -x & \text{if } |x| \geq \frac{1}{t+1} \end{cases}$$

are unbounded, see [4].

One of the reasons for this negative result is that in (2) the constants k and γ are allowed to depend on t_0, i.e., for each value of t_0 different constants k and γ may be chosen. By requiring that those constants are independent of t_0, we obtain *uniform* asymptotic stability. As shown in [2, Lemma 5.3] uniform asymptotic stability gives rise to some robustness that is not guaranteed by asymptotic stability.

2 How to obtain uniform asymptotic stability

Typically when proving stability of a nonlinear dynamical system, one can find a Lyapunov function candidate of which the time-derivative along solutions is not negative definite, but only negative semi-definite. Often, the Lemma of Barbălat [1] is used to complete the proof and show asymptotic stability.

Though this approach works to show asymptotic stability, it does not yield *uniform* asymptotic stability, which from a robustness point of view is important as mentioned in the previous section. An alternative way to complete the proof is by usings Matrosov's Theorem, or one of its generalisations:

Theorem 2.1 (cf. [3, Theorem 1]) *Consider the dynamical system*

$$\dot{x} = f(t,x) \qquad\qquad x(t_0) = x_0 \tag{3}$$

with $f(t,0) = 0$, $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ locally bounded, continuous and locally uniformly continuous in t.

If there exist j differentiable functions $V_i : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$, bounded in t, and continuous functions $Y_i : \mathbb{R}^n \to \mathbb{R}$ for $i \in \{1,2,\dots j\}$ such that

- *V_1 is positive definite,*
- *$\dot{V}_i(t,x) \leq Y_i(x)$, for all $i \in \{1,2,\dots,j\}$,*
- *$Y_i(x) = 0$ for $i \in \{1,2,\dots,k-1\}$ implies $Y_k(x) \leq 0$, for all $k \in \{1,2,\dots,j\}$,*
- *$Y_i(x) = 0$ for all $i \in \{1,2,\dots,j\}$ implies $x = 0$,*

then the origin $x = 0$ of (3) *is uniformly globally asymptotically stable (UGAS).*

So in addition to the Lyapunov function V_1, auxiliary functions V_i need to be found to complete the proof. We will present candidates for the functions V_i that often work to complete a stability proof and show *uniform* asymptotic stability.

References

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[3] A. Loría, E. Panteley, and A. Teel, "A nested Matrosov theorem and persistency of excitation for uniform convergence in stable nonautonomous systems," *IEEE Transactions on Automatic Control*, vol. 50, no. 2, pp. 183-198, 2005.

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