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Almost Global Tracking Control of a Quadrotor UAV on SE(3)

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Abstract— **In this paper we present a controller which achieves uniform almost global asymptotic stability of the tracking error dynamics for a quadrotor on SE(3). By considering the tracking control of a quadrotor UAV on SE(3) we avoid singularities of Euler angles and ambiguity of quaternions and by explicitly taking into account the constraint of non-zero total thrust in our controller design, we do not achieve a local result but almost global asymptotic stability of the tracking controller. Second, we consider the position tracking error in the body-frame of the reference UAV. As a result, contrary to most existing tracking controllers, our control action becomes independent of the definition of the inertial frame. We illustrate by simulations that even in the presence of small disturbances and sampled, delayed, and noisy measurements the controller achieves stable tracking error dynamics for which errors converge to some region near the origin.**

I. INTRODUCTION

During the last decade, the interest in control of Unmanned Aerial Vehicles (UAVs) has increased considerably. Many papers deal with hovering based on linearisation, but there is also a vast amount of literature considering trajectory tracking using nonlinear control techniques. Many approaches have been considered, such as, amongst others, feedback linearisation [1], [2], backstepping [3]–[6], MPC [7], LMIs [8], and sliding mode control [9], [10]. See also [11] and references therein.

All of these controllers are based on Euler angles, exhibiting singularities when describing rotational motions, significantly reducing their applicability for achieving large angular maneuvers. To overcome this issue, in [12] attitude stabilization of a UAV has been considered using quaternions. However, as quaternions have ambiguities in representing an attitude (two quaternions can be associated with an attitude), these should be carefully resolved since otherwise the system becomes sensitive to small measurement noises [13] and may exhibit unwinding behavior [14].

To overcome the singularities of Euler angles and the ambiguity of quaternions, the tracking control of UAVs has also been considered directly on the special Euclidian group SE(3), [15]–[20]. A crucial assumption in these papers is that the total thrust is non-zero. This assumption is required for having a well-defined controller. However, as this total thrust results from the controller, an ambiguity in the stability proof arises. This ambiguity can be overcome by assuming that the reference total thrust is bounded away from zero and initial errors are sufficiently small. The price to pay for this repair is that only a local result remains, with a relatively small region of attraction. In this paper we only need to assume that the total thrust for the *reference* stays away from zero. Our controller design then guarantees that the total thrust for the UAV itself stays away from zero. The only paper that we are aware of with a similar result is [21]. In that paper, angular velocities are considered as input, and almost global asymptotic stability has been proven. In this paper we consider torques as input and we show *uniform* almost global asymptotic stability.

A common property of all the above mentioned papers on tracking control of UAVs is the definition of the tracking error in the inertial frame, leading to an undesired property of the resulting controllers. Consider a reference trajectory to be tracked by the UAV, as well as an initial condition. Apply the proposed controller and determine the resulting system behavior. Next, consider the same reference trajectory and initial condition, but take a different inertial frame, e.g., rotated by an angle of 90° about the z-axis. Since in this new inertial frame the error is defined differently, the input and therefore the resulting system behavior is different.

The main contributions of this paper are twofold. First, we present a controller which achieves *uniform* almost global asymptotic stability of the tracking error dynamics for a quadrotor UAV on SE(3). By considering the tracking control of a quadrotor UAV on SE(3) we avoid singularities of Euler angles and ambiguity of quaternions and by explicitly taking into account the constraint of non-zero total thrust in our controller design, we do not achieve a local result but almost global asymptotic stability of the tracking controller (note that a global result can not be achieved on SE(3), cf. [14]). Second, we consider the position tracking error in the body-frame of the reference UAV. As a result, contrary to most existing tracking controllers, our control action becomes *independent of the definition of the inertial frame*.

The outline of this paper is as follows. In Section II we introduce definitions and theorems used in the remainder of the paper. In Section III we introduce the quadrotor dynamics and the problem formulation. In Section IV we derive a position tracking controller under the assumption that we can use the body-fixed linear accelerations as (virtual) input. In Section V we aim to realize this virtual input by controlling the rotor thrusts. In Section VI we analyze the stability of the cascaded system that we obtained. Section VII contains simulation results with our proposed controller and Section VIII concludes the paper.

II. PRELIMINARIES

In this section we introduce notation, definitions and theorems used in the remainder of this paper.

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Let e_i for $i \in \{1,2,3\}$ denote the standard unit vector.

Let $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ denote a vector-function $\sigma(e) = \frac{s(e^T e)}{e^T e} e$, where $s : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable monotone function satisfying $s(0) = 0$ and $\lim_{x\to 0} s(x)/x = s'(0) > 0$. Furthermore, let $V_\sigma(e) = \int_0^{e^T e} s(x)/x \mathrm{d}x$, which is positive definite and radially unbounded. Possible candidates are $\sigma(e) = e$ and $\sigma(e) = \frac{e}{\sqrt{1+e^T e}}$, where the latter function is bounded. In the remainder we use the functions σ_1, σ_2, and σ_3.

Definition 1: A function σ_i for which $\|\sigma_i(e)\| \leq M$ for all e is called a *saturation function*.

Theorem 2 (cf. [23, Theorem 1]): Consider the dynamical system

$$\dot{x} = f(t,x) \qquad x(t_0) = x_0 \tag{1}$$

with $f(t,0) = 0$, $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ locally bounded, continuous and locally uniformly continuous in t.

If there exist j differentiable functions $V_i : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$, bounded in t, and continuous functions $Y_i : \mathbb{R}^n \to \mathbb{R}$ for $i \in \{1,2,\ldots j\}$ such that

- V_1 is positive definite,
- $\dot{V}_i(t,x) \leq Y_i(x)$, for all $i \in \{1,2,\ldots,j\}$,
- $Y_i(x) = 0$ for $i \in \{1,2,\ldots,k-1\}$ implies $Y_k(x) \leq 0$, for all $k \in \{1,2,\ldots,j\}$,
- $Y_i(x) = 0$ for all $i \in \{1,2,\ldots,j\}$ implies $x = 0$,

then the origin $x = 0$ of (1) is uniformly globally asymptotically stable (UGAS).

Proof: Since V_1 is positive definite and $\dot{V}_1 \leq Y_1(x) \leq 0$ the origin of (1) is uniformly globally stable. Since $V_i(t,x)$ are continuous and bounded in t, $V_i(t,x)$ are bounded for $i \in \{2,3,\ldots,j\}$. Next, apply [23, Theorem 1]. ■

For definitions of uniform global (or local) asymptotic (or exponential) stability (UGAS, UGES, ULES), see [22].

Definition 3: The origin of (1) is *uniformly almost globally asymptotically stable (UaGAS)* if it is UGAS, except for initial conditions in a set of measure zero.

Theorem 4 (cf. [15], [17]): Consider the system

$$\dot{R} = RS(\omega) \tag{2a}$$

$$J\dot{\omega} = -K_\omega \omega + K_R \sum_{i=1}^{3} k_i (e_i \times R^T e_i), \tag{2b}$$

where $R \in \mathrm{SO}(3) = \{R \in \mathbb{R}^{3\times 3} \mid R^T R = I, \det R = 1\}$, $\omega \in \mathbb{R}^3$, $J = J^T > 0$ and

$$S(a) = -S(a)^T = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}. \tag{3}$$

If $K_\omega = K_\omega^T > 0$, $K_R = K_R^T > 0$, and $k_i > 0$ are distinct (e.g., $0 < k_1 < k_2 < k_3$), then the equilibrium point $(I,0)$ of (2) is uniformly locally exponentially stable (ULES) and uniformly almost global asymptotic stable (UaGAS). That is, let $E_c = \{I, \mathrm{diag}(1,-1,-1), \mathrm{diag}(-1,1,-1), \mathrm{diag}(-1,-1,1)\}$. Then R converges to E_c and ω converges to zero. The equilibria $(R,0)$ of (2), where $R \in E_c \setminus \{I\}$ are unstable and the set of all initial conditions converging to the equilibrium $(R,0)$, where $R \in E_c \setminus \{I\}$ form a lower dimensional manifold.

Theorem 5 (cf. [24]): Consider a system $\dot{x} = f(t,x)$ with $f(t,0) = 0$ that can be written as

$$\dot{x}_1 = f_1(t,x_1) + g(t,x_1,x_2)x_2 \tag{4a}$$

$$\dot{x}_2 = f_2(t,x_2) \tag{4b}$$

where $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^m$, $f_1(t,x_1)$ is continuously differentiable in (t,x_1) and $f_2(t,x_2)$, $g(t,x_1,x_2)$ are continuous in their arguments, and locally Lipschitz in x_2 and (x_1,x_2) respectively. This system is a cascade of the systems

$$\dot{x}_1 = f_1(t,x_1) \tag{5}$$

and (4b). If the origins of the systems (5) and (4b) are UGAS and solutions of (4) remain bounded, then the origin of the system (4) is UGAS. In addition, if the systems (5) and (4b) are ULES, then the system (4) is ULES.

Theorem 6 (cf. [25, Corollary 2.4.6]): If the origin of (5) is uniformly globally exponentially stable (UGES), the origin of the system (4b) is ULES and UGAS, and

$$\|g(t,x_1,x_2)\| \leq k_1(\|x_2\|) \; + \; k_2(\|x_2\|)\|x_1\|$$

then the origin of the system (4) is ULES and UGAS.

III. DYNAMICS AND PROBLEM FORMULATION

A. Quadrotor dynamics

Let $\rho \in \mathbb{R}^3$ denote the position of the centre of mass relative to a North-East-Down (NED) inertial frame. Let $R \in$ SO(3) denote the rotation matrix from the body-fixed frame to the inertial frame. Furthermore, let $\nu \in \mathbb{R}^3$ and $\omega \in \mathbb{R}^3$ denote the body-fixed linear and angular velocities. Then the dynamics of a UAV can be described as:

$$\dot{\rho} = R\nu \tag{6a}$$

$$\dot{\nu} = -S(\omega)\nu + gR^T e_3 - \frac{f}{m} e_3 \tag{6b}$$

$$\dot{R} = RS(\omega) \tag{6c}$$

$$J\dot{\omega} = S(J\omega)\omega + \tau, \tag{6d}$$

where m denotes the total mass, $J = J^T > 0$ the inertia matrix with respect to the body-fixed frame, the matrix S is given by (3), and $f \in \mathbb{R}$ and $\tau \in \mathbb{R}^3$ denote respectively the total thrust magnitude and the total moment vector in the body-fixed frame, which are assumed to be the inputs.

B. Problem

Assume that a feasible reference trajectory is given, i.e., a trajectory $(\rho_r, R_r, \nu_r, \omega_r, f_r, \tau_r)$ satisfying

$$\dot{\rho}_r = R_r \nu_r \tag{7a}$$

$$\dot{\nu}_r = -S(\omega_r)\nu_r + gR_r^T e_3 - \frac{f_r}{m} e_3 \tag{7b}$$

$$\dot{R}_r = R_r S(\omega_r) \tag{7c}$$

$$J\dot{\omega}_r = S(J\omega_r)\omega_r + \tau_r, \tag{7d}$$

where $0 < f_r^{\min} \leq f_r(t)$.

Define the following error coordinates on SE(3):

$$\tilde{\rho} = R_r^T(\rho - \rho_r) \qquad \tilde{R} = R_r^T R$$
$$\tilde{\nu} = -\tilde{R}^T S(\omega_r)\tilde{\rho} + \nu - \tilde{R}^T \nu_r \qquad \tilde{\omega} = \omega - \tilde{R}^T \omega_r$$

with corresponding error measure:

$$\varepsilon(\tilde{\rho}, \tilde{R}, \tilde{\nu}, \tilde{\omega}) = \|\tilde{\rho}\| + \|\log \tilde{R}\| + \|\tilde{\nu}\| + \|\tilde{\omega}\|. \tag{8}$$

Then we can define the tracking control problem as follows.

Problem 7: For $(\rho_r, R_r, \nu_r, \omega_r, f_r, \tau_r)$ being a given feasible reference trajectory, find appropriate control laws

$$\begin{aligned} f &= f(\rho, R, \nu, \omega, \rho_r, R_r, \nu_r, \omega_r) \\ \tau &= \tau(\rho, R, \nu, \omega, \rho_r, R_r, \nu_r, \omega_r) \end{aligned} \tag{9}$$

such that for the resulting closed-loop system (6), (7), (9)

$$\lim_{t\to\infty} \varepsilon\big(\tilde{\rho}(t), \tilde{R}(t), \tilde{\nu}(t), \tilde{\omega}(t)\big) = 0.$$

IV. POSITION TRACKING CONTROL

We separate the design of the tracking controller into two parts. In this section we consider the derivation of a position tracking controller under the assumption that we can use the body-fixed linear accelerations as (virtual) input. In the next section we consider the problem of realizing this virtual input by means of the actual inputs.

As mentioned in the introduction, the commonly used position tracking error is given by $\rho - \rho_r$. A drawback of this definition of the tracking error is its dependence on the choice of the inertial frame. Though translating the inertial frame leaves this error definition invariant, rotating the inertial frame does not. By alternatively defining the error in a body fixed frame (of either the drone of the reference), rotation of the inertial frame leaves the alternative error definition invariant. We propose to express the tracking error in the body-fixed frame of the reference:

$$\begin{bmatrix} \rho_e \\ \nu_e \end{bmatrix} = \begin{bmatrix} R_r^T(\rho_r - \rho) \\ \nu_r - R_r^T R\nu \end{bmatrix}. \tag{10}$$

Using this definition the tracking error dynamics become

$$\dot{\rho}_e = -S(\omega_r)\rho_e + \nu_e$$
$$\dot{\nu}_e = -S(\omega_r)\nu_e + \frac{f}{m}R_r^T R e_3 - \frac{f_r}{m}e_3.$$

For stabilizing these time-varying tracking error dynamics we assume $\frac{f}{m}R_r^T R e_3 - \frac{f_r}{m}e_3$ to be a virtual input which we want to achieve by controlling the thrust magnitude and the attitude.

Remark 8: Note that by selecting f and R we have full control over $\frac{f}{m}R_r^T R e_3 - \frac{f_r}{m}e_3$ and therefore can consider it to be a virtual input. However, if we would have expressed the tracking error in the body-fixed frame of the UAV by taking $\rho_e = R^T(\rho - \rho_r)$ and $\nu_e = \nu - R^T R_r \nu_r$, then by selecting f and R we would have had to have full control over $\frac{f_r}{m}R^T R_r e_3 - \frac{f}{m}e_3$, which we do not have. This explains our choice for defining the error in the body-fixed frame of the reference as in (10).

Proposition 9: Consider the dynamics

$$\dot{\rho}_e = -S(\omega_r)\rho_e + \nu_e \tag{12a}$$
$$\dot{\nu}_e = -S(\omega_r)\nu_e + u \tag{12b}$$

in closed loop with the dynamic state feedback

$$\begin{aligned} u = R_r^T(K_P P_e + K_p p_e) - R_r^T K_P R_r \sigma_1(\rho_e + R_r^T P_e) \\ - k_\rho \sigma_2(\rho_e + R_r^T P_e) - K_\nu \sigma_3(\nu_e + R_r^T p_e) \end{aligned} \tag{13a}$$
$$\dot{P}_e = p_e \tag{13b}$$
$$\dot{p}_e = -K_P P_e - K_p p_e + K_P R_r \sigma_1(\rho_e + R_r^T P_e), \tag{13c}$$

where $K_P = K_P^T > 0$, $K_p = K_p^T > 0$, $K_\nu = K_\nu^T > 0$, and $k_\rho > 0$. If $\omega_r(t)$ is bounded and continuous, then the origin of the closed-loop system (12),(13) is UGAS.

Proof: Define the change of coordinates

$$\bar{\rho}_e = \rho_e + R_r^T P_e \qquad \bar{\nu}_e = \nu_e + R_r^T p_e. \tag{14}$$

Then the closed-loop system (12),(13) can be described as

$$\dot{P}_e = p_e \tag{15a}$$
$$\dot{p}_e = -K_P P_e - K_p p_e + K_P R_r \sigma_1(\bar{\rho}_e) \tag{15b}$$
$$\dot{\bar{\rho}}_e = -S(\omega_r)\bar{\rho}_e + \bar{\nu}_e \tag{15c}$$
$$\dot{\bar{\nu}}_e = -S(\omega_r)\bar{\nu}_e - k_\rho \sigma_2(\bar{\rho}_e) - K_\nu \sigma_3(\bar{\nu}_e). \tag{15d}$$

Consider the positive definite function

$$V_1(\bar{\rho}_e, \bar{\nu}_e) = k_\rho V_{\sigma_2}(\bar{\rho}_e) + \frac{1}{2}\bar{\nu}_e^T \bar{\nu}_e.$$

Along solutions of (15c),(15d) we have

$$\dot{V}_1(\bar{\rho}_e, \bar{\nu}_e) = -\bar{\nu}_e^T K_\nu \sigma_3(\bar{\nu}_e) = Y_1(\bar{\nu}_e) \leq 0.$$

Define $V_2(\bar{\rho}_e, \bar{\nu}_e) = \bar{\nu}_e^T \bar{\rho}_e$. Then

$$\dot{V}_2(\bar{\rho}_e, \bar{\nu}_e) = \bar{\nu}_e^T \bar{\nu}_e - \bar{\rho}_e^T k_\nu \sigma_3(\bar{\nu}_e) - \bar{\rho}_e^T k_\rho \sigma_2(\bar{\rho}_e) = Y_2(\bar{\rho}_e, \bar{\nu}_e).$$

Using Theorem 2 implies that (15c),(15d) is UGAS.

Furthermore, since the system

$$\begin{bmatrix} \dot{P}_e \\ \dot{p}_e \end{bmatrix} = \begin{bmatrix} 0 & I \\ -K_P & -K_p \end{bmatrix} \begin{bmatrix} P_e \\ p_e \end{bmatrix}$$

is UGES for $K_p = K_p^T > 0$, $K_P = K_P^T > 0$, we can use Theorem 6 to conclude that (15) is ULES and UGAS. ■

Remark 10: For $\sigma_1(\bar{\rho}_e) = \bar{\rho}_e$ we have

$$\dot{p}_e = -K_p p_e + K_P R_r \rho_e.$$

So p_e is a filtered version of $R_r \rho_e = \rho_r - \rho$, and from (15a) P_e is the integral of that signal. So $R_r^T P_e$ is a natural integrated version of our (filtered) tracking error coordinate.

If we furthermore take $k_\rho \sigma_2(\bar{\rho}_e) = K_I \bar{\rho}_e$, $\sigma_3(\bar{\nu}_e) = \bar{\nu}_e$, $K_P = R_r(K_\rho - K_I)R_r^T$, and $K_p = R_r K_\nu R_r^T$ we have

$$u = -K_I R_r^T P_e - K_\rho \rho_e - K_\nu \nu_e,$$

which is a PID controller.

Remark 11: Let K_P and K_p be given. For σ_1 we can take a saturation function to guarantee that $\|K_P R_r \sigma_1(\bar{\rho}_e)\| \leq \kappa$ for some $\kappa > 0$. Let

$$x = \begin{bmatrix} P_e \\ p_e \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & I \\ -K_P & -K_p \end{bmatrix}.$$

Then there exist k and λ such that $\|e^{(t-t_0)A}\| \leq ke^{-\lambda(t-t_0)}$ and we have for (15a),(15b)

$$\|x(t)\| \leq k\|x(t_0)\| + \frac{k}{\lambda}\kappa.$$

By taking κ and $x(t_0)$ sufficiently small, we can guarantee

$$\|R_r^T(K_P P_e + K_p p_e)\| \leq \epsilon$$

for an arbitrary given $\epsilon > 0$. Therefore, by also taking a saturation function for σ_2 and σ_3 we can guarantee to meet an arbitrary given upperbound on $\|u\|$ in (13a).

V. ATTITUDE CONTROL

In the previous section we showed that if we would have $\frac{f}{m}R_r^T Re_3 - \frac{f_r}{m}e_3$ equal to u given by (13), we can stabilize the position tracking error dynamics. Therefore, in this section we aim to use our inputs f and τ to let $fR_r^T Re_3$ converge to the vector $f_r e_3 + mu$, where u is given by (13). Since for a feasible reference trajectory $0 < f_r^{\min} \leq f_r(t)$ we know from Remark 11 that by properly selecting σ_1, σ_2, σ_3, $P_e(t_0)$ and $p_e(t_0)$ we can guarantee that $\|u\| \leq \frac{f_r^{\min}-\epsilon}{m}$ for any $0 < \epsilon < f_r^{\min}$. As a result, we have for

$$f = \|f_r e_3 + mu\|, \tag{16}$$

where u is given by (13), that $0 < \epsilon \leq f(t)$.

By taking f as in (16) we have determined one of our four inputs, and we can use τ for achieving the desired attitude.

Since $0 < \epsilon \leq f$, we can define

$$f_d = \begin{bmatrix} f_{d1} \\ f_{d2} \\ f_{d3} \end{bmatrix} = \frac{f_r e_3 + mu}{\|f_r e_3 + mu\|} \tag{17a}$$

as the desired thrust direction, satisfying $f_{d3} > 0$. We let

$$R_d = \begin{bmatrix} 1 - \frac{f_{d1}^2}{1+f_{d3}} & -\frac{f_{d1}f_{d2}}{1+f_{d3}} & f_{d1} \\ -\frac{f_{d1}f_{d2}}{1+f_{d3}} & 1 - \frac{f_{d2}^2}{1+f_{d3}} & f_{d2} \\ -f_{d1} & -f_{d2} & f_{d3} \end{bmatrix} \in \mathrm{SO}(3) \tag{17b}$$

denote the rotation matrix which rotates the desired thrust vector to the thrust vector of the reference (i.e., e_3) in the plane containing both vectors. This also gives

$$\omega_d = \begin{bmatrix} -\dot{f}_{d2} + \frac{f_{d2}\dot{f}_{d3}}{1+f_{d3}} \\ \dot{f}_{d1} - \frac{f_{d1}\dot{f}_{d3}}{1+f_{d3}} \\ \frac{f_{d2}\dot{f}_{d1} - f_{d1}\dot{f}_{d2}}{1+f_{d3}} \end{bmatrix}.$$

Using (16) and (17), we can write $f_r e_3 + mu = fR_d e_3$, so our goal to determine τ which makes $fR_r^T Re_3$ converge to $f_r e_3 + mu$ can be replaced by the goal to determine τ which makes $R_r^T R$ converge to R_d.

We define the following attitude error in the body-fixed frame of the drone:

$$R_e = R_d^T(R_r^T R). \tag{18}$$

The associated angular velocity tracking error is given by

$$\omega_e = \omega - R^T R_r \omega_r - R_e^T \omega_d, \tag{19}$$

that is, we have $\dot{R}_e = R_e S(\omega_e)$. Furthermore,

$$\begin{aligned} J\dot{\omega}_e = {} & S(J\omega)\omega + \tau - JR^T R_r J^{-1}[S(J\omega_r)\omega_r + \tau_r] \\ & + JS(\omega_e)[\omega - \omega_e] + JR_e^T[S(\omega_d)R_d^T\omega_r - \dot{\omega}_d]. \end{aligned}$$

The input

$$\begin{aligned} \tau = {} & -K_\omega \omega_e + K_R \sum_{i=1}^{3} k_i(e_i \times R_e^T e_i) - S(J\omega)\omega \\ & - JR_e^T[S(\omega_d)R_d^T\omega_r - \dot{\omega}_d] - JS(\omega_e)[\omega - \omega_e] \\ & + JR^T R_r J^{-1}[S(J\omega_r)\omega_r + \tau_r] \end{aligned} \tag{20}$$

results in the closed-loop system

$$\begin{aligned} \dot{R}_e &= R_e S(\omega_e) \\ J\dot{\omega}_e &= -K_\omega \omega_e + K_R \sum_{i=1}^{3} k_i(e_i \times R_e^T e_i), \end{aligned}$$

for which $(I, 0)$ is ULES and UaGAS for distinct $k_i > 0$ and $K_\omega = K_\omega^T > 0$ and $K_R = K_R^T > 0$ according to Theorem 4.

VI. CASCADE ANALYSIS

In the previous two sections we determined respectively a desired control action for the position tracking error dynamics, and a controller for f and τ which asymptotically achieves this desired control action for the position tracking error dynamics. As a final step in our analysis we need to analyse stability of the cascaded system of attitude controller and desired position controller.

Consider the system dynamics (6) and the reference dynamics (7) in closed loop with the inputs (13),(16) and (20). The resulting closed-loop system can be written as

$$\dot{P}_e = p_e \tag{21a}$$
$$\dot{p}_e = -K_P P_e - K_p p_e + K_P R_r \sigma_1(\bar{\rho}_e) \tag{21b}$$
$$\dot{\bar{\rho}}_e = -S(\omega_r)\bar{\rho}_e + \bar{\nu}_e \tag{21c}$$
$$\dot{\bar{\nu}}_e = -S(\omega_r)\bar{\nu}_e - k_\rho \sigma_2(\bar{\rho}_e) - K_\nu \sigma_3(\bar{\nu}_e) + \frac{f}{m}R_r^T R(I - R_e^T)e_3 \tag{21d}$$
$$\dot{R}_e = R_e S(\omega_e) \tag{21e}$$
$$J\dot{\omega}_e = -K_\omega \omega_e + K_R \sum_{i=1}^{3} k_i(e_i \times R_e^T e_i). \tag{21f}$$

Proposition 12: If the functions σ_1, σ_2, σ_3, and $P_e(t_0)$ and $p_e(t_0)$ are properly chosen guaranteeing that $\|u\| \leq \frac{f_r^{\min}-\epsilon}{m}$ for some $0 < \epsilon < f_r^{\min}$, cf. Remark 11, then the origin $(P_e, p_e, \bar{\rho}, \bar{\nu}, R_e, \omega_e) = (0, 0, 0, 0, I, 0)$ of (21) is ULES and UaGAS.

Proof: Notice that (21) can be seen as a cascade of the systems ((21a), (21b), (21c), (21d)) and ((21e),(21f)).

Since we have UaGAS of ((21e),(21f)), we consider our stability analysis on $\mathbb{R}^{12} \times G$ where $G \subset \mathrm{SO}(3) \times \mathbb{R}^3$ denotes the almost global region of attraction of ((21e),(21f)).

A first observation is that due to the saturation function σ_1, we have that P_e and p_e remain bounded, cf. Remark 11. It remains to show that $\bar{\rho}_e$ and $\bar{\nu}_e$ remain bounded. Consider

$$V = k_\rho V_{\sigma_2}(\bar{\rho}_e) + \frac{1}{2}\bar{\nu}_e^T \bar{\nu}_e. \qquad (22)$$

Differentiating (22) along solutions of (21) results in

$$\dot{V} \leq \bar{\nu}_e^T \frac{f}{m} R_r^T R(I - R_e^T)e_3 \leq c_1\sqrt{V}\|I - R_e\|.$$

Since ((21e),(21f)) is ULES we have

$$\sqrt{V(t)} - \sqrt{V(t_0)} \leq c_2(t_0).$$

So V is bounded, and therefore solutions of (21) are bounded. The result follows from Theorem 5. ■

Corollary 13: The controller (13), (16), (20) solves the reference trajectory tracking problem, Problem 7.

Proof: From $P_e \to 0$, $p_e \to 0$, $\bar{\rho}_e \to 0$, $\bar{\nu}_e \to 0$ we have using (14) that $\rho_e \to 0$ and $\nu_e \to 0$, so using (10) we have $\tilde{\rho} \to 0$ and $\tilde{\nu} \to 0$. Furthermore, we have $u \to 0$, so from (17) we obtain $f_d \to e_3$ and $R_d \to I$. Using $R_e \to I$ and (18) we have $\tilde{R} \to 0$. Finally, since $\dot{u} \to 0$ also $\omega_d \to 0$, and therefore from $\omega_e \to 0$ and (19), we have $\tilde{\omega} \to 0$. ■

VII. SIMULATION RESULTS

In this section a simple case study is presented to validate our theoretical results. We consider the dynamics (6) with $m = 0.456$[kg], $J = \text{diag}(0.0022, 0.0025, 0.0045)$[kgm^2], and $g = 9.81$[m/s^2]. For the reference (7) we assume a discrepancy of 5% between the actual mass of the quadrotor m and the expected mass $m_r = 0.48$[kg]. For the reference trajectory we take

$$\rho_r(t) = \begin{bmatrix} \cos t \\ \sin t \\ 1.5 + \sin t \end{bmatrix}.$$

Using (7) this also determines f_r, R_r, (along the lines of (17)), ω_r, τ_r, and ν_r. The initial conditions are set to

$$\rho(t_0) = \begin{bmatrix} -1 \\ 0.7 \\ 4 \end{bmatrix} \qquad R(t_0) = \begin{bmatrix} -0.25 & -0.433 & 0.866 \\ 0.533 & -0.808 & -0.25 \\ 0.808 & 0.34 & 0.433 \end{bmatrix}$$
$$\nu(t_0) = \begin{bmatrix} 0.1 \\ -0.8 \\ 0.7 \end{bmatrix} \qquad \omega(t_0) = \begin{bmatrix} -1 \\ 0.3 \\ -2 \end{bmatrix}.$$

For the controller we use the functions $\sigma_i(x) = \frac{x}{\sqrt{1+x^T x}}$ for $i \in \{1, 2, 3\}$ and the gains $K_P = 0.4I$, $K_p = I$, $k_\rho = 2.6$, $K_\nu = 2I$, $K_\omega = 30J$, $K_R = 70J$, $k_1 = 0.9$, $k_2 = 1$, $k_3 = 1.1$, which guarantee that $\|u\| < \min_t f_r(t)$ as explained in Remark 11.

To more closely resemble an AR.Drone 2.0, we include noise and sensor models, include a time-delay of 0.1 seconds for obtaining position measurements, and include physical limitations on the admissible thrust f and torques τ. For more details about these models and the noise levels, see [26] and [27, Chapter 6].

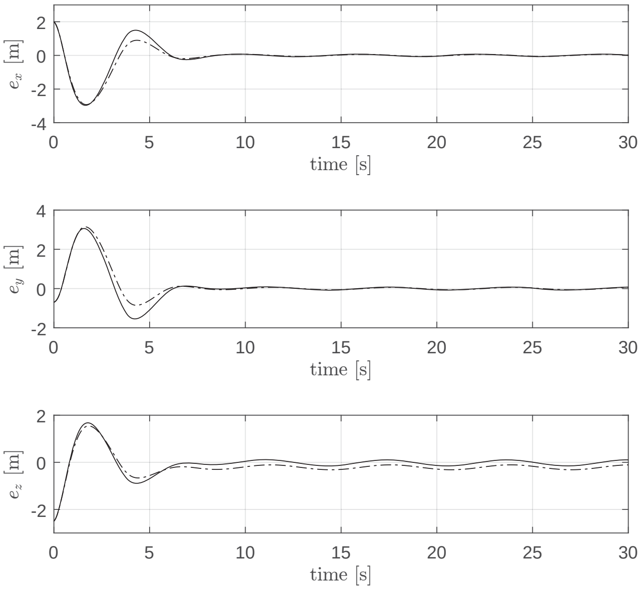

Fig. 1. Filtered errors $e = \rho_r(t) - \rho(t)$ in the inertial frame without integral control (dashed) and with integral control (solid).

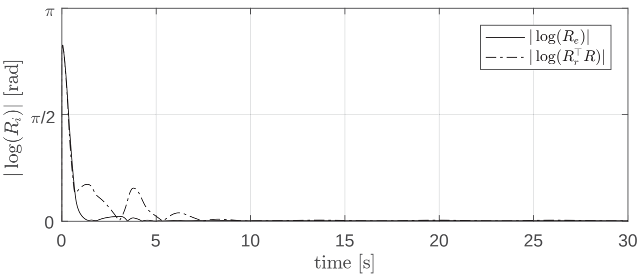

Fig. 2. Attitude-error of the quadrotor with respect to the desired attitude, R_e, (solid), and with respect to the reference attitude, $R_r^T R$, (dashed), characterized by the metric which describes the angle of rotation.

To illustrate the effect of the integral action we also performed simulations without integral action, that is, $K_P = 0$, $K_p = 0$, and without (13b) and (13c).

The results are presented in Fig. 1 which shows the filtered version of the position error in the inertial frame (in x, y and z direction). The effect of the integrating action can clearly be seen in the altitude. In case no integral action is added to the controller, the quadrotor consistently remains at a larger altitude than desired. This results from the fact that since it is assumed that the quadrotor is heavier than in reality, an excessive reference thrust is generated by the system.

As explained in Section V, we try to align the normalized vectors $R_r^T R e_3$ and $R_d e_3$ by making $R_r^T R$ approach R_d, which is represented by the matrix R_e that should converge to I. However, the ultimate goal is to have the attitude R converge to R_r, which is represented by the matrix $R_r^T R$ that should converge to I. In Fig. 3 the distance of both R_e and $R_r^T R_r$ to I are depicted, where we use the metric which makes the distance between two elements of SO(3) the length of the geodesic between them using the natural Riemannian metric, i.e., $\text{d}(R_1, R_2) = \|\log(R_1 R_2^T)\|$. From this figure we observe that in the presence of noisy measurements, the quadrotor is still able to track the attitude accurately.

These simulations illustrate that in case the system is subject to small constant disturbances (e.g., difference in actual and expected mass) and sampled, delayed, and noisy measurements, closed-loop stability is maintained and the corresponding tracking errors converge to some region near the origin.

VIII. CONCLUSIONS

In this paper we presented a controller which achieves uniform almost global asymptotic stability of the tracking error dynamics for a quadrotor on SE(3). By considering the tracking control of a quadrotor UAV on SE(3), we avoid singularities of Euler angles and ambiguity of quaternions, allowing for large angular maneuvers. Furthermore, by explicitly taking into account the constraint of non-zero total thrust in our controller design, our presented controller achieves an almost global result instead of only a local result.

We validated our controller by simulations in which we added small constant disturbances (e.g., difference in actual and expected mass) and used sampled, delayed, and noisy measurements. The simulations show that despite these disturbances the controller achieves stable tracking error dynamics for which errors converge to some region near the origin.

Our next steps are to implement the controller on an experimental setup with an AR.Drone 2.0. Some first experimental results are available in [26], [27]. Furthermore, we want to extend the state feedback controller to an output feedback controller, since the body-fixed velocity ν is not available for measurement.

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