Netherlands Organisation for Scientific Research

Control of multi-class queueing networks with infinite virtual queues

Erjen Lefeber (TU/e)

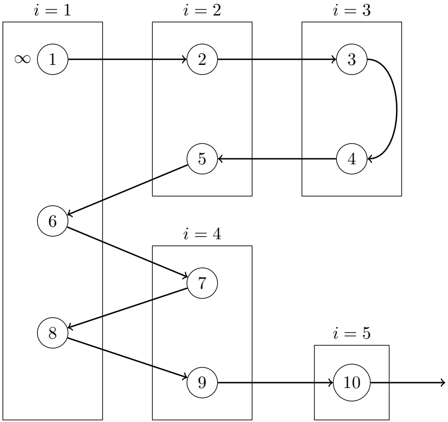
Workshop on Optimization, Scheduling and Queues
Honoring Gideon Weiss on his Retirement

Technische Universiteit
Eindhoven
University of Technology

June 8, 2012

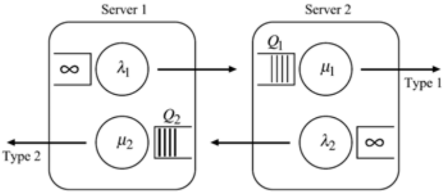
Where innovation starts

Multi-class queueing network with IVQs



Example: Push pull queueing system

Kopzon, Weiss (2002); Kopzon, Nazarathy, Weiss (2009); Nazarathy, Weiss (2010)



Static production planning problem

$$\max_{u,\alpha} w'\alpha$$

α_1, α_2 nominal input rates

u_i fraction of time spent on class i

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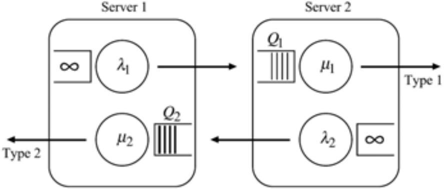

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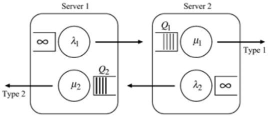


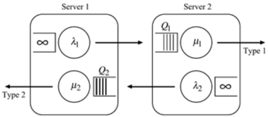
$$\max_{u,\alpha} \; w_1\alpha_1 + w_2\alpha_2$$

$$\text{s.t.} \quad \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ \lambda_1 & -\mu_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 & -\mu_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ 0 \\ \alpha_2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$u, \alpha \geq 0$$

Example: Push pull queueing system



Three possible solutions (exclusing singular cases $\lambda_1 = \mu_1$ or $\lambda_2 = \mu_2$):

1. $\alpha_1 = \min\{\lambda_1, \mu_1\}, \alpha_2 = 0$,
2. $\alpha_1 = 0, \alpha_2 = \min\{\lambda_2, \mu_2\}$,
3. $\alpha_1 = \frac{\lambda_1 \mu_1 (\lambda_2 - \mu_2)}{\lambda_1 \lambda_2 - \mu_1 \mu_2}, \alpha_2 = \frac{\lambda_2 \mu_2 (\lambda_1 - \mu_1)}{\lambda_1 \lambda_2 - \mu_1 \mu_2}$.

Interesting solution: solution 3

- $\rho_1 = \rho_2 = 1$ (full utilization of servers)
- $\tilde{\rho}_1 = \frac{\lambda_2 (\lambda_1 - \mu_1)}{\lambda_1 \lambda_2 - \mu_1 \mu_2} < 1, \tilde{\rho}_2 = \frac{\lambda_1 (\lambda_2 - \mu_2)}{\lambda_1 \lambda_2 - \mu_1 \mu_2} < 1$.

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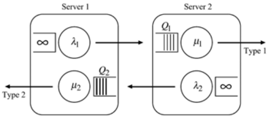

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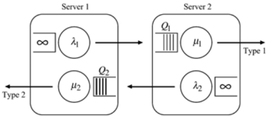

Question

Can we stabilize system with $\rho_i = 1$ and $\tilde{\rho}_i < 1$?

Two cases

inherently stable case: $\lambda_1 < \mu_1$ and $\lambda_2 < \mu_2$

inherently unstable case: $\lambda_1 > \mu_1$ and $\lambda_2 > \mu_2$

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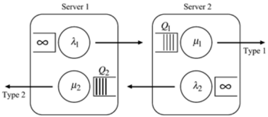

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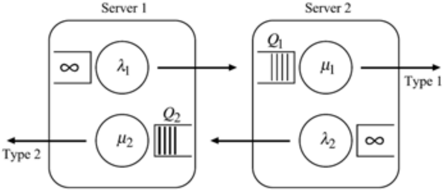


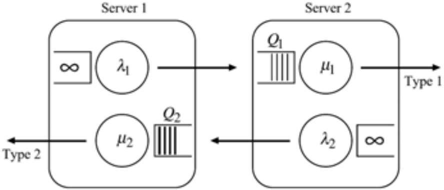
Positive result

Pull priority stabilizes network

Observation

For inherently unstable case: pull priority is not stabilizing.

Inherently stable case: $\lambda_1 > \mu_1, \lambda_2 > \mu_2$



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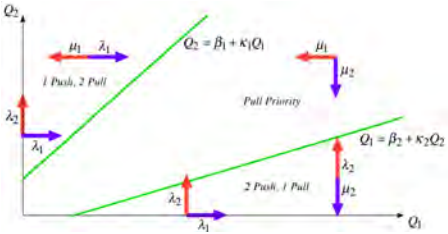
Kopzon, Nazaraty, Weiss (2009); Nazarathy, Weiss (2010):

Positive result

Threshold policy stabilizes network



Problem

Guo, Lefeber, Nazarathy, Weiss, Zhang (2011):

Key research question

Can we stabilize a MCQN-IVQ with $\tilde{\rho}_i < 1$ for all servers?

Some positive results

- IVQ re-entrant line (LBFS stable; FBFS not necessarily)
- Two re-entrant lines on two servers (pull priority)
- Ring of machines (pull priority)

Fluid model framework for verifying stability

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Fluid model framework for verifying stability

Problem setting

Consider a MCQN-IVQ with s servers.

Server i serves 1 IVQ and $n_i \geq 1$ standard queues. Assumption: $\rho = 1$.

Let $n = \sum_{i=1}^{s} n_i$; $s \times n$ constituency matrix C, rank $C = s \leq n$.

IVQ at server i served at rate λ_i, Standard queue j served at rate $\mu_j > 0$.

Let $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_s)$, $M = \mathrm{diag}(\mu_1, \ldots, \mu_n)$

$n \times n$ Routing matrix P: p_{jk} fraction of class j routed to k.

Assumption: P has spectral radius < 1, i.e. $(I - P')$ invertible.

$s \times n$ matrix P_{IVQ}. p_{ij}^{IVQ} fraction of IVQ at server i routed to j

Dynamics fluid model ($u_j(t)$ fraction of time spent on std. queue j)

$$\dot{Q}(t) = P'_{\mathrm{IVQ}}\Lambda[1 - Cu(t)] - (I - P')Mu(t) \qquad Q(0) = Q_0$$

subject to

$$0 \leq Q(t) \qquad 0 \leq u(t) \qquad Cu(t) \leq 1$$

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Problem setting

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$$\dot{Q}(t) = P'_{\text{IVQ}}\Lambda[1 - Cu(t)] - (I - P')Mu(t) \qquad Q(0) = Q_0$$
$$= \underbrace{P'_{\text{IVQ}}\Lambda 1}_{\alpha} - \underbrace{[P'_{\text{IVQ}}\Lambda C + (I - P')M]}_{R}\, u(t)$$

subject to

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Additional assumptions

- Controllable system, i.e. R is invertable.
- $\tilde{\rho} < 1$, i.e. $CR^{-1}\alpha < 1$.
- All standard queues are served: $u^* = R^{-1}\alpha > 0$

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Problem setting

To summarize:

$$\dot{Q}(t) = \alpha - Ru(t) \qquad\qquad Q(0) = Q_0$$

subject to

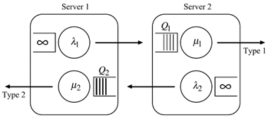
$$0 \leq Q(t) \qquad\qquad 0 \leq u(t) \qquad\qquad Cu(t) \leq 1$$

Furthermore

- C full rank 0-1 matrix, i.e., CC' is invertible
- $I - P'$ and R are invertible (also $(I - P')^{-1} \geq 0$)
- $0 < R^{-1}\alpha = u^*$
- $CR^{-1}\alpha < 1$

Problem: Determine stabilizing u

Example



Dynamics:

$$\begin{bmatrix} \dot{Q}_1(t) \\ \dot{Q}_2(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} - \begin{bmatrix} \mu_1 & \lambda_1 \\ \lambda_2 & \mu_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Constraints

$$0 \leq Q(t) \qquad 0 \leq u(t) \qquad u(t) \leq 1$$

Assumptions:

R invertible: $\mu_1\mu_2 \neq \lambda_1\lambda_2$ or $\rho_1\rho_2 \neq 1$

$CR^{-1}\alpha < 1$, $R^{-1}\alpha = 0$: $\frac{1-\rho_1}{1-\rho_1\rho_2} > 0$, $\frac{1-\rho_2}{1-\rho_1\rho_2} > 0$.

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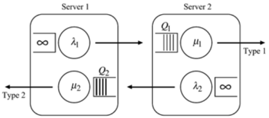

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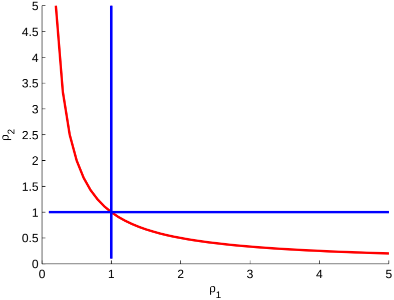
Example

Conditions: $\rho_1\rho_2 \neq 1$, $\frac{1-\rho_1}{1-\rho_1\rho_2} > 0$, $\frac{1-\rho_2}{1-\rho_1\rho_2} > 0$



Example: uncontrollable case

Some words about case $\lambda_1 = \mu_1$, $\lambda_2 = \mu_2$, i.e., R not invertible

Uncontrollable dynamics

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Define change of coordinates:

$$z_1(t) = Q_1(t) + Q_2(t) \qquad z_2(t) = \lambda_2 Q_1(t) - \lambda_1 Q_2(t)$$

Then we have

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 + \lambda_2 \\ 0 \end{bmatrix} - \begin{bmatrix} \lambda_1 + \lambda_2 & \lambda_1 + \lambda_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

In particular the variable $z_2(t)$ evolves independent of the policy chosen.

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Example: controller design

System

$$\dot{Q}(t) = \alpha - Ru(t) \qquad Q(0) = Q_0$$

$$0 \leq Q(t) \qquad 0 \leq u(t) \leq 1$$

Basic idea

Decouple state from input, i.e. what does u_i control?

Define change of coordinates $z(t) = R^{-1}Q(t)$:

Transformed system

$$z(t) = R^{-1}\alpha - u(t) = u^* - u(t) \qquad z(0) = z_0 = R^{-1}Q_0$$

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$$0 \leq Q(t) \qquad 0 \leq u(t) \leq 1$$

Basic idea

Decouple state from input, i.e. what does u_i control?

Define change of coordinates $z(t) = R^{-1}Q(t)$:

Transformed system

$$\dot{z}(t) = R^{-1}\alpha - u(t) = u^* - u(t) \qquad z(0) = z_0 = R^{-1}Q_0$$
$$0 \leq Rz(t) \qquad 0 \leq u(t) \leq 1$$

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Example: controller design

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Example

Change of coordinates

$$z_1(t) = \frac{\mu_2}{\mu_1\mu_2 - \lambda_1\lambda_2} Q_1(t) - \frac{\lambda_1}{\mu_1\mu_2 - \lambda_1\lambda_2} Q_2(t)$$

$$z_2(t) = \frac{-\lambda_2}{\mu_1\mu_2 - \lambda_1\lambda_2} Q_1(t) + \frac{\mu_1}{\mu_1\mu_2 - \lambda_1\lambda_2} Q_2(t)$$

Resulting control problem

$$z_1(t) = u_1^* - u_1(t) \qquad\qquad 0 \leq u_1(t) \leq 1$$

$$z_2(t) = u_2^* - u_2(t) \qquad\qquad 0 \leq u_2(t) \leq 1$$

while making sure that

$$0 \leq \begin{bmatrix} \mu_1 & \lambda_1 \\ \lambda_2 & \mu_2 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

Example

Neglecting the latter constraint, the problem of controlling

$$z_1(t) = u_1^* - u_1(t) \qquad 0 \le u_1(t) \le 1$$
$$z_2(t) = u_2^* - u_2(t) \qquad 0 \le u_2(t) \le 1$$

becomes easy:

$$u_1(t) = \begin{cases} 1 & \text{if } z_1(t) > 0 \\ u_1^* & \text{if } z_1(t) = 0 \\ 0 & \text{if } z_1(t) < 0 \end{cases} \qquad u_2(t) = \begin{cases} 1 & \text{if } z_2(t) > 0 \\ u_2^* & \text{if } z_2(t) = 0 \\ 0 & \text{if } z_2(t) < 0 \end{cases}$$

Observations

- Above controller also solves problem with constraint
- Optimal controller for minimizing $\int_0^\infty \|z(t)\|_1 \, dt$.
- Minimal time controller

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Example: Controller

Controller for stochastic queueing network

$$u_1(t) = \begin{cases} 1 & \text{if } \frac{\mu_2}{\mu_1\mu_2-\lambda_1\lambda_2} Q_1(t) > \frac{\lambda_1}{\mu_1\mu_2-\lambda_1\lambda_2} Q_2(t) \text{ and } Q_1(t) > 0 \\ 0 & \text{if } \frac{\mu_2}{\mu_1\mu_2-\lambda_1\lambda_2} Q_1(t) < \frac{\lambda_1}{\mu_1\mu_2-\lambda_1\lambda_2} Q_2(t) \text{ or } Q_1(t) = 0 \end{cases}$$

$$u_2(t) = \begin{cases} 1 & \text{if } \frac{\lambda_2}{\mu_1\mu_2-\lambda_1\lambda_2} Q_1(t) < \frac{\mu_1}{\mu_1\mu_2-\lambda_1\lambda_2} Q_2(t) \text{ and } Q_2(t) > 0 \\ 0 & \text{if } \frac{\lambda_2}{\mu_1\mu_2-\lambda_1\lambda_2} Q_1(t) > \frac{\mu_1}{\mu_1\mu_2-\lambda_1\lambda_2} Q_2(t) \text{ or } Q_2(t) = 0 \end{cases}$$

Lyapunov function: cost-to-go from optimal control problem

$$V(z) = \begin{cases} z_1^2/(1-u_1^*) + z_2^2/(1-u_2^*) & \text{if } z_1 \geq 0 \text{ and } z_2 \geq 0 \\ z_1^2/u_1^* + z_2^2/(1-u_2^*) & \text{if } z_1 \leq 0 \text{ and } z_2 \geq 0 \\ z_1^2/(1-u_1^*) + z_2^2/u_2^* & \text{if } z_1 \geq 0 \text{ and } z_2 \leq 0 \\ z_1^2/u_1^* + z_2^2/u_2^* & \text{if } z_1 \leq 0 \text{ and } z_2 \leq 0 \end{cases}$$

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mpSCLP

Multi parametric Separated Continuous Linear Program:

$$\min_{u(t)} \int_0^\infty \|z_1(t)\| dt$$

subject to

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Multi parametric since we want solution as function of z_0.

Conjecture

mpSCLP can be solved explicitely and solution has nice structure

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mpSCLP: structure