Netherlands Organisation for Scientific Research

Model-based Predictive Control (MPC)

Erjen Lefeber (TU/e), Marco Laumanns (IBM,ETHZ)

SCM-MPC Workshop, München

Technische Universiteit **Eindhoven** University of Technology

Where innovation starts

May 7, 2012

My background

(Nonlinear) Control Theory

What I like

Apply ideas/concepts from control theory in other fields

Four major problems

1. Generate feasible reference trajectory
2. Design (static) state feedback controller
3. Design observer
4. Design (dynamic) output feedback controller

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Example: Linear dynamics, discrete time

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) \qquad\qquad x(0) = x_0 \\
y(k) &= Cx(k)
\end{aligned}
$$

Problem 1: Generate feasible reference trajectory

Determine $(u_r(k), x_r(k), y_r(k))$ satisfying

$$
\begin{aligned}
x_r(k+1) &= Ax_r(k) + Bu_r(k) \\
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Typical approach: solve optimization problem

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Typical approach: solve optimization problem

Design (static) state feedback controller

Given dynamics and reference

$$x(k+1) = Ax(k) + Bu(k) \qquad x_r(k+1) = Ax_r(k) + Bu_r(k)$$

Define tracking error and change of input:

$$\tilde{x}(k) = x(k) - x_r(k) \qquad \tilde{u}(k) = u(k) - u_r(k)$$

Resulting in error dynamics:

$$\begin{aligned}\tilde{x}(k+1) &= A\tilde{x}(k) + B\tilde{u}(k)\\ \tilde{y}(k) &= C\tilde{x}(k)\end{aligned}$$

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Introduction to Control Theory: Problem 2

Design (static) state feedback controller

Problem: Determine $\tilde{u}([\tilde{x}(k)])$ such that $\lim_{k\to\infty} \tilde{x}(k) = 0$, where

$$\tilde{x}(k+1) = A\tilde{x}(k) + B\tilde{u}(k) \qquad \tilde{x}(0) = \tilde{x}_0$$

Solution

Use $\tilde{u}(k) = -L\tilde{x}(k)$. Resulting closed loop dynamics:

$$\tilde{x}(k+1) = (A - BL)\tilde{x}(k) \qquad x(0) = \tilde{x}_0$$

Lemma: If $\operatorname{rank}\begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix} = n$ then eigenvalues of $A - BL$ can be placed arbitrarily.

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Design (static) state feedback controller

Given dynamics and reference

$$x(k+1) = Ax(k) + Bu(k) \qquad x_r(k+1) = Ax_r(k) + Bu_r(k)$$

we use the following controller:

$$u(k) = u_r(k) - L[x(k) - x_r(k)]$$

which guarantees that $\lim_{k\to\infty} x(k) - x_r(k) = 0$, provided L is properly chosen.

Design observer

Given dynamics

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\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) \qquad\qquad x(0) = x_0 \\
y(k) &= Cx(k)
\end{aligned}
$$

Is it possible to reconstruct $x(k)$ from $u(k)$ and $y(k)$?

Solution

Let $\hat{x}(k)$ denote our estimate for $x(k)$:

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\begin{aligned}
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Let $\hat{x}(k)$ denote our estimate for $x(k)$:

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + Bu(k) + K[y(k) - \hat{y}(k)] \\ \hat{y}(k) &= C\hat{x}(k) \end{aligned}$$

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Design observer

Given dynamics and observer

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For observer error $\bar{x}(k) = x(k) - \hat{x}(k)$, we obtain

$$\begin{aligned} \bar{x}(k+1) &= A\bar{x}(k) - K\bar{y}(k) = [A - KC]\bar{x}(k) \\ \bar{y}(k) &= C\bar{x}(k) \end{aligned}$$

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Design (dynamic) output feedback controller

Given dynamics and reference

$$x(k+1) = Ax(k) + Bu(k) \qquad x_r(k+1) = Ax_r(k) + Bu_r(k)$$

we use the following controller:

$$\begin{aligned} u(k) &= u_r(k) - L[\hat{x}(k) - x_r(k)] \\ \hat{x}(k+1) &= A\hat{x}(k) + Bu(k) + K[y(k) - \hat{y}(k)] \\ \hat{y}(k) &= C\hat{x}(k) \end{aligned}$$

which guarantees that $\lim_{k\to\infty} x(k) - x_r(k) = 0$, provided K and L are properly chosen.

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Linear Quadratic Gaussian control (Problem 4)

System

$$X(k+1) = A(k)X(k) + B(k)U(k) + V(k)$$
$$Y(k) = C(k)X(k) + W(k)$$

with $V(k), W(k)$ Gaussian white noise (cov. matrices $\Sigma_v(k), \Sigma_w(k)$).

Objective

Minimize

$$J = E\left(X(N)^T Q(N) X(N) + \sum_{k=0}^{N-1} X(k)^T Q(k) X(k) + U(k)^T R(k) U(k)\right)$$

where $Q(k) \geq 0$, $R(k) > 0$.

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where $Q(k) \geq 0$, $R(k) > 0$.

Solution

Controller

$$\begin{aligned} U(k) &= -L(k)\hat{X}(k) \\ \hat{X}(k+1) &= A(k)\hat{X}(k) + B(k)U(k) + K(k)[Y(k) - C(k)\hat{X}(k)] \end{aligned}$$

Linear Quadratic Gaussian control (Problem 4)

Solution

Controller

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\begin{aligned}
U(k) &= -L(k)\hat{X}(k) \\
\hat{X}(k+1) &= A(k)\hat{X}(k) + B(k)U(k) + K(k)[Y(k) - C(k)\hat{X}(k)] \\
K(k) &= A(k)P(k)C(k)^T[C(k)P(k)C(k)^T + \Sigma_w(k)]^{-1} \\
P(k+1) &= A(k)\Gamma[C(k), P(k), \Sigma_w(k)]A(k)^T + \Sigma_v(k) \\
L(k) &= [B(k)^T S(k+1)B(k) + R(k)]^{-1}B(k)^T S(k+1)A(k) \\
S(k) &= A(k)^T\Gamma[B(k)^T, S(k+1), R(k)]A(k) + Q(k)
\end{aligned}
$$

$S(N) = Q(N)$, $P(0) = E(X_0X_0^T)$, $\hat{x}(0) = E(X_0)$. Furthermore,

$$
\Gamma[F, G, H] = G - GF^T(FGF^T + H)^{-1}FG
$$

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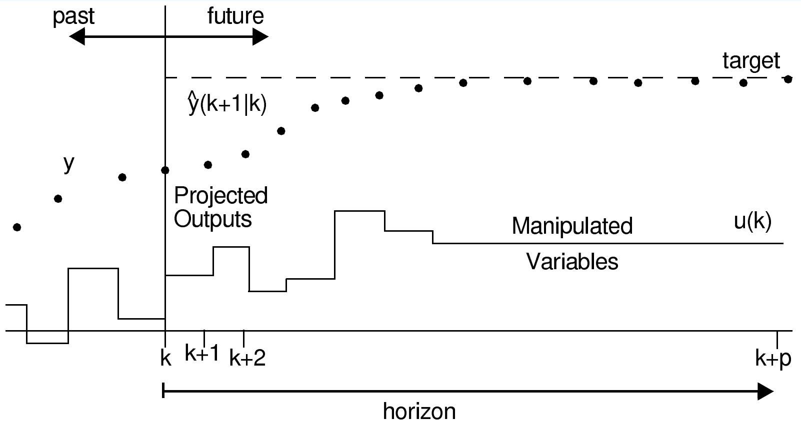
Model based predictive control

Receding horizon



First era: industrial success stories

1950s Various oil and petrochemical industries: optimal process settings computed every 15-20 minutes, implemented by manual operators.

60s and 70s Feedback controller from repeatedly using recomputed open loop controllers (Lee and Markus). Repeatedly solving Problem 1.

- Deterministic (without any disturbance model)
- Lack of stability guarantees
- Lack of systematic tuning guidelines

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Model based predictive control

Second era: founding of MPC theory

Consider system $x(k+1) = Ax(k) + Bu(k)$.
At each time: measure (or estimate) state x_0 and solve

$$\min_{u(0),\ldots u(N-1)} \hat{x}(N)^T Q_N x(N) + \sum_{k=0}^{N-1} \left[x(k)^T Q x(k) + u(k)^T R u(k) \right]$$

$$\begin{aligned} \text{s.t.}\quad & x(k+1) = Ax(k) + Bu(k) \qquad x(0) = x_0 \\ & u(k) \in \mathbb{U} \\ & x(k) \in \mathbb{X} \qquad x_(N) \in \mathbb{X}_N \end{aligned}$$

where $\mathbb{U}$, $\mathbb{X}$, $\mathbb{X}(p)$ convex compact sets containing 0.

Result: feedback $u(x_0)$ (online calculation).

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Example

Consider dynamics

$$x(k+1) = 4x(k) + u(k) \qquad x(k|k) = x_0$$

Horizon of 1: $\min_{u(k|k)} x(k+1|k)^2 + u(k|k)^2$

$$\min_{u(k|k)} [4x_0 + u(k|k)]^2 + u(k|k)^2 = 16x_0^2 + 8x_0 u(k|k) + 2u(k|k)^2$$

Optimal solution: $u(k|k) = \frac{-8x_0}{2 \cdot 2} = -2x_0$

Closed-loop system: $x(k+1) = 4x(k) - 2x(k) = 2x(k)$ Unstable!

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Model based predictive control

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Second era: founding of MPC theory

Observation 1: Infinite horizon results in stabilizing controller

Observation 2: After finite amount of time: solution remains unconstrained

Idea: Properly select terminal costs and horizon

Main steps

1. Solve infinite horizon LQR problem: $u = Kx$, $V = x^T P x$
2. Determine *maximal output admissible set*: $\mathbb{X}_N$
3. Determine N s.t. $x(N) \in \mathbb{X}_N$

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Model based predictive control

Second era: founding of MPC theory

- Robust MPC (next slide)
- Nonlinear MPC

Third era: Diversification through fast MPC

- MPC for hybrid systems and systems with logical constraints
- Explicit MPC (mpLP,mpQP)
- Fast optimization
- Application (mechanical and electronic systems)

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Model based predictive control

Robust MPC

Dynamics

$$x(k+1) = Ax(k) + Bu(k) + Ed(k) \qquad d(k) \in \mathbb{D} = \{d : Ld \le l\}$$

$$J^{*(k)}(x^{(k)}) = \min_{u^{(k)}} J^{(k)}(x^{(k)}, u^{(k)})$$

$$\text{s.t.} \left\{ \begin{array}{l} Fx^{(k)} + Gu^{(k)} \le g \\ Ax^{(k)} + Bu^{(k)} \in \mathbb{X}^{(k)} \end{array} \right\} \forall d^{(k)} \in \mathbb{D}$$

$$J^{(k)}(x^{(k)}, u^{(k)}) = \max_{d^{(k)} \in \mathbb{D}} \|Qx^{(k)}\|_1 + \|Ru^{(k)}\|_1 + J^{*(k+1)}(Ax^{(k)} + Bu^{(k)} + Ed^{(k)})$$

$$\mathbb{X}^{(k)} = \{x \in \mathbb{R}^n \; : \forall d \in \mathbb{D} \;\; \exists u \in \mathbb{R}^{n_u} \text{ with } Fx + Gu \le g \text{ and } Ax + Bu + Ev \in \mathbb{X}^{(k+1)}\}.$$

where $J^{*K}(x^{(K)}) = 0$ and $\mathbb{X}^{(K)} = \{x \in \mathbb{R}^n : Fx \le g\}$

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Dynamics

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$$J^{*(k)}(x^{(k)}) = \min_{u^{(k)}} J^{(k)}(x^{(k)}, u^{(k)})$$

$$\text{s.t.} \left\{ \begin{array}{l} Fx^{(k)} + Gu^{(k)} \leq g \\ Ax^{(k)} + Bu^{(k)} \in \mathbb{X}^{(k)} \end{array} \right\} \forall d^{(k)} \in \mathbb{D}$$

$$J^{(k)}(x^{(k)}, u^{(k)}) = \max_{d^{(k)} \in \mathbb{D}} \|Qx^{(k)}\|_1 + \|Ru^{(k)}\|_1 + J^{*(k+1)}(Ax^{(k)} + Bu^{(k)} + Ed^{(k)})$$

$$\mathbb{X}^{(k)} = \{x \in \mathbb{R}^n \; : \forall d \in \mathbb{D} \;\; \exists u \in \mathbb{R}^{n_u} \text{ with}$$

$$Fx + Gu \leq g \text{ and } Ax + Bu + Ev \in \mathbb{X}^{(k+1)} \}.$$

where $J^{*K}(x^{(K)}) = 0$ and $\mathbb{X}^{(K)} = \{x \in \mathbb{R}^n : Fx \leq g\}$

Robust MPC

Dynamics

$$x(k+1) = Ax(k) + Bu(k) + Ed(k) \qquad d(k) \in \mathbb{D} = \{d : Ld \leq l\}$$

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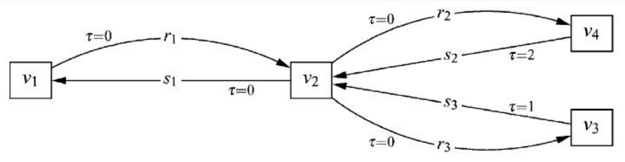
Model based predictive control

Robust MPC: Example



Retailer v_2: uncertain demand $d(t) \in [0, 8]$

- Order $u_1(t) \in [0, 6]$ from supplier v_3: cost 4, delay 1
- Order $u_2(t) \in [0, 6]$ from supplier v_4: cost 1, delay 2

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Robust MPC: Example

$$x(t+1) = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(t) + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} d(t).$$

Resulting (dual base stock) policy:

$$u_1^*(x) = \min\{\max\{20 - x_1 - x_2 - x_3 - x_4, 0\}, 4\},$$
$$u_2^*(x) = \max\{16 - x_1 - x_2 - x_3 - x_4, 0\}.$$

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