Output feedback tracking of ships

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Abstract

In this paper we consider output feedback tracking of ships with position and orientation measurements only. Ship dynamics are highly nonlinear, and for tracking control, as opposed to dynamic positioning, these nonlinearities have to be taken into account in the control design. We propose an observer-controller scheme which takes into account the complete ship dynamics, including Coriolis and centripetal forces and nonlinear damping, and prove that the closed-loop system is semi-globally uniformly stable. Furthermore, a gain tuning procedure for the observer-controller scheme is developed.

Experimental results are presented where the observer-controller scheme is implemented onboard a Froude scaled 1:70 model supply ship. The experimentally obtained results are compared with simulation results under ideal conditions and both support the theoretical results on semi-global exponential stability of the closed-loop system.

1 Introduction

Marine control systems [5] can be used to make operations more accurate, more cost effective and safer, e.g. operations where a trajectory must be tracked with a certain accuracy like dredging operations, towing operations and cable laying operations. The ships used for this kind of operations are typically fully actuated ships.

In general only the position and orientation of the ship are measured. Ship velocities are reconstructed from the measured position and orientation by means of an observer [10], since in any tracking controller also the ship velocities are necessary. This paper considers the problem of output feedback tracking of a fully actuated ship with only the position and orientation measurements available.

The development of observers and observer-controller schemes for fully actuated ships stems for an important part from the issue of dynamic positioning (DP) and position mooring of ships. Traditional DP systems are designed by linearizing the kinematic equations of motion about a set of predefined constant yaw angles such that linear control strategies can be applied. The kinematic equations are typically linearized about 36 different yaw angles (36 steps of 10 degrees) to cover the whole heading envelope. The first DP systems were designed using conventional PID controllers in cascade with low-pass and/or notch filters to suppress the wave-induced motion components. From the 1970s more advanced control techniques based on optimal control and Kalman-filter theory were used, see for an overview [5] and references therein. But still these techniques use the 36 linearized systems, from which there is no guarantee for global stability of the total system. In addition, controlling the total system by a set of linearized systems will decrease the performance of the total system. Nonlinear observers and controllers are used to remove these assumptions and make it possible to prove global stability of the total system.

In [8, 6, 12, 2, 7] the dynamic positioning problem and position mooring problem for ships are considered, where the developed observer and controller are based on a ship model not including the Coriolis and centripetal forces and moments nor nonlinear damping. Because the velocities during position keeping and mooring are close to zero, both the Coriolis and centripetal and the nonlinear damping terms can be disregarded. However during trajectory tracking this assumption is not valid anymore and both the Coriolis and centripetal and the nonlinear damping forces and moments must be considered in the observer and controller. Output feedback tracking control for fully actuated ships is considered in [11] and [1]. Also in [13] a system with Coriolis and centripetal forces is considered.

In [11] the proposed approach is mainly based on the work of [4] and is based on passivity in the sense that both the controller and observer are designed such that the closed-loop system matches a predefined desired energy function. The ship model includes the Coriolis and centripetal term, but does not include the nonlinear damping term. The error dynamics is proven to be semi-globally exponentially stable, while the error dynamics is globally exponentially stable if the Coriolis and centripetal forces and moments are negligible.

In [1] an observer-controller scheme is proposed for an Euler-Lagrange system not including the Coriolis and centripetal term, but including a nonlinear damping term. It is assumed that the nonlinear damping term satisfies the monotone damping property, which in general is not satisfied in marine systems. For appropriate choices of the output injection terms, the error dynamics is globally uniformly asymptotically stable.

In [13] an observer-controller scheme is proposed for another class of Euler-Lagrange systems. It is assumed that only linear damping is included and a rather special form of the Coriolis and centripetal term is considered there. Notice that in general in marine systems the Coriolis and centripetal term is not of this form.

In this paper our aim is to propose an observer-controller scheme for tracking control of fully actuated ships with only position and orientation measurements available. Therefore we have to take into account the full tracking model, including both the nonlinear damping and the Coriolis and centripetal forces and moments in the ship dynamics.

The approach of our study can be summarized in the following ways:

- In the observer design our approach is to consider the dynamic ship model in the Earth-fixed frame, where we can use the properties of the Coriolis and centripetal matrix written in Christoffel symbols.
- In the controller design we consider the dynamic ship model in the body-fixed frame, so the stabilizing terms can be chosen with respect to the forward, sideward and orientation errors. We use an existing controller [9], which can be tuned like a second-order system due to the definition of the control errors.
- An observer-controller scheme is proposed where the dynamic ship model for the observer and controller is considered in the Earth-fixed frame and the body-fixed frame respectively. Using a Lyapunov approach we are able to prove semi-global uniform exponential stability of the closed-loop system.
- Experiments with a model ship in a basin are performed. The experimentally obtained results are compared with computer simulations under ideal conditions and both support the theoretical results.

The paper is organized as follows. In the next section the dynamic ship model is presented and its properties are discussed. This dynamic ship model is used in Section 3, where the observer, controller and observer-controller scheme are proposed. The observer-controller scheme has been tested in simulations and experiments and the corresponding results are presented in Section 4. Finally conclusions and recommendations for future developments are drawn in Section 5.

In Appendix I some mathematical preliminaries used throughout the paper are presented. The proofs of the propositions are shown in appendices II-IV.

2 Ship dynamics

The nonlinear manoeuvring model for surface ships is considered [5]

$$\mathbf{M}(\eta)\ddot{\eta} + \mathbf{C}(\eta, \dot{\eta})\dot{\eta} + \mathbf{d}(\eta, \dot{\eta}) = \tau, \tag{1}$$

where

$$
\begin{aligned}
\mathbf{M}(\eta) &= \mathbf{J}(\psi)\mathbf{M}\mathbf{J}^T(\psi)\\
\mathbf{C}(\eta,\dot\eta) &= \mathbf{J}(\psi)\left(\mathbf{C}(\mathbf{J}^T(\psi)\dot\eta) - \mathbf{M}\mathbf{S}(\dot\psi)\right)\mathbf{J}^T(\psi)\\
\mathbf{d}(\eta,\dot\eta) &= \mathbf{J}(\psi)\mathbf{D}\mathbf{J}^T(\psi)\dot\eta + \mathbf{J}(\psi)D_n(\mathbf{J}^T(\psi)\dot\eta)\\
\mathbf{J}(\psi) &= \begin{bmatrix} \cos\psi & -\sin\psi & 0\\ \sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{bmatrix}\\
\mathbf{S}(\dot\psi) = \mathbf{S}(r) &= \begin{bmatrix} 0 & -r & 0\\ r & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.
\end{aligned} \tag{2}
$$

The vector η represents the position and orientation in the Earth-fixed frame, i.e. $\eta = [x\ y\ \psi]^T$. The transformation matrix $\mathbf{J}(\psi)$ transforms the velocities in the body fixed frame to the velocities in the Earth-fixed frame, i.e. $\dot\eta = \mathbf{J}(\psi)\nu$, where $\nu = [u\ v\ r]^T$.

The matrix $\mathbf{M}$ is the system inertia matrix including added mass, $\mathbf{C}(\nu)$ corresponds to the Coriolis and centripetal forces and moments and also includes some added mass, $\mathbf{D}$ is the linear damping matrix, the vector $D_n(\nu)$ includes the nonlinear damping terms and τ is the vector of inputs. Notice that the matrices $\mathbf{M}$, $\mathbf{C}(\nu)$, $\mathbf{D}$ and the vector $D_n(\nu)$ are described in the body-fixed frame.

The dynamic model (1) satisfies the following properties [5].

1. The system inertia matrix satisfies $\mathbf{M}(\eta) = \mathbf{M}^T(\eta) > 0$.
2. The Coriolis and centripetal term is written in terms of Christoffel symbols and satisfies
$$\mathbf{C}(q,x)y = \mathbf{C}(q,y)x, \quad \forall\, x, y. \tag{3}$$
3. Since the rotation matrix $\mathbf{J}(\psi)$ is singularity free, the matrices $\mathbf{M}^{-1}(\eta)$ and $\mathbf{C}(\eta,\dot\eta)$ are bounded in η, i.e.
$$\|\mathbf{M}^{-1}(\eta)\| \le M_M,\ \|\mathbf{C}(\eta,x)\| \le C_M\|x\|\ \forall\,\eta, x. \tag{4}$$

Hydrodynamic damping for marine vessels is mainly caused by: potential damping, skin friction, wave drift damping and damping due to vortex shedding. The different damping terms contribute to both linear and quadratic damping. Therefore, it is assumed that:
Assumption 1. The total damping term $\mathbf{d}(\eta,\dot\eta)$ satisfies the following property

$$\|\mathbf{d}(q,x) - \mathbf{d}(q,y)\| \le \left(d_{M1} + d_{M2}\|x-y\|\right)\|x-y\|. \tag{5}$$

3 Observer-controller scheme

3.1 Observer

In this section an observer is proposed for output feedback tracking of a ship with only position and orientation measurements. Due to the chosen observer structure, which is opposed to the structure of the observers proposed for dynamic positioning [8] [6] [12] [2] [7], we include nonlinear damping and the Coriolis and centripetal forces and moments. The nonlinear manoeuvring model is considered in the Earth-fixed frame in order to use the nice property of the Coriolis and centripetal term written in the Christoffel symbols (3), which we do not have in the nonlinear manoeuvring model expressed in the body-fixed frame. The following observer is proposed

$$\begin{aligned}
\dot{\hat{\eta}} &= \dot{\hat{\eta}} + \mathbf{L}_1\tilde{\eta} \\
\ddot{\hat{\eta}} &= -\mathbf{M}^{-1}(\eta)\mathbf{C}(\eta,\dot{\hat{\eta}})\dot{\hat{\eta}} - \mathbf{M}^{-1}(\eta)\mathbf{d}(\eta,\dot{\hat{\eta}}) + \mathbf{M}^{-1}(\eta)\tau + \mathbf{L}_2\tilde{\eta}
\end{aligned} \tag{6}$$

with the observer errors defined as

$$\tilde{\eta} = \eta - \hat{\eta},\ \dot{\tilde{\eta}} = \dot{\eta} - \dot{\hat{\eta}},\ \ddot{\tilde{\eta}} = \ddot{\eta} - \ddot{\hat{\eta}} \tag{7}$$

and the observer gains $\mathbf{L}_1$ and $\mathbf{L}_2$ are chosen symmetric and positive definite. Then the observer error is given by

$$\begin{aligned}
\ddot{\tilde{\eta}} = &- \mathbf{M}^{-1}(\eta)(\mathbf{C}(\eta,\dot{\eta})\dot{\eta} - \mathbf{C}(\eta,\dot{\hat{\eta}})\dot{\hat{\eta}}) \\
&- \mathbf{M}^{-1}(\eta)(\mathbf{d}(\eta,\dot{\eta}) - \mathbf{d}(\eta,\dot{\hat{\eta}})) - \mathbf{L}_2\tilde{\eta} - \mathbf{L}_1\dot{\tilde{\eta}}.
\end{aligned} \tag{8}$$

Notice that we assume that the velocity of the ship $\dot{\eta}$ is bounded, i.e.

$$\|\dot{\eta}\| \leq V_M, \tag{9}$$

which is a reasonable assumption because of the physical limitations of the ship. Notice that in the observer-controller scheme this assumption is replaced by bounds on the reference trajectory.

Proposition 3.1 *Consider the ship described by the nonlinear model* (1) *in combination with the observer* (6). *Given the initial estimates,* $\hat{\eta}_0$ *and* $\dot{\hat{\eta}}_0$, *chosen the observer gains* $\mathbf{L}_1$ *and* $\mathbf{L}_2$ *symmetric and positive definite such that*

$$\lambda_{\min}(\mathbf{L}_1) > 1, \quad \lambda_{\min}(\mathbf{L}_2) \geq 1 \tag{10}$$

$$\lambda_{\min}(\mathbf{L}_2) > \frac{1}{2}M_M C_M V_M + \frac{1}{4}M_M d_{M1}(V_M) \tag{11}$$

$$\lambda_{\max}(\mathbf{L}_2) \geq \lambda_{\min}(\mathbf{L}_2) \tag{12}$$

$$\lambda_{\min}(\mathbf{L}_1) > \frac{(2\alpha_{11}+\gamma_2)+\sqrt{(2\alpha_{11}+\gamma_2)^2-4(\alpha_{11}^2-\gamma_1)}}{2} \tag{13}$$

$$\lambda_{\max}(\mathbf{L}_1) \leq \frac{\lambda_{\min}^2(\mathbf{L}_1) - 2\alpha_{11}\lambda_{\min}(\mathbf{L}_1) + \alpha_{11}^2 - \gamma_1}{\gamma_2} \tag{14}$$

where

$$\begin{aligned}
\alpha_{11} &= \frac{1}{2} + 3M_M C_M V_M + \frac{3}{2}M_M d_{M1}(V_M) \\
\gamma_1 &= (3M_M C_M + 3M_M d_{M2})^2\big(\lambda_{\max}(\mathbf{L}_2)\|\tilde{\eta}_0\|^2 + \|\dot{\tilde{\eta}}_0\|^2\big) \\
\gamma_2 &= (3M_M C_M + 3M_M d_{M2})^2\,\|\tilde{\eta}_0\|^2 \\
\tilde{\eta}_0 &= \eta_0 - \hat{\eta}_0, \quad \dot{\tilde{\eta}}_0 = \dot{\eta}_0 - \dot{\hat{\eta}}_0.
\end{aligned} \tag{15}$$

If Assumption 1 is satisfied and $\|\dot{\eta}\| \leq V_M$ *then the observer error dynamics* (8) *is semi-globally uniform exponential stable.*

For the proof, see Appendix II.

The observer gains can be chosen according to the following procedure:

1. Choose $\lambda_{\min}(\mathbf{L}_2)$ such that (10) and (11) are satisfied.
2. Choose $\lambda_{\max}(\mathbf{L}_2)$ such that (12) is satisfied.
3. Choose $\lambda_{\min}(\mathbf{L}_1)$ such that (10) and (13) are satisfied.
4. Choose $\lambda_{\max}(\mathbf{L}_1)$ such that (14) is satisfied and $\lambda_{\max}(\mathbf{L}_1) \geq \lambda_{\min}(\mathbf{L}_1)$. This is possible since $\lambda_{\min}(\mathbf{L}_1)$ satisfies (13).

3.2 Controller

In this section the controller proposed in [9] is discussed. Notice that a new stability proof is developed, which assumes a bounded reference yaw velocity opposed to an assumed physical bound on the ship's yaw velocity in [9].

From a practical point of view it is important that the tuning procedure for the controller is intuitive in the sense that the gains are tuned with respect to the body-fixed errors. The price to be paid is that the stability can only be guaranteed for bounded yaw rates. The control errors are defined such that disregarding the rotations the closed loop system can be tuned like a second-order system.

The nonlinear manoeuvring model (1) is considered, however now the dynamics described in the body-fixed frame is considered, i.e.

$$\begin{aligned} \dot{\eta} &= \mathbf{J}(\psi)\nu \\ \mathbf{M}\dot{\nu} + \mathbf{C}(\nu)\nu + \mathbf{d}(\nu) &= \tau, \end{aligned} \tag{16}$$

where $\mathbf{d}(\nu) = \mathbf{D}\nu + D_n(\nu)$. Here $\mathbf{J}(\eta)$ is the transformation matrix between the Earth-fixed frame and the body-fixed frame. The vector $\nu = [u\ v\ r]^T$ includes the forward velocity, the sideward velocity and the angular velocity respectively. Notice that the dynamics in the body-fixed frame is independent of the orientation of the ship, i.e. the mass matrix is constant and $\mathbf{C}(\nu)$ consists of constants and products of constants and the ship velocities.

The tracking errors are defined as [9]

$$\begin{aligned} e_\eta &= \eta - \eta_{\text{ref}} \\ e_\nu &= \nu - \mathbf{J}^T(e_\psi)\nu_{\text{ref}} \\ \dot{e}_\nu &= \dot{\nu} - \mathbf{S}^T(\dot{e}_\psi)\mathbf{J}^T(e_\psi)\nu_{\text{ref}} - \mathbf{J}^T(e_\psi)\dot{\nu}_{\text{ref}}, \end{aligned} \tag{17}$$

where η_{ref} is the reference position, ν_{ref} the reference velocity and $\dot{\nu}_{\text{ref}}$ the reference acceleration, $e_\psi = \psi - \psi_{\text{ref}}$ and notice that $\dot{e}_\psi = e_r$.

The following controller is proposed

$$\begin{aligned} \tau = {} & \mathbf{M}\left(\mathbf{S}^T(\dot{e}_\psi)\mathbf{J}^T(e_\psi)\nu_{\text{ref}} + \mathbf{J}^T(e_\psi)\dot{\nu}_{\text{ref}}\right) + \mathbf{C}(\nu)\nu \\ & + D_n(\nu) + \mathbf{D}\mathbf{J}^T(e_\psi)\nu_{\text{ref}} - \mathbf{K}_d e_\nu - \mathbf{K}_p \mathbf{J}^T(\psi)e_\eta, \end{aligned} \tag{18}$$

where the gain matrices $\mathbf{K}_p$ and $\mathbf{K}_d$ are chosen positive definite. This is a passivity-based controller, cf. [11, 4]. Then the error dynamics is given by

$$\begin{aligned} \dot{e}_\eta &= \mathbf{J}(\psi)e_\nu \\ \dot{e}_\nu &= -\mathbf{M}^{-1}(\mathbf{D} + \mathbf{K}_d)e_\nu - \mathbf{M}^{-1}\mathbf{K}_p\mathbf{J}^T(\psi)e_\eta. \end{aligned} \tag{19}$$

Notice that the proposed controller is the controller proposed in [9] without integral action. Also notice that we assume that $|r_{\text{ref}}| \leq r_{\text{ref}}^{\max}$ instead of a physical bound on the yaw velocity.

Proposition 3.2 *Consider the nonlinear manoeuvring model for surface ships* (1) *in combination with the controller* (18). *Given the initial position and velocity of the ship and* $|r_{ref}| \leq r_{ref}^{\max}$, *chosen the controller gains according to the following gain tuning procedure*

1. *Choose*

$$\begin{aligned} \mathbf{K}_d &= 2\mathbf{M}\mathbf{\Lambda}\mathbf{\Omega} - \mathbf{D} \\ \mathbf{K}_p &= \mathbf{M}\mathbf{\Omega}^2. \end{aligned} \tag{20}$$

2. *Choose* $\mathbf{\Lambda} = \mathbf{\Lambda}^T > 0$ *and further choose* $\mathbf{\Omega} = \mathbf{\Omega}^T > 0$ *such that*

$$\overline{\alpha} = \lambda_{\min}(\mathbf{Q}) - r_{ref}^{\max} - \sqrt{\frac{\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{P})}}\|x_{e0}\| > 0, \tag{21}$$

where

$$\begin{aligned} &x_{e0} = [e_{\eta 0}\ e_{\nu 0}]^T, \\ &e_{\eta 0} = \eta_0 - \eta_{ref0},\ e_{\nu 0} = \nu_0 - \mathbf{J}^T(e_{\psi 0})\nu_{ref0}, \\ &\mathbf{Q} = \begin{bmatrix} 2\mathbf{\Omega}^2 & 0 \\ 0 & 2\mathbf{\Omega}^2 \end{bmatrix}, \mathbf{P} = \begin{bmatrix} \frac{(2\mathbf{\Lambda}\mathbf{\Omega})^2 + \mathbf{\Omega}^2 + \mathbf{\Omega}^4}{2\mathbf{\Lambda}\mathbf{\Omega}} & \mathbf{I} \\ \mathbf{I} & -\frac{\mathbf{I} + \mathbf{\Omega}^2}{2\mathbf{\Lambda}\mathbf{\Omega}} \end{bmatrix}. \end{aligned} \tag{22}$$

If the gains are chosen according to this procedure then the error dynamics (19) *is semi-globally uniform exponential stable.*

For the proof, see Appendix III.

The controller gains can be chosen according to the following procedure:

1. Choose $\mathbf{\Lambda} = \mathbf{\Lambda}^T > 0$.
2. Choose $\mathbf{\Omega} = \mathbf{\Omega}^T > 0$ such that (21) is satisfied. This is possible since $\lambda_{\min}(\mathbf{Q}) \sim \mathcal{O}(\mathbf{\Omega^2})$, $\sqrt{\frac{\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{P})}} \sim \mathcal{O}(\mathbf{\Omega})$ and $\mathbf{\Lambda}$ influences the ratio $\frac{\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{P})}$

3.3 Observer-controller scheme

In this section an observer-controller scheme is proposed based on the observer and controller proposed in the previous sections.

For the reasons mentioned earlier, the dynamic ship model for the observer design is considered in the Earth-fixed frame, while for the controller design the dynamic ship model in the body-fixed frame is considered. It is assumed that the orientation of the ship is exactly measured, so $\hat{\nu} = \mathbf{J}^T(\psi)\dot{\hat{\eta}}$. Since the orientation of a ship is usually measured by a gyro compass, which has an accuracy better than 1 degree and measurement noise typically less than 0.1 degree [11], this is a reasonable assumption.

The observer (6) is combined with the controller (18), which results in the following observer-controller scheme

$$
\begin{aligned}
&\dot{\hat{\eta}} = \hat{\dot{\eta}} + \mathbf{L}_1\tilde{\eta} \\
&\dot{\hat{\dot{\eta}}} = -\mathbf{M}^{-1}(\eta)\mathbf{C}(\eta,\hat{\dot{\eta}})\hat{\dot{\eta}} - \mathbf{M}^{-1}(\eta)\mathbf{d}(\eta,\hat{\dot{\eta}}) + \mathbf{M}^{-1}(\eta)\tau + \mathbf{L}_2\tilde{\eta} \\
&\hat{\nu} = \mathbf{J}^T(\psi)\hat{\dot{\eta}} \\
&\tau = \mathbf{M}\left(\mathbf{S}^T(\dot{\hat{e}}_\psi)\mathbf{J}^T(\hat{e}_\psi)\nu_{\text{ref}} + \mathbf{J}(\hat{e}_\psi)^T\dot{\nu}_{\text{ref}}\right) + \mathbf{C}(\hat{\nu})\hat{\nu} \\
&\quad + D_n(\hat{\nu}) + \mathbf{D}\mathbf{J}(\hat{e}_\psi)^T\nu_{\text{ref}} - \mathbf{K}_d\hat{e}_\nu - \mathbf{K}_p\mathbf{J}(\hat{\psi})^T\hat{e}_\eta.
\end{aligned}
\tag{23}
$$

Notice that the reference velocity ν_{ref} and the reference acceleration $\dot{\nu}_{\text{ref}}$ are assumed to be bounded.

Proposition 3.3 *Consider the ship modelled by the nonlinear manoeuvring model* (1) *in combination with the observer-controller scheme* (23)*. Given the initial position and velocity of the ship, the initial estimates and* $|r_{ref}| \leq r_{ref}^{\max}$*, chosen the controller gains according to the following gain tuning procedure*

1. *Choose*

$$
\begin{aligned}
\mathbf{K}_d &= 2\mathbf{M}\mathbf{\Lambda}\mathbf{\Omega} - \mathbf{D} \\
\mathbf{K}_p &= \mathbf{M}\mathbf{\Omega}^2.
\end{aligned}
\tag{24}
$$

2. *Choose* $\mathbf{\Lambda} = \mathbf{\Lambda}^T > 0$ *and further choose* $\mathbf{\Omega} = \mathbf{\Omega}^T > 0$ *such that*

$$
\overline{\alpha}_{1c} = \lambda_{\min}(\mathbf{Q}) - r_{ref}^{\max} - \sqrt{\frac{\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{P})}}\|x_{e0}\| > 0,
\tag{25}
$$

where

$$
\begin{aligned}
&x_{e0} = [e_{\eta 0}\ e_{\nu 0}]^T, \\
&e_{\eta 0} = \eta_0 - \eta_{ref0},\ e_{\nu 0} = \nu_0 - \mathbf{J}^T(e_{\psi 0})\nu_{ref0}, \\
&\mathbf{Q} = \begin{bmatrix} 2\mathbf{\Omega}^2 & 0 \\ 0 & 2\mathbf{\Omega}^2 \end{bmatrix}, \mathbf{P} = \begin{bmatrix} \frac{(2\mathbf{\Lambda}\mathbf{\Omega})^2 + \mathbf{\Omega}^2 + \mathbf{\Omega}^4}{2\mathbf{\Lambda}\mathbf{\Omega}} & \mathbf{I} \\ \mathbf{I} & -\frac{\mathbf{I}+\mathbf{\Omega}^2}{2\mathbf{\Lambda}\mathbf{\Omega}} \end{bmatrix},
\end{aligned}
\tag{26}
$$

while ϵ_3 *is chosen such that*

$$
\begin{aligned}
&\overline{\alpha}_1 = \lambda_{\min}(\mathbf{Q}) - r_{ref}^{\max} - \epsilon_3^2\lambda_{\max}(\mathbf{P})(M_M\lambda_{\max}(\mathbf{K}_p) \\
&+ M_M\lambda_{\max}(\mathbf{K}_d) + 2M_M\lambda_{\max}(\mathbf{D}+\mathbf{K}_d)\|\nu_{ref}\| + \|\nu_{ref}\| + \\
&2C_M\|\nu_{ref}\| + d_{M3}) - \left(1 + \epsilon_3^2\lambda_{\max}(\mathbf{P})(2C_M + 2M_M\right. \\
&\left.\lambda_{\max}(\mathbf{K}_p) + 2\|\nu_{ref}\| + 2\|\dot{\nu}_{ref}\|)\right)\sqrt{\frac{V_0}{\lambda_{\min}(\mathbf{P})}} > 0
\end{aligned}
\tag{27}
$$

$$
V_0 = \lambda_{\max}(\mathbf{P})\|x_{e0}\|^2 + (\lambda_{\max}(\mathbf{L}_2) + \lambda_{\max}(\mathbf{L}_1))\|\tilde{\eta}_0\|^2 + \|\dot{\tilde{\eta}}_0\|^2.
\tag{28}
$$

while the observer gains $\mathbf{L}_1$ and $\mathbf{L}_2$ are chosen symmetric and positive definite such that

$$\lambda_{\min}(\mathbf{L}_1) > 1, \quad \lambda_{\min}(\mathbf{L}_2) \geq 1 \tag{29}$$

$$\begin{aligned}\lambda_{\min}(\mathbf{L}_2) > &\frac{1}{2}M_M C_M \|\nu_{ref}\| + \frac{1}{4}M_M d_{M1} + 2\epsilon_3^{-2}\lambda_{\max}(\mathbf{P})M_M \\ &\times(\lambda_{\max}(\mathbf{K}_p) + \lambda_{\max}(\mathbf{K}_d) + 2\lambda_{\max}(\mathbf{D}+\mathbf{K}_d)\|\nu_{ref}\|) \\ &+\Big(\frac{1}{2}M_M C_M + 2\epsilon_3^{-2}\lambda_{\max}(\mathbf{P})\big(2M_M\lambda_{\max}(\mathbf{K}_p) \\ &+2\|\nu_{ref}\| + 2\|\dot{\nu}_{ref}\|\big)\Big)\sqrt{\frac{V_0}{\lambda_{\min}(\mathbf{P})}}\end{aligned} \tag{30}$$

$$\lambda_{\max}(\mathbf{L}_2) \geq \lambda_{\min}(\mathbf{L}_2) \tag{31}$$

$$\lambda_{\min}(\mathbf{L}_1) > \frac{(2\alpha_{11}+\gamma_2) + \sqrt{(2\alpha_{11}+\gamma_2)^2 - 4(\alpha_{11}^2 - \gamma_1)}}{2} \tag{32}$$

$$\lambda_{\max}(\mathbf{L}_1) \leq \frac{\lambda_{\min}^2(\mathbf{L}_1) - 2\alpha_{11}\lambda_{\min}(\mathbf{L}_1) + \alpha_{11}^2 - \gamma_1}{\gamma_2} \tag{33}$$

where

$$\begin{aligned}\alpha_{11} = &\frac{1}{2} + 3M_M C_M\|\nu_{ref}\| + \frac{3}{2}M_M d_{M1} \\ &+\epsilon_3^{-2}\lambda_{\max}(\mathbf{P})(\|\nu_{ref}\| + 2C_M\|\nu_{ref}\| + d_{M3})\end{aligned} \tag{34}$$

$$\begin{aligned}\gamma = &\ 2C_M + 2\lambda_{\max}(\mathbf{P})C_M + 2\lambda_{\max}(\mathbf{P})d_{M4}\sqrt{\frac{1}{\lambda_{\min}(\mathbf{P})}} \\ &+3M_M C_M + 3M_M d_{M2}\end{aligned} \tag{35}$$

$$\gamma_1 = \gamma^2\big(\lambda_{\max}(\mathbf{P})\|x_{e0}\|^2 + \lambda_{\max}(\mathbf{L}_2)\|\tilde{\eta}_0\|^2 + \|\dot{\tilde{\eta}}_0\|^2\big) \tag{36}$$

$$\gamma_2 = \gamma^2\|\tilde{\eta}_0\|^2 \tag{37}$$

$$\tilde{\eta}_0 = \eta_0 - \hat{\eta}_0, \quad \dot{\tilde{\eta}}_0 = \dot{\eta}_0 - \dot{\hat{\eta}}_0. \tag{38}$$

If Assumption 1 is satisfied, the reference velocity ν_{ref} and the reference acceleration $\dot{\nu}_{ref}$ are bounded, then the observer-controller scheme is semi-globally uniform exponential stable.

For the proof, see Appendix IV.

The controller gains can be chosen according to the following procedure:

1. Choose $\mathbf{\Lambda} = \mathbf{\Lambda}^T > 0$.
2. Choose $\mathbf{\Omega} = \mathbf{\Omega}^T > 0$ such that (25) is satisfied. This is possible since $\lambda_{\min}(\mathbf{Q}) \sim \mathcal{O}(\mathbf{\Omega^2})$, $\sqrt{\frac{\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{P})}} \sim \mathcal{O}(\mathbf{\Omega})$ and $\mathbf{\Lambda}$ influences the ratio $\frac{\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{P})}$
3. Choose ϵ_3 such that (27) is satisfied.
4. Choose $\lambda_{\min}(\mathbf{L}_2)$ such that (29) and (30) are satisfied.
5. Choose $\lambda_{\max}(\mathbf{L}_2)$ such that (31) is satisfied.
6. Choose $\lambda_{\min}(\mathbf{L}_1)$ such that (29) and (32) are satisfied.
7. Choose $\lambda_{\max}(\mathbf{L}_1)$ such that (33) is satisfied and $\lambda_{\max}(\mathbf{L}_1) \geq \lambda_{\min}(\mathbf{L}_1)$. This is possible since $\lambda_{\min}(\mathbf{L}_1)$ satisfies (32).

4 Computer simulations and experiment

4.1 Marine Cybernetics Laboratory

Experiments with the proposed observer-controller scheme are carried out at the Marine Cybernetics Laboratory (MClab) located at Tyholt, Trondheim. The MClab consist of a 40 m $\times$ 6.45 m concrete basin, a measurement system, a wave generator, a laptop running a user interface to control the experiments and a model supply ship, see Figure 1 and 2. The model ship used during the experiments is Cybership II.

The 6 DOF position of the ship is measured by a Proreflex motion capture system. This system consists of 4 Earth-fixed mounted cameras, 4 active/passive responders onboard of the ship and a position measurement program NyPOS running on a computer. The measurement frequency is set at 9 Hz. The area where the position measurements are available is restricted to 8 m $\times$ 5 m.

A Dell Latitude D800 laptop with a 1.60 GHz Intel Prentium M processor and 512 MB RAM, working under Microsoft Windows XP Professional ver. 2002, is used to run the user interface. The user interface is build in Labview ver. 6.2 and allows us to control the ship by manual inputs, a joystick or an automatic controller. The laptop is also used to build the observer-controller scheme in Matlab ver. 6.5.0 Release 13 and Simulink ver. 5.0. OPAL-RT ver. 6.2. is used to generate makefiles and transmits these files over a wireless network to the computer onboard the ship. Eventually the system build in Simulink is running onboard the ship. The differential equations are solved with a fixed step solver using the Euler algorithm. Since the error is 2nd order w.r.t. time, a small step size is required. A step size of 0.05 s is used, or 20 Hz.

The basin is also equipped with a DHI wave maker system. This system can generate all kind of predefined, regular and irregular, 2D waves in the basin.

Cybership II is a model supply ship, Froude scaled 1:70. The length of the ship is 1.3 m and the weight about 24 kg. Five actuators actuate the ship: at the stern two screw-rudder pairs and in the bow a two blade tunnel thruster. The maximum actuated surge force is 2 N, the maximum sway force is 1.5 N and the maximum yaw moment is 1.5 Nm. Thereafter the growth rate of the actuated forces is limited. Onboard Cybership II a 300 MHz computer is located which runs the QNX 6.2 real-time operating system. This computer runs the observer-controller scheme and communicates with the steppermotors of the rudders and the servomotors controlling the rpms of the screws and the tunnel thruster through an H bridge circuit.

The model matrices of the nonlinear manoeuvring model in the body-fixed frame (16) are defined as

$$
\begin{aligned}
\mathbf{M} &= \begin{bmatrix} 25.8 & 0 & 0 \\ 0 & 33.8 & 1.0115 \\ 0 & 1.0115 & 2.76 \end{bmatrix} \\
\mathbf{C}(\nu) &= \begin{bmatrix} 0 & 0 & -33.8v - 1.0115r \\ 0 & 0 & 25.8u \\ 33.8v + 1.0115r & -25.8u & 0 \end{bmatrix} \\
\mathbf{D}(\nu) &= \begin{bmatrix} 0.72 + 1.33|u| & 0 & 0 \\ 0 & 0.86 + 36.28|v| & -0.11 \\ 0 & -0.11 - 5.04|v| & 0.5 \end{bmatrix}.
\end{aligned} \tag{39}
$$

4.2 Trajectory and tuning

The ship is expected to track an elliptic trajectory with constant surge velocity. The elliptic trajectory is chosen since the yaw velocity is not constant, the whole heading envelope is covered and the trajectory can be repeatedly followed. The ship is supposed to track 4 times an elliptic reference trajectory with a constant forward speed. The ship starts with a forward speed of 0.05 m/s, which is increased after every round to 0.1 m/s, 0.15 m/s and 0.2 m/s respectively.

The ship starts in position $\eta_0 = [3.8 \ -1.2 \ \frac{1}{2}\pi]^T$ with velocity $\nu_0 = [0 \ 0 \ 0]^T$, while the initial estimates are set as $\hat{\eta}_0 = [4 \ -1 \ 2.94]^T$ and $\hat{\eta}_0 = [0.5 \ 0.5 \ -0.5]^T$. The gains of the observer are set as

$$\begin{aligned} \mathbf{L}_1 &= \mathrm{diag}(15,\ 15,\ 15) \\ \mathbf{L}_2 &= \mathrm{diag}(50,\ 50,\ 50), \end{aligned} \tag{40}$$

while the controller gains are set as

$$\begin{aligned} \mathbf{K}_d &= 2\mathbf{M\Lambda\Omega} - \mathbf{D} \\ \mathbf{K}_p &= \mathbf{M\Omega}^2. \end{aligned} \tag{41}$$

with

$$\begin{aligned} \mathbf{\Omega} &= \mathrm{diag}(0.72,\ 0.42,\ 0.64) \\ \mathbf{\Lambda} &= \mathrm{diag}(1,\ 1,\ 1). \end{aligned} \tag{42}$$

4.3 Computer simulation and experimental results

The working of the observer-controller scheme is verified by both computer simulations and experimentally obtained results.

The computer simulation is performed under ideal conditions, which implies that the ship is simulated with the dynamic model used in the lab experiment, there is no simulated measurement noise added and no effects of environmental disturbances like wind, waves and current are included. The computer simulation serve as a ideal reference and is used to do a back to back comparison with the experimentally obtained results.

Experiments are performed with a model ship in a closed basin. The lab experiment is performed with and without waves added to assess the robustness of the scheme. The results are influenced by external disturbances and measurement noise.

Only linear damping is implemented in the observer-controller scheme, since numerical problems occurred by implementing the nonlinear damping term in the observer-controller scheme. Although disregarding nonlinear damping decreases the quality of the ship model and therefore decrease the performance of the observer, it can be accepted since nonlinear damping is a stabilizing effect and has the same effect as increasing the controller gain $\mathbf{K}_d$.

In this paper we present only some representative results. For more results, the interested reader is refered to [15].

4.3.1 Computer simulations

Computer simulations are performed with the same settings as in the lab experiment. For the computer simulation a higher order solver and smaller time step have been used.

The top graph in Figure 3 shows XY plots of the reference (gray) and tracked trajectory (black) in the computer simulation. Convergence of the trajectory towards the reference is clearly seen.

Figure 4 shows the errors in the body-fixed x- and y-position and the heading ψ in this simulation by means of the dashed line. The errors converge to exactly 0. Even though the nonlinear damping is included in the observer controller scheme the errors converge to exactly 0. This is clear from the graphs on the right, where we zoom in on the error and let the simulation run longer

4.3.2 Lab experiments without waves

The bottom graph in Figure 3 shows XY plots of the reference (gray) and tracked trajectory (black) in the lab experiments without waves. The tracked path converges during the transient response towards the reference with only small deviations around $y = 0$, for both extremal values of x.

Figure 4 shows the errors in the body-fixed x- and y-position and the heading ψ in the lab experiment by means of the solid line. Similar to the numerical simulations, the errors converge to 0. However, when we zoom in on the error to the centimeter scale, we see some small deviations in the x and y direction occuring.

As mentioned before, the speed of the ship is increased every round. It appears that the size of the deviations is correlated to the speed of the ship. This is most obvious in the y-direction. Opposed to the directly actuated body-fixed x-direction (screw-rudder pair) and heading (tunnel thruster), the body-fixed y-direction is indirectly actuated by a combination of the two aft screw-rudder pairs and the tunnel thruster. The stronger correlation between the ships forward velocity and the error in the body-fixed y-direction as seen in the experiment is possibly a combination of higher loaded screws and tunnel thruster, the complicated way to actuate the body-fixed y-direction and the increasing Coriolis and centripetal forces on the ship.

The stepwise increase of the ships forward velocity is also shown in Figure 5. Only the 6 DOF position, $\eta = [x\ y\ z\ \phi\ \theta\ \psi]^T$, of the ship can be measured. To obtain a quantity for the real ship velocity to compare with the reference velocity, the measured position is differentiated with a numerical differentiator. If measurements are lost, the system takes the latest available measurement. The differentiated position is then equal to 0 and the quantity for the real ship velocity becomes unrealistic if after some time a position measurement is available. Because measurement failure and of course also measurement noise disturb the differentiated signal, the velocities obtained by the numerical differentiator and the reference velocity are shown against time in Figure 5. The velocities seem to tend to their reference values. The peaks around for instance 60, 360, 530, and 650 seconds respectively are results of measurement failure. This is supported by the large peaks in the velocities obtained by the numerical differentiator.

Allthough there is a small deviation between the numerical simulations and the lab experiment, these differences are negligable. In particular when the measurement failures are taken into account. Therefore, the experimental results in Figure 4 compare well with the numerical simulations, and thus support the theoretical result of semi-global uniform exponential stability of the closed-loop errors.

4.3.3 Lab experiments with waves

The robustness of the observer-controller scheme is explored by introducing waves to the model ship in the experiment. To compare the performance of the scheme between the experiments with and without waves and the computer simulation an error index is defined

$$\textit{error index} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} (e_{x\text{body}}(t)^2 + e_{y\text{body}}(t)^2 + e_{\psi}(t)^2) dt. \qquad (43)$$

The waves are generated using the JONSWAP (Joint North Sea Wave Project) distribution [5] with time mean period of 0.75 seconds, $\gamma = 3.3$ and a significant wave height of 0.01 m. The JONSWAP distribution is commonly used to model non-fully developed seas and is therefore more peaked then those representing fully developed seas. The situation considered corresponds with WMO sea state code 3 (moderate sea swell) in reality.

For the graphs resulting from these experiments the reader is refered to [15]. These are comparable to the results presented above for the lab experiments without waves. The corresponding error indices are presented in Table 1. The results show only small changes in the calculated error indices from which we can conclude some robustness against external disturbances in the scheme.

Table 1: Performance of observer-controller scheme

	t_1	t_2	Error index
Simulation	0	714.2	9.305610^{-2}
Experiment without waves	0	715.1	$1.1145\ 10^{-1}$
Experiment with waves	0	730.6	$1.2686\ 10^{-1}$

5 Conclusions

An observer-controller scheme is proposed to track a trajectory in real-time using the position and heading measurements of the ship.

In the observer design the dynamic ship model in the Earth-fixed frame is considered, which has the advantage that the properties of the Coriolis and centripetal matrix written in Christoffel symbols can be used.

In the controller design the dynamic ship model in the body-fixed frame is considered, so that the stabilizing terms can be chosen with respect to the forward, sideward and orientation error. Disregarding the rotations, the closed-loop system can be tuned like a second-order system.

In the observer-controller design the dynamic ship model for the observer and controller is considered in the Earth-fixed frame and the body-fixed frame respectively. The closed-loop system is proven to be semi-globally uniform exponential stable.

Experimental results from tests with a model ship are compared with simulation results under ideal conditions. In ideal simulations the errors converge exactly to 0, while the experimental results tend to 0. Both experimental and simulation results are comparable with the theoretical results on exponential convergence of the closed loop-errors.

The experiments also show that the observer-controller scheme is robust with respect to environmental disturbances.

Notice that the presented observer-controller scheme can also be used for other Euler-Lagrange systems including nonlinear damping and Coriolis and centripetal forces and moments.

References

[1] O.M. Aamo, M. Arcak, T.I. Fossen, and P.V. Kokotović. Global output tracking control of a class of Euler-Lagrange systems. In *Proceedings of the 39th IEEE Conference on Decision and Control*, Sydney, Australia, December 2000.

[2] M.F. Aarset, J.P. Strand, and T.I. Fossen. Nonlinear vectorial observer backstepping with integral action and wave filtering. In *Proceedings of the IFAC Conference on Control Applications in Marine Systems (CAMS'98)*, Fukuoka, Japan, October 1998.

[3] H. Berghuis, P. Löhnberg, and H. Nijmeijer. Tracking control of robots using only position measurements. In *Proceedings of the 30st Conference on Decision and Control*, pages 1039–1040, Brighton, England, December 1991.

[4] H. Berghuis and H. Nijmeijer. A passivity approach to controller-observer design for robots. *IEEE Transactions on Robotics and Automation*, 6(6):740–754, 1993.

[5] T.I. Fossen. *Marine Control Systems. Guidance, Navigation, and Control of Ships, Rigs and Underwater Vehicles.* Marine Cybernetics AS, Trondheim, Norway, 2002.

[6] T.I. Fossen and Å. Grøvlen. Nonlinear output feedback control of dynamically positioned ships using vectorial observer backstepping. *IEEE Transactions on Control Systems Technology*, TCST-6(1):121–128, 1998.

[7] T.I. Fossen and J.P. Strand. Passive nonlinear observer design for ships using lyapunov methods:full-scale experiments with a supply vessel. *Automatica*, AUT-35(1), 1999.

[8] Å Grøvlen and T.I. Fossen. Nonlinear control of dynamic positioned ships using only position feedback: An observer backstepping approach. In *Proceedings of the 35th Conference on Decision and Control*, pages 3388–3393, Kobe, Japan, December 1996.

[9] K.-P. Lindegaard. *Acceleration Feedback in Dynamic Positioning*. PhD thesis, Norwegian University of Science and Technology, Trondheim, Norway, 2003.

[10] H. Nijmeijer and T.I. Fossen. *New Directions in Nonlinear Observer Design*, volume 244 of *Lecture Notes in Control and Information Sciences*. Springer, London, 1999.

[11] K. Y. Pettersen and H. Nijmeijer. Output feedback tracking control for ships. In *Output Feedback Tracking Control for Ships* [10], chapter 7, pages 311–334.

[12] A. Robertsson and R. Johansson. Comments on "nonlinear output feedback control of dynamically positioned ships using vectorial backstepping". *IEEE Transactions on Control Systems Technology*, TCST-6(3):439–441, 1998.

[13] R. Skjetne and H. Shim. A systematic nonlinear observer design for a class of Euler-Lagrange systems. In *Proc. Mediterranean Conf. Contr. and Automation*, Dubrovnik, Croatia, June 2001.

[14] J.T. Wen and D.S. Bayard. New class of control laws for robotic manupilators:non-adaptive case. *International Journal of Control*, 47:1361–1385, 1988.

[15] M. Wondergem. Output feedback tracking of a fully actuated ship. MSc thesis, Eindhoven University of Technology, Eindhoven, The Netherlands, July 2005.

A Mathematical preliminaries

In this paper the following notations are used: The norm of a vector or matrix is denoted as $\|.\|$ and the norm of a scalar is denoted as $|.|$. The minimum eigenvalue of a matrix $\mathbf{A}$ is denoted as $\lambda_{\min}(\mathbf{A})$, while the maximum eigenvalue is denoted by $\lambda_{\max}(\mathbf{A})$.

The systems considered in this paper are of the form

$$\dot{x} = f(x) \tag{44}$$

where $f : \mathbb{R}_+ \times \mathcal{D} \to \mathbb{R}^n$ is piecewise continuous on $\mathbb{R}_+ \times \mathcal{D}$ and locally Lipschitz in x on $\mathbb{R}_+ \times \mathcal{D}$, and $\mathcal{D} \subset \mathbb{R}^n$ is a domain that contains the origin $x = 0$.

Definition A.1 *The equilibrium point $x = 0$ is said to be semi-globally uniform exponential stable if for each $r > 0$ and for all $(t_0, x(t_0)) \in \mathbb{R}_+ \times \mathcal{B}$, a function $\beta \in \mathcal{KL}$ exists such that*

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \ \forall\, t \geq t_0 \geq 0, \ \forall\, x(t_0) \in \mathcal{B}_r \tag{45}$$

where β

$$\beta = k\|x(t_0)\| \exp^{-\gamma(t-t_0)} \qquad k > 0, \ \gamma > 0 \tag{46}$$

A modified version of the β-ball lemma plays a central role in the stability proof of the systems. This modified version of the β-ball lemma [14] is presented in [3].

Lemma A.1 *Given a dynamical system*

$$\begin{aligned} &\dot{x}_i = f_i(x_1, \ \ldots, x_m) \text{ and } f_i(0, \ldots, 0) = 0, \\ &x_i \in \mathbb{R}^n, \ t \geq 0, \ i = 1, \ldots, m \end{aligned} \tag{47}$$

Let $f_i(.)$ be locally Lipschitz with respect to x_1, ..., x_m. Suppose a function $V(.)$: $\mathbb{R}^{n \times m} \to \mathbb{R}^+$ is given such that

$$V(x_1, \ldots, x_m) = \sum_{i,j=1}^{m} x_i^T \mathbf{P}_{ij} x_j \qquad \mathbf{P}_{ij} = \mathbf{P}_{ij}^T > 0 \tag{48}$$

where for each $i = 1, \ldots, m$ *there exists a* $\xi_i > 0$ *such that*

$$\xi_i \|x_i\|^2 \leq V(x_1, \ldots, x_m) \tag{49}$$

$$\dot{V}(x_1, \ldots, x_m) \leq -\sum_{i \in I_1} (\alpha_i - \sum_{j \in I_2} \gamma_{ij} \|x_j\|) \|x_i\|^2 \tag{50}$$

where α_i*,* $\gamma_{ij} > 0$*,* $I_2 \subset I_1 \subset \{1, \ldots, m\}$*. Define* $V_0 = V(x_1(0), \ldots, x_m(0))$*. If for all* $i \in I_1$

$$\overline{\alpha}_i = \alpha_i - \sum_{j \in I_2} \gamma_{ij} V_0^{\frac{1}{2}} \xi_j^{-\frac{1}{2}} > 0 \tag{51}$$

then for all $\kappa_i \in [0, \overline{\alpha}_i]$ *the following inequality holds*

$$\dot{V}(x_1, \ldots, x_m) \leq -\sum_{i \in I_1} \kappa_i \|x_i\|^2, \; \forall \, t \geq 0 \tag{52}$$

B Proof of Proposition 4.1.

The following candidate Lyapunov function, which satisfies (48), is proposed

$$V(\tilde{\eta}, \dot{\tilde{\eta}}) = \frac{1}{2} \tilde{\eta}^T (\mathbf{L}_2 + \lambda \mathbf{L}_1) \tilde{\eta} + \lambda \dot{\tilde{\eta}}^T \tilde{\eta} + \frac{1}{2} \dot{\tilde{\eta}}^T \dot{\tilde{\eta}}, \tag{53}$$

and $\lambda > 0$.

Note that the cross terms can be rewritten by completion of squares, i.e.

$$\begin{aligned} \|\lambda\tilde{\eta}\|\|\dot{\tilde{\eta}}\| &= -\frac{1}{2}(\epsilon\lambda\|\tilde{\eta}\| - \epsilon^{-1}\|\dot{\tilde{\eta}}\|)^2 + \frac{1}{2}\epsilon^2\lambda^2\|\tilde{\eta}\|^2 + \frac{1}{2}\epsilon^{-2}\|\dot{\tilde{\eta}}\|^2 \\ &\leq \frac{1}{2}\epsilon^2\lambda^2\|\tilde{\eta}\|^2 + \frac{1}{2}\epsilon^{-2}\|\dot{\tilde{\eta}}\|^2, \end{aligned} \tag{54}$$

where the parameter $\epsilon > 0$. The proposed candidate Lyapunov function can thus be lower- and upperbounded such that (49) is satisfied, i.e.

$$\begin{aligned} &\underbrace{\frac{1}{2}(\lambda_{\min}(\mathbf{L}_2) + \lambda\lambda_{\min}(\mathbf{L}_1) - \lambda^2\epsilon_1^2)}_{\xi_1} \|\tilde{\eta}\|^2 + \underbrace{\frac{1}{2}(1 - \epsilon_1^{-2})}_{\xi_2} \|\dot{\tilde{\eta}}\|^2 \\ &\leq V(\tilde{\eta}, \dot{\tilde{\eta}}) \leq \\ &\frac{1}{2}(\lambda_{\max}(\mathbf{L}_2) + \lambda\lambda_{\max}(\mathbf{L}_1) + \lambda^2\epsilon_1^2)\|\tilde{\eta}\|^2 + \frac{1}{2}(1 + \epsilon_1^{-2})\|\dot{\tilde{\eta}}\|^2. \end{aligned} \tag{55}$$

The derivative of the proposed candidate Lyapunov function is

$$\begin{aligned} &\dot{V}(\tilde{\eta}, \dot{\tilde{\eta}}) = -\lambda\tilde{\eta}^T\mathbf{L}_2\tilde{\eta} - \dot{\tilde{\eta}}^T(\mathbf{L}_1 - \lambda\mathbf{I})\dot{\tilde{\eta}} + (\lambda\tilde{\eta} + \dot{\tilde{\eta}})^T \\ &\Big(-\mathbf{M}^{-1}(\eta)(\mathbf{C}(\eta, \dot{\eta})\dot{\eta} - \mathbf{C}(\eta, \dot{\hat{\eta}})\dot{\hat{\eta}}) - \mathbf{M}^{-1}(\eta)(d(\eta, \dot{\eta}) - d(\eta, \dot{\hat{\eta}}))\Big). \end{aligned} \tag{56}$$

Note that using property (3) it can be shown that

$$\begin{aligned} &\mathbf{C}(\eta, \dot{\eta})\dot{\eta} - \mathbf{C}(\eta, \dot{\hat{\eta}})\dot{\hat{\eta}} = \mathbf{C}(\eta, \dot{\eta})\dot{\eta} - \mathbf{C}(\eta, \dot{\eta})\dot{\eta} + \mathbf{C}(\eta, \dot{\eta})\dot{\tilde{\eta}} \\ &+ \mathbf{C}(\eta, \dot{\tilde{\eta}})\dot{\eta} - \mathbf{C}(\eta, \dot{\tilde{\eta}})\dot{\tilde{\eta}} = 2\mathbf{C}(\eta, \dot{\eta})\dot{\tilde{\eta}} - \mathbf{C}(\eta, \dot{\tilde{\eta}})\dot{\tilde{\eta}}. \end{aligned} \tag{57}$$

Now using (57), (9), (4), (5) and rewriting the cross term by completion of squares (54), upperbounds can be found for

$$
\begin{aligned}
&(\lambda\tilde{\eta}+\dot{\tilde{\eta}})^T\left(-\mathbf{M}^{-1}(\eta)(\mathbf{C}(\eta,\dot{\eta})\dot{\eta}-\mathbf{C}(\eta,\dot{\hat{\eta}})\dot{\hat{\eta}})\right)\\
&\leq(\lambda\|\tilde{\eta}\|+\|\dot{\tilde{\eta}}\|)(2M_MC_MV_M\|\dot{\tilde{\eta}}\|+M_MC_M\|\dot{\tilde{\eta}}\|^2)\\
&\leq\frac{1}{2}\epsilon_2^2\lambda^2 2M_MC_MV_M\|\tilde{\eta}\|^2+\frac{1}{2}\epsilon_2^{-2}2M_MC_MV_M\|\dot{\tilde{\eta}}\|^2\\
&+2M_MC_MV_M\|\dot{\tilde{\eta}}\|^2+(\lambda\|\tilde{\eta}\|+\|\dot{\tilde{\eta}}\|)M_MC_M\|\dot{\tilde{\eta}}\|^2
\end{aligned}
\tag{58}
$$

and

$$
\begin{aligned}
&(\lambda\tilde{\eta}+\dot{\tilde{\eta}})\left(-\mathbf{M}^{-1}(\eta)(\mathbf{d}(\eta,\dot{\eta})-\mathbf{d}(\eta,\dot{\hat{\eta}}))\right)\\
&\leq(\lambda\|\tilde{\eta}\|+\|\dot{\tilde{\eta}}\|)M_M(d_{M1}\|\dot{\tilde{\eta}}\|+d_{M2}\|\dot{\tilde{\eta}}\|^2)\\
&\leq\frac{1}{2}\epsilon_2^2\lambda^2M_Md_{M1}\|\tilde{\eta}\|^2+\frac{1}{2}\epsilon_2^{-2}M_Md_{M1}\|\dot{\tilde{\eta}}\|^2\\
&+M_Md_{M1}\|\dot{\tilde{\eta}}\|^2+(\lambda\|\tilde{\eta}\|+\|\dot{\tilde{\eta}}\|)M_Md_{M2})\|\dot{\tilde{\eta}}\|^2.
\end{aligned}
\tag{59}
$$

Now using (58) and (59) the derivative of the proposed candidate Lyapunov function (56) can be upperbounded by

$$
\begin{aligned}
\dot{V}(\tilde{\eta},\dot{\tilde{\eta}})\leq&-\underbrace{\left(\lambda\lambda_{\min}(\mathbf{L}_2)-\frac{1}{2}\epsilon_2^2\lambda^2(2M_MC_MV_M+M_Md_{M1})\right)}_{\alpha_1}\|\tilde{\eta}\|^2\\
&-\underbrace{\left(\lambda_{\min}(\mathbf{L}_1)-\lambda-(2+\epsilon_2^{-2})M_MC_MV_M-(1+\frac{1}{2}\epsilon_2^{-2})M_Md_{M1}\right)}_{\alpha_2}\|\dot{\tilde{\eta}}\|^2\\
&+\underbrace{(\lambda M_MC_M+\lambda M_Md_{M2})}_{\gamma_{21}}\|\tilde{\eta}\|\|\dot{\tilde{\eta}}\|^2\\
&+\underbrace{(M_MC_M+M_Md_{M2})}_{\gamma_{22}}\|\dot{\tilde{\eta}}\|\|\dot{\tilde{\eta}}\|^2
\end{aligned}
\tag{60}
$$

We choose the observer gains $\mathbf{L}_1$ and $\mathbf{L}_2$ such that $\lambda_{\min}(\mathbf{L}_1)>1$, $\lambda_{\min}(\mathbf{L}_2)\geq 1$. In addition we choose $\epsilon_1=2$ and $\lambda=\frac{1}{2}$ in (55), such that

$$
\begin{aligned}
&\frac{1}{4}(\|\tilde{\eta}\|^2+\|\dot{\tilde{\eta}}\|^2)\leq\frac{1}{4}\|\tilde{\eta}\|^2+\frac{3}{8}\|\dot{\tilde{\eta}}\|^2\leq V(\tilde{\eta},\dot{\tilde{\eta}})\\
&\leq(\lambda_{\max}(\mathbf{L}_2)+\lambda_{\max}(\mathbf{L}_1))\|\tilde{\eta}\|^2+\|\dot{\tilde{\eta}}\|^2.
\end{aligned}
\tag{61}
$$

We choose $\epsilon_2=1$ in (60), which in analogy with (51) in Lemma 2.1 results in

$$
\begin{aligned}
&\overline{\alpha}_1=\frac{1}{2}\lambda_{\min}(\mathbf{L}_2)-\frac{1}{4}M_MC_MV_M-\frac{1}{8}M_Md_{M1}\\
&\overline{\alpha}_2=\lambda_{\min}(\mathbf{L}_1)-\alpha_{11}-\gamma\sqrt{\lambda_{\max}(\mathbf{L}_2)\|\tilde{\eta}_0\|^2+\|\dot{\tilde{\eta}}_0\|^2+\lambda_{\max}(\mathbf{L}_1)\|\tilde{\eta}_0\|^2},
\end{aligned}
\tag{62}
$$

where

$$
\begin{aligned}
\alpha_{11}&=\frac{1}{2}+3M_MC_MV_M+\frac{3}{2}M_Md_{M1}\\
\gamma&=3M_MC_M+3M_Md_{M2}
\end{aligned}
\tag{63}
$$

From (62) and (63) a gain tuning procedure can be developed to guarantee $\overline{\alpha}_1, \overline{\alpha}_2 > 0$. We can choose $\lambda_{\min}(\mathbf{L}_2)$ such that $\overline{\alpha}_1 > 0$, i.e.

$$\lambda_{\min}(\mathbf{L}_2) > \frac{1}{2} M_M C_M V_M + \frac{1}{4} M_M d_{M1}, \tag{64}$$

and without any further restrictions we can choose $\lambda_{\max}(\mathbf{L}_2) \geq \lambda_{\min}(\mathbf{L}_2)$. Given the observer gain $\mathbf{L}_2$, α_{11} and γ are known and a lowerbound for $\lambda_{\min}(\mathbf{L}_1)$ can be found, i.e.

$$(\lambda_{\min}(\mathbf{L}_1) - \alpha_{11})^2 > \gamma^2 \big((\lambda_{\max}(\mathbf{L}_1) + \lambda_{\max}(\mathbf{L}_2)) \|\tilde{\eta}_0\|^2 + \|\dot{\tilde{\eta}}_0\|^2\big). \tag{65}$$

We define

$$\begin{aligned} \gamma_1 &= \gamma^2 \left(\lambda_{\max}(\mathbf{L}_2) \|\tilde{\eta}_0\|^2 + \|\dot{\tilde{\eta}}_0\|^2\right) \\ \gamma_2 &= \gamma^2 \|\tilde{\eta}_0\|^2 \end{aligned} \tag{66}$$

It follows from

$$(\lambda_{\min}(\mathbf{L}_1) - \alpha_{11})^2 - \gamma_1 > \gamma_2 \lambda_{\max}(\mathbf{L}_1) \tag{67}$$

that necessarily

$$(\lambda_{\min}(\mathbf{L}_1) - \alpha_{11})^2 - \gamma_1 > \gamma_2 \lambda_{\min}(\mathbf{L}_1), \tag{68}$$

which results in

$$\lambda_{\min}(\mathbf{L}_1) > \frac{(2\alpha_{11} + \gamma_2) + \sqrt{(2\alpha_{11} + \gamma_2)^2 - 4(\alpha_{11}^2 - \gamma_1)}}{2}. \tag{69}$$

Finally to guarantee (65) we have to choose $\lambda_{\max}(\mathbf{L}_1)$ such that

$$\lambda_{\max}(\mathbf{L}_1) \leq \frac{(\lambda_{\min}(\mathbf{L}_1) - \alpha_{11})^2 - \gamma_1}{\gamma_2}. \tag{70}$$

Now, we have proven that the observer gains $\mathbf{L}_1$ and $\mathbf{L}_2$ can be chosen such that $\overline{\alpha}_1, \overline{\alpha}_2 > 0$. Then

$$\dot{V}(\tilde{\eta}, \dot{\tilde{\eta}}) \leq -\overline{\alpha}_1 \|\tilde{\eta}\|^2 - \overline{\alpha}_2 \|\dot{\tilde{\eta}}\|^2 \tag{71}$$

and from Lemma I.I it follows that

$$\dot{V}(\tilde{\eta}, \dot{\tilde{\eta}}) \leq -\kappa_1 \|\tilde{\eta}\|^2 - \kappa_2 \|\dot{\tilde{\eta}}\|^2 \;\forall\; \kappa_1 \in [0, \overline{\alpha}_1] \text{ and } \kappa_2 \in [0, \overline{\alpha}_2] \tag{72}$$

which shows that

$$\dot{V}(\tilde{\eta}, \dot{\tilde{\eta}}) \leq -\beta V(\tilde{\eta}, \dot{\tilde{\eta}}) \tag{73}$$

for some $\beta > 0$, and the observer error dynamics is locally uniformly exponentially stable. Since for every initial condition the observer gains can be chosen such that the observer error dynamics is locally uniformly exponentially stable, we can claim a semi-global result. $\diamond$

C Proof of Proposition 4.2.

Writing $x_e = [e_\eta \; e_\nu]^T$, the closed loop system can be written as

$$\dot{x}_e = \mathbf{T}^T(\psi)\mathbf{A}_c\mathbf{T}(\psi)x_e, \tag{74}$$

where $\mathbf{T}(\psi)$ is defined as $\mathbf{T}(\psi) = \text{diag}(\mathbf{J}^T(\psi),\ \mathbf{I})$ and

$$\begin{aligned}
\mathbf{A}_c &= \mathbf{A} - \mathbf{B}\mathbf{K} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K}_p & -\mathbf{M}^{-1}(\mathbf{D} + \mathbf{K}_d) \end{bmatrix} \\
\mathbf{B} &= \begin{bmatrix} 0 \\ \mathbf{M}^{-1} \end{bmatrix}, \qquad \mathbf{K} = \begin{pmatrix} \mathbf{K}_p & \mathbf{K}_d \end{pmatrix}.
\end{aligned} \tag{75}$$

Notice that the eigenvalues of $\mathbf{T}^T(\psi)\mathbf{A}_c\mathbf{T}(\psi)$ are equal to the eigenvalues of $\mathbf{A}_c$, since $\mathbf{T}^T(\psi) = \mathbf{T}^{-1}(\psi)$.

Define $z = \mathbf{T}(\psi)x_e$, then

$$\dot{z} = \dot{\mathbf{T}}(\psi)\mathbf{T}^T(\psi)z + \mathbf{A}_c z = (\mathbf{A}_c + \mathbf{S}_T(r))z, \tag{76}$$

where $\mathbf{S}_T(r) = \text{diag}(\mathbf{S}(\mathbf{r}),\ \mathbf{0})$.

Choosing the gains

$$\begin{aligned}
\mathbf{K}_d &= 2\mathbf{M}\mathbf{\Lambda}\mathbf{\Omega} - \mathbf{D} \\
\mathbf{K}_p &= \mathbf{M}\mathbf{\Omega}^2,
\end{aligned} \tag{77}$$

$\mathbf{\Lambda} = \mathbf{\Lambda}^T > 0$ and $\mathbf{\Omega} = \mathbf{\Omega}^T > 0$
results in

$$\mathbf{A}_c = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{\Omega}^2 & -2\mathbf{\Lambda}\mathbf{\Omega} \end{bmatrix}, \tag{78}$$

The following candidate Lyapunov function is proposed

$$V = x_e^T\mathbf{T}^T(\psi)\mathbf{P}\mathbf{T}(\psi)x_e = z^T\mathbf{P}z, \tag{79}$$

where

$$\mathbf{P} = \begin{bmatrix} \frac{(2\mathbf{\Lambda}\mathbf{\Omega})^2 + \mathbf{\Omega}^2 + \mathbf{\Omega}^4}{2\mathbf{\Lambda}\mathbf{\Omega}} & \mathbf{I} \\ \mathbf{I} & -\frac{\mathbf{I} + \mathbf{\Omega}^2}{2\mathbf{\Lambda}\mathbf{\Omega}} \end{bmatrix}. \tag{80}$$

The corresponding derivative is

$$\begin{aligned}
\dot{V} &= z^T \left(\underbrace{\begin{bmatrix} 2\mathbf{\Omega}^2 & 0 \\ 0 & 2\mathbf{\Omega}^2 \end{bmatrix}}_{\mathbf{Q}} + \begin{bmatrix} 0 & \mathbf{S}^T(r) \\ \mathbf{S}(r) & 0 \end{bmatrix} \right) z \\
&\leq -(\lambda_{\min}(\mathbf{Q}) - r_{\text{ref}}^{\max} - |e_r|)\|x_e\|^2.
\end{aligned} \tag{81}$$

Since (79) satisfies (49)

$$\lambda_{\min}(\mathbf{P})\|x_e\|^2 \leq V \leq \lambda_{\max}(\mathbf{P})\|x_e\|^2 \tag{82}$$

and (81) satisfies (50)

$$\dot{V} \leq -(\underbrace{\lambda_{\min}(\mathbf{Q}) - r_{\text{ref}}^{\max}}_{\alpha} - \underbrace{1}_{\gamma} |e_r|)\|x_e\|^2, \tag{83}$$

Lemma 2.1 is used to prove stability. In analogy with (51)

$$\overline{\alpha} = \lambda_{\min}(\mathbf{Q}) - r_{\text{ref}}^{\max} - \sqrt{\frac{\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{P})}} \|x_{e0}\| > 0 \tag{84}$$

Since $\lambda_{\min}(\mathbf{Q}) \sim \mathcal{O}(\mathbf{\Omega^2})$, $\sqrt{\frac{\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{P})}} \sim \mathcal{O}(\mathbf{\Omega})$ and $\mathbf{\Lambda}$ influences the ratio $\frac{\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{P})}$, we can choose $\mathbf{\Omega}$ and $\mathbf{\Lambda}$ such that (84) is positive. Therefore we first choose the relative damping ratio $\mathbf{\Lambda}$ and further choose $\mathbf{\Omega}$ such that (84) is positive. Then using Lemma 1.1 it follows that

$$\dot{V} \leq -\beta V \tag{85}$$

for some $\beta > 0$, and the controller error dynamics is locally exponentially stable. Since for every initial condition the controller gains can be chosen such that $\overline{\alpha} > 0$, we can claim a semi-global result. $\diamond$

D Proof of Proposition 4.3.

The closed-loop errors can be written in the form

$$\begin{aligned}
&\dot{e}_\eta = \mathbf{J}(\psi) e_\nu \\
&\dot{e}_\nu = -\mathbf{M}^{-1}(\mathbf{D} + \mathbf{K}_d) e_\nu - \mathbf{M}^{-1}\mathbf{K}_p \mathbf{J}^T(\psi) e_\eta + g(e_\eta, e_\nu, \tilde{\eta}, \dot{\tilde{\eta}}) \\
&\dot{\tilde{\eta}} = \dot{\eta} - \dot{\hat{\eta}} \\
&\ddot{\tilde{\eta}} = -\mathbf{M}^{-1}(\eta)(\mathbf{C}(\eta, \dot{\eta})\dot{\eta} - \mathbf{C}(\eta, \dot{\hat{\eta}})\dot{\hat{\eta}}) \\
&\quad - \mathbf{M}^{-1}(\eta)(\mathbf{d}(\eta, \dot{\eta}) - \mathbf{d}(\eta, \dot{\hat{\eta}})) - \mathbf{L}_2 \tilde{\eta} - \mathbf{L}_1 \dot{\tilde{\eta}},
\end{aligned} \tag{86}$$

where

$$\begin{aligned}
&g(e_\eta, e_\nu, \tilde{\eta}, \dot{\tilde{\eta}}) = -\mathbf{M}^{-1}\mathbf{K}_p \mathbf{J}^T(\psi)(\mathbf{J}(\tilde{\psi}) - \mathbf{I}) e_\eta + \mathbf{M}^{-1}\mathbf{K}_p \mathbf{J}^T(\psi)\mathbf{J}(\tilde{\psi})\tilde{\eta} \\
&+ \mathbf{M}^{-1}\mathbf{K}_d \tilde{\nu} + (\mathbf{M}^{-1}(\mathbf{D} + \mathbf{K}_d)\mathbf{J}^T(e_\psi)(\mathbf{J}(\tilde{\psi}) - \mathbf{I}) - \mathbf{S}^T(\tilde{r})\mathbf{J}^T(e_\psi))\nu_{\text{ref}} \\
&+ \mathbf{S}^T(e_r)\mathbf{J}^T(e_\psi)(\mathbf{J}(\tilde{\psi}) - \mathbf{I})\nu_{\text{ref}} + \mathbf{J}^T(e_\psi)(\mathbf{J}(\tilde{\psi}) - \mathbf{I})\dot{\nu}_{\text{ref}} - \mathbf{C}(\nu)\tilde{\nu} \\
&+ \mathbf{C}(\tilde{\nu})(\nu - \tilde{\nu}) + D_n(\nu) - D_n(\hat{\nu}).
\end{aligned} \tag{87}$$

By collecting the states $x_e = [e_\eta \; e_\nu]^T$, we propose the following candidate Lyapunov function

$$\begin{aligned}
&V(x_e, \tilde{\eta}, \dot{\tilde{\eta}}) = x_e^T \mathbf{T}^T(\psi)\mathbf{P}\mathbf{T}(\psi) x_e \\
&+ \frac{1}{2}\tilde{\eta}^T(\mathbf{L}_2 + \frac{1}{2}\mathbf{L}_1)\tilde{\eta} + \frac{1}{2}\dot{\tilde{\eta}}^T \tilde{\eta} + \frac{1}{2}\dot{\tilde{\eta}}^T \dot{\tilde{\eta}},
\end{aligned} \tag{88}$$

which is a linear combination of the Lyapunov functions of the separate observer and separate controller. From (61) and (82) a lowerbound and an upperbound for

$V(x_e, \tilde{\eta}, \dot{\tilde{\eta}})$ can be found, i.e.

$$\begin{aligned}&\lambda_{\min}(\mathbf{P})\|x_e\|^2 + \frac{1}{4}(\|\tilde{\eta}\|^2 + \|\dot{\tilde{\eta}}\|^2) \leq V(x_e, \tilde{\eta}, \dot{\tilde{\eta}}) \leq \\ &\lambda_{\max}(\mathbf{P})\|x_e\|^2 + (\lambda_{\max}(\mathbf{L}_2) + \lambda_{\max}(\mathbf{L}_1))\|\tilde{\eta}\|^2 + \|\dot{\tilde{\eta}}\|^2.\end{aligned} \tag{89}$$

From (2) and Assumption 1 it can be seen that the Coriolis and centripetal term can be upperbouned, i.e.

$$\begin{aligned}&\mathbf{C}(\nu)\tilde{\nu} + \mathbf{C}(\tilde{\nu})(\nu - \tilde{\nu}) = \mathbf{C}(e_\nu + \mathbf{J}^T(e_\psi)\nu_{\text{ref}})\tilde{\nu} + \mathbf{C}(\tilde{\nu})(e_\nu + \mathbf{J}^T(e_\psi)\nu_{\text{ref}}) \\ &-\mathbf{C}(\tilde{\nu})\tilde{\nu} \leq 2C_M\|e_\nu\|\|\tilde{\nu}\| + 2C_M\|\nu_{\text{ref}}\|\|\tilde{\nu}\| + C_M\|\tilde{\nu}\|^2.\end{aligned} \tag{90}$$

Using (90), Assumption 2 and $\|\tilde{\nu}\| = \|\dot{\tilde{\eta}}\|$, the term $g(e_\eta, e_\nu, \tilde{\eta}, \dot{\tilde{\eta}})$ can be upperbounded, i.e.

$$\begin{aligned}&g(x_e, \tilde{\eta}, \dot{\tilde{\eta}}) \leq M_M\,\big(2\lambda_{\max}(\mathbf{K}_p)\|x_e\|\|\tilde{\eta}\| + \lambda_{\max}(\mathbf{K}_p)\|\tilde{\eta}\| \\ &+\lambda_{\max}(\mathbf{K}_d)\|\dot{\tilde{\eta}}\| + 2\lambda_{\max}(\mathbf{D} + \mathbf{K}_d)\|\nu_{\text{ref}}\|\big) + 2\|x_e\|\|\nu_{\text{ref}}\|\|\tilde{\eta}\| \\ &+\|\nu_{\text{ref}}\|\|\dot{\tilde{\eta}}\| + 2\|\dot{\tilde{\eta}}\|\|\dot{\nu}_{\text{ref}}\|\|\tilde{\eta}\| + 2C_M\|e_\nu\|\|\tilde{\nu}\| \\ &+2C_M\|\nu_{\text{ref}}\|\|\tilde{\nu}\| + C_M\|\tilde{\nu}\|^2 + (d_{M3} + d_{M4}\|\dot{\tilde{\eta}}\|)\|\dot{\tilde{\eta}}\|.\end{aligned} \tag{91}$$

Now using (60), (81) and (91) the derivative of the proposed candidate Lyapunov function (88) can be upperbounded by

$$\begin{aligned}&\dot{V}(x_e, \tilde{\eta}, \dot{\tilde{\eta}}) \leq -(\lambda_{\min}(\mathbf{Q}) - r_{\text{ref}}^{\max} - |e_r|)\|x_e\|^2 \\ &-\left(\frac{1}{2}\lambda_{\min}(\mathbf{L}_2) - \frac{1}{4}M_M C_M\|\nu_{\text{ref}}\| - \frac{1}{8}M_M d_{M1}\right)\|\tilde{\eta}\|^2 \\ &-\Bigg(\lambda_{\min}(\mathbf{L}_1) - \frac{1}{2} - 2M_M C_M\|\nu_{\text{ref}}\| - M_M d_{M1} - M_M C_M\|\nu_{\text{ref}}\| \\ &-\frac{1}{2}M_M d_{M1}\Bigg)\|\dot{\tilde{\eta}}\|^2 + \left(\frac{1}{2}M_M C_M + \frac{1}{2}M_M d_{M2}\right)\|\tilde{\eta}\|\|\dot{\tilde{\eta}}\|^2 \\ &+(M_M C_M + M_M d_{M2})\|\dot{\tilde{\eta}}\|\|\dot{\tilde{\eta}}\|^2 + \frac{1}{4}M_M C_M\|x_e\|\|\tilde{\eta}\|^2 + 2M_M C_M \\ &\|x_e\|\|\dot{\tilde{\eta}}\|^2 + 2\lambda_{\max}(\mathbf{P})M_M\,\big(2\lambda_{\max}(\mathbf{K}_p)\|\tilde{\eta}\|\|x_e\|^2 + \lambda_{\max}(\mathbf{K}_p)\|\tilde{\eta}\| \\ &\|x_e\| + \lambda_{\max}(\mathbf{K}_d)\|\dot{\tilde{\eta}}\|\|x_e\| + 2\lambda_{\max}(\mathbf{D} + \mathbf{K}_d)\|\nu_{\text{ref}}\|\|x_e\|\|\tilde{\eta}\|\big) + \\ &4\lambda_{\max}(\mathbf{P})\|\nu_{\text{ref}}\|\|\tilde{\eta}\|\|x_e\|^2 + 2\lambda_{\max}(\mathbf{P})\|\nu_{\text{ref}}\|\|\dot{\tilde{\eta}}\|\|x_e\| + 4\lambda_{\max}(\mathbf{P}) \\ &\|\dot{\nu}_{\text{ref}}\|\|\tilde{\eta}\|\|x_e\|^2 + 4\lambda_{\max}(\mathbf{P})C_M\|\nu_{\text{ref}}\|\|\dot{\tilde{\eta}}\|\|x_e\| + 4\lambda_{\max}(\mathbf{P})C_M \\ &\|\dot{\tilde{\eta}}\|\|x_e\|^2 + 2\lambda_{\max}(\mathbf{P})C_M\|x_e\|\|\dot{\tilde{\eta}}\|^2 + 2\lambda_{\max}(\mathbf{P})d_{M3}\|x_e\|\|\dot{\tilde{\eta}}\| \\ &+2\lambda_{\max}(\mathbf{P})d_{M4}\|x_e\|\|\dot{\tilde{\eta}}\|^2.\end{aligned} \tag{92}$$

Rewriting the cross terms in (92) by completion of squares (54),

$$
\begin{aligned}
&\dot{V}(x_e,\tilde{\eta},\dot{\tilde{\eta}}) \leq -(\lambda_{\min}(\mathbf{Q}) - r_{\text{ref}}^{\max} - |e_r|)\|x_e\|^2 \\
&+\epsilon_3^2\lambda_{\max}(\mathbf{P})(M_M\lambda_{\max}(\mathbf{K}_p)+M_M\lambda_{\max}(\mathbf{K}_d)+2M_M\lambda_{\max}(\mathbf{D}+ \\
&\mathbf{K}_d)\|\nu_{\text{ref}}\|+\|\nu_{\text{ref}}\|+2C_M\|\nu_{\text{ref}}\|+2C_M\|x_e\|+d_{M3}+ \\
&2M_M\lambda_{\max}(\mathbf{K}_p)\|x_e\|+2\|\nu_{\text{ref}}\|\|x_e\|+2\|\dot{\nu}_{\text{ref}}\|\|x_e\|)\|x_e\|^2 \\
&-\left(\frac{1}{2}\lambda_{\min}(\mathbf{L}_2)-\frac{1}{4}M_MC_M\|\nu_{\text{ref}}\|-\frac{1}{8}M_Md_{M1}\right)\|\tilde{\eta}\|^2 \\
&+\frac{1}{4}M_MC_M\|x_e\|\|\tilde{\eta}\|^2 \\
&+\epsilon_3^{-2}\lambda_{\max}(\mathbf{P})(M_M\lambda_{\max}(\mathbf{K}_p)+M_M\lambda_{\max}(\mathbf{K}_d)+2M_M\lambda_{\max}(\mathbf{D}+\mathbf{K}_d) \\
&\|\nu_{\text{ref}}\|+2M_M\lambda_{\max}(\mathbf{K}_p)\|x_e\|+2\|\nu_{\text{ref}}\|\|x_e\|+2\|\dot{\nu}_{\text{ref}}\|\|x_e\|)\|\tilde{\eta}\|^2 \\
&-\Bigg(\lambda_{\min}(\mathbf{L}_1)-\frac{1}{2}-2M_MC_M\|\nu_{\text{ref}}\|-M_Md_{M1}-M_MC_M\|\nu_{\text{ref}}\| \\
&-\frac{1}{2}M_Md_{M1}\Bigg)\|\dot{\tilde{\eta}}\|^2+\left(\frac{1}{2}M_MC_M+\frac{1}{2}M_Md_{M2}\right)\|\tilde{\eta}\|\|\dot{\tilde{\eta}}\|^2 \\
&+(M_MC_M+M_Md_{M2})\|\dot{\tilde{\eta}}\|\|\dot{\tilde{\eta}}\|^2 \\
&+\epsilon_3^{-2}\lambda_{\max}(\mathbf{P})(\|\nu_{\text{ref}}\|+2C_M\|\nu_{\text{ref}}\|+2C_M\|x_e\|+d_{M3})\|\dot{\tilde{\eta}}\|^2 \\
&+(2\lambda_{\max}(\mathbf{P})C_M+2\lambda_{\max}(\mathbf{P})d_{M4})\|x_e\|\|\dot{\tilde{\eta}}\|^2
\end{aligned} \tag{93}
$$

In analogy with (51), we obtain from (93) $\overline{\alpha}_1$, $\overline{\alpha}_2$ and $\overline{\alpha}_3$, i.e.

$$
\begin{aligned}
&\overline{\alpha}_1=\lambda_{\min}(\mathbf{Q})-r_{\text{ref}}^{\max}-\epsilon_3^2\lambda_{\max}(\mathbf{P})(M_M\lambda_{\max}(\mathbf{K}_p) \\
&+M_M\lambda_{\max}(\mathbf{K}_d)+2M_M\lambda_{\max}(\mathbf{D}+\mathbf{K}_d)\|\nu_{\text{ref}}\|+\|\nu_{\text{ref}}\|+ \\
&2C_M\|\nu_{\text{ref}}\|+d_{M3})-\big(1+\epsilon_3^2\lambda_{\max}(\mathbf{P})(2C_M+2M_M\lambda_{\max}(\mathbf{K}_p) \\
&+2\|\nu_{\text{ref}}\|+2\|\dot{\nu}_{\text{ref}}\|)\big)\sqrt{\frac{V_0}{\lambda_{\min}(\mathbf{P})}} \\
&\overline{\alpha}_2=\frac{1}{2}\lambda_{\min}(\mathbf{L}_2)-\frac{1}{4}M_MC_M\|\nu_{\text{ref}}\|-\frac{1}{8}M_Md_{M1}-\epsilon_3^{-2}\lambda_{\max}(\mathbf{P}) \\
&(M_M\lambda_{\max}(\mathbf{K}_p)+M_M\lambda_{\max}(\mathbf{K}_d)+2M_M\lambda_{\max}(\mathbf{D}+\mathbf{K}_d)\|\nu_{\text{ref}}\|) \\
&-\left(\frac{1}{4}M_MC_M+\epsilon_3^{-2}\lambda_{\max}(\mathbf{P})(2M_M\lambda_{\max}(\mathbf{K}_p)+2\|\nu_{\text{ref}}\|+2\|\dot{\nu}_{\text{ref}}\|)\right) \\
&\sqrt{\frac{V_0}{\lambda_{\min}(\mathbf{P})}} \\
&\overline{\alpha}_3=\lambda_{\min}(\mathbf{L}_1)-\frac{1}{2}-3M_MC_M\|\nu_{\text{ref}}\|-M_Md_{M1} \\
&-\frac{1}{2}M_Md_{M1}-\epsilon_3^{-2}\lambda_{\max}(\mathbf{P})(\|\nu_{\text{ref}}\|+2C_M\|\nu_{\text{ref}}\|+d_{M3}) \\
&-\Bigg((2C_M+2\lambda_{\max}(\mathbf{P})C_M+2\lambda_{\max}(\mathbf{P})d_{M4})\sqrt{\frac{1}{\lambda_{\min}(\mathbf{P})}} \\
&+3M_MC_M+3M_Md_{M2}\Bigg)\sqrt{V_0}
\end{aligned} \tag{94}
$$

where

$$V_0 \leq \lambda_{\max}(\mathbf{P})\|x_{e0}\|^2 + (\lambda_{\max}(\mathbf{L}_2) + \lambda_{\max}(\mathbf{L}_1))\|\tilde{\eta}_0\|^2 + \|\dot{\tilde{\eta}}_0\|^2 \tag{95}$$

From (94) and (95) a gain tuning procedure is developed to guarantee $\overline{\alpha}_1, \overline{\alpha}_2, \overline{\alpha}_3 > 0$. In analogy with the gain tuning procedure for the controller in Section 4.2, we choose

$$\begin{aligned} \mathbf{K}_d &= 2\mathbf{M}\mathbf{\Lambda}\mathbf{\Omega} - \mathbf{D} \\ \mathbf{K}_p &= \mathbf{M}\mathbf{\Omega}^2. \end{aligned} \tag{96}$$

Then choose $\mathbf{\Lambda}$ and given $\mathbf{\Lambda}$, $\mathbf{\Omega}$ is chosen such that

$$\overline{\alpha}_{1c} = \lambda_{\min}(\mathbf{Q}) - r_{\text{ref}}^{\max} - \sqrt{\frac{V_0}{\lambda_{\min}(\mathbf{P})}} > 0, \tag{97}$$

where

$$\mathbf{Q} = \begin{bmatrix} 2\mathbf{\Omega}^2 & 0 \\ 0 & 2\mathbf{\Omega}^2 \end{bmatrix}, \mathbf{P} = \begin{bmatrix} \frac{(2\mathbf{\Lambda}\mathbf{\Omega})^2 + \mathbf{\Omega}^2 + \mathbf{\Omega}^4}{2\mathbf{\Lambda}\mathbf{\Omega}} & \mathbf{I} \\ \mathbf{I} & -\frac{\mathbf{I}+\mathbf{\Omega}^2}{2\mathbf{\Lambda}\mathbf{\Omega}} \end{bmatrix}, \tag{98}$$

while ϵ_3 is chosen such that $\overline{\alpha}_1 > 0$.

Since ϵ_3 is known, we can choose $\lambda_{\min}(\mathbf{L}_2)$ such that $\overline{\alpha}_2 > 0$, i.e.

$$\begin{aligned} &\lambda_{\min}(\mathbf{L}_2) > \frac{1}{2}M_M C_M\|\nu_{\text{ref}}\| + \frac{1}{4}M_M d_{M1} + 2\epsilon_3^{-2}\lambda_{\max}(\mathbf{P}) \\ &(M_M\lambda_{\max}(\mathbf{K}_p) + M_M\lambda_{\max}(\mathbf{K}_d) + 2M_M\lambda_{\max}(\mathbf{D}+\mathbf{K}_d)\|\nu_{\text{ref}}\|) \\ &+ \left(\frac{1}{2}M_M C_M + 2\epsilon_3^{-2}\lambda_{\max}(\mathbf{P})(2M_M\lambda_{\max}(\mathbf{K}_p) + 2\|\nu_{\text{ref}}\| + 2\|\dot{\nu}_{\text{ref}}\|)\right) \\ &\sqrt{\frac{V_0}{\lambda_{\min}(\mathbf{P})}} \end{aligned} \tag{99}$$

and without any further restriction we can choose $\lambda_{\max}(\mathbf{L}_2) \geq \lambda_{\min}(\mathbf{L}_2)$. In analogy with the gain tuning procedure for the observer in Section 4.1, we define

$$\begin{aligned} &\alpha_{11} = \frac{1}{2} + 3M_M C_M\|\nu_{\text{ref}}\| + \frac{3}{2}M_M d_{M1} \\ &+ \epsilon_3^{-2}\lambda_{\max}(\mathbf{P})(\|\nu_{\text{ref}}\| + 2C_M\|\nu_{\text{ref}}\| + d_{M3}) \\ &\gamma = 2C_M + 2\lambda_{\max}(\mathbf{P})C_M + 2\lambda_{\max}(\mathbf{P})d_{M4}\sqrt{\frac{1}{\lambda_{\min}(\mathbf{P})}} + 3M_M C_M \\ &+ 3M_M d_{M2} \\ &\gamma_1 = \gamma^2\left(\lambda_{\max}(\mathbf{P})\|x_{e0}\|^2 + \lambda_{\max}(\mathbf{L}_2)\|\tilde{\eta}_0\|^2 + \|\dot{\tilde{\eta}}_0\|^2\right) \\ &\gamma_2 = \gamma^2\|\tilde{\eta}_0\|^2 \end{aligned} \tag{100}$$

Given the already known controller gains $\mathbf{K}_p$ and $\mathbf{K}_d$ and the observer gain $\mathbf{L}_2$, a lower bound for $\lambda_{\min}(\mathbf{L}_1)$ can be found, i.e.

$$\begin{aligned} &(\lambda_{\min}(\mathbf{L}_1) - \alpha_{11})^2 > \\ &\gamma^2\left(\lambda_{\max}(\mathbf{P})\|x_{e0}\|^2 + (\lambda_{\max}(\mathbf{L}_1) + \lambda_{\max}(\mathbf{L}_2))\|\tilde{\eta}_0\|^2 + \|\dot{\tilde{\eta}}_0\|^2\right). \end{aligned} \tag{101}$$

It follows from

$$(\lambda_{\min}(\mathbf{L}_1) - \alpha_{11})^2 - \gamma_1 > \gamma_2 \lambda_{\max}(\mathbf{L}_1) \tag{102}$$

that necessarily

$$(\lambda_{\min}(\mathbf{L}_1) - \alpha_{11})^2 - \gamma_1 > \gamma_2 \lambda_{\min}(\mathbf{L}_1), \tag{103}$$

which results in

$$\lambda_{\min}(\mathbf{L}_1) > \frac{(2\alpha_{11} + \gamma_2) + \sqrt{(2\alpha_{11} + \gamma_2)^2 - 4(\alpha_{11}^2 - \gamma_1)}}{2}. \tag{104}$$

Finally to guarantee (101) we have to choose $\lambda_{\max}(\mathbf{L}_1)$ such that

$$\lambda_{\max}(\mathbf{L}_1) \leq \frac{(\lambda_{\min}(\mathbf{L}_1) - \alpha_{11})^2 - \gamma_1}{\gamma_2}. \tag{105}$$

Now, we have proven that the observer gains $\mathbf{L}_1$ and $\mathbf{L}_2$ can be chosen such that $\overline{\alpha}_2, \overline{\alpha}_3 > 0$.

Then using Lemma 1.1 it follows that

$$\dot{V} \leq -\beta V \tag{106}$$

for some $\beta > 0$, and the observer-controller error dynamics is locally exponential stable for bounded $\|\nu_{\text{ref}}\|$ and $\|\dot{\nu}_{\text{ref}}\|$. Since for every initial condition the observer and controller gains can be chosen such that $\overline{\alpha}_1, \overline{\alpha}_2, \overline{\alpha}_3 > 0$, we can claim a semi-globally result. $\diamond$

Figure 1: Basin at the Marine Cybernetics Laboratory located at Tyholt, Trondheim.

Figure 2: Cybership II, a model supply ship, Froude scaled 1:70.

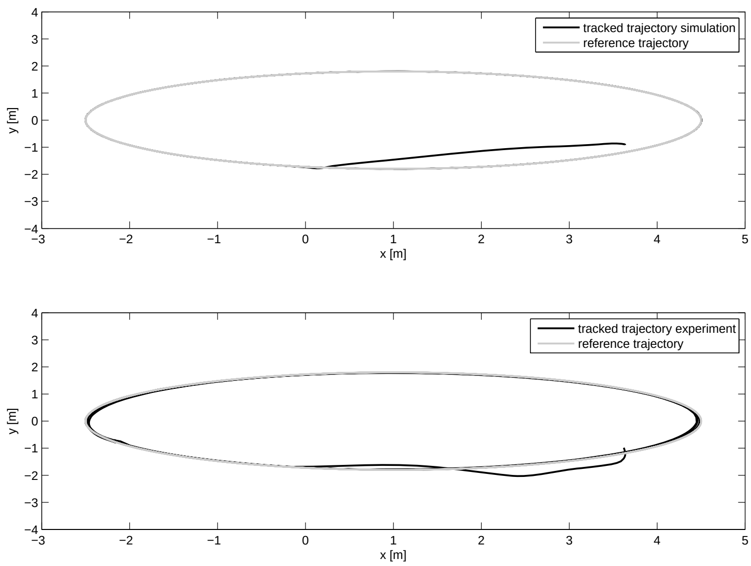

Figure 3: XY plot of the reference and tracked trajectories in the experiment without waves and computer simulation.

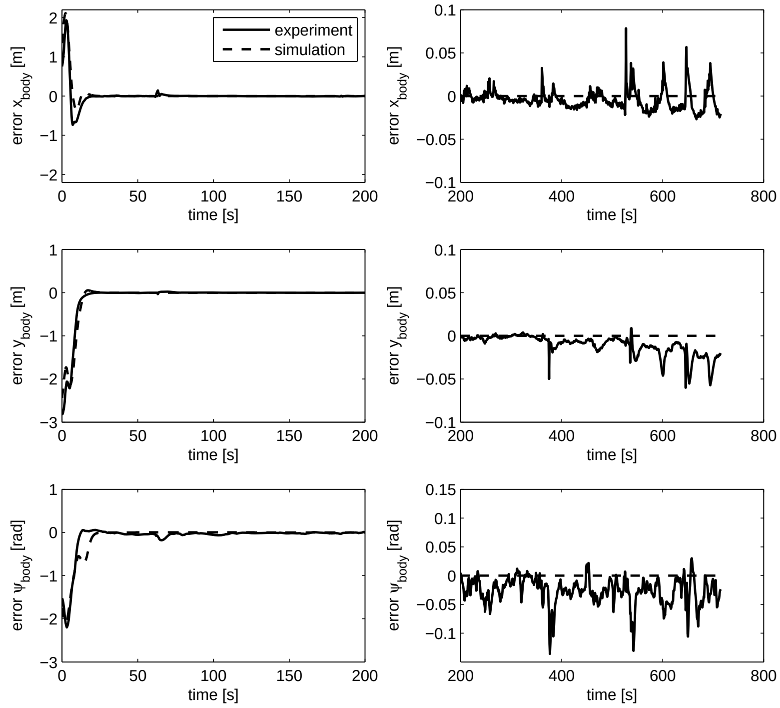

Figure 4: Position errors of the observer-controller scheme: $e_{x\text{body}}$, $e_{y\text{body}}$ and e_{ψ}.

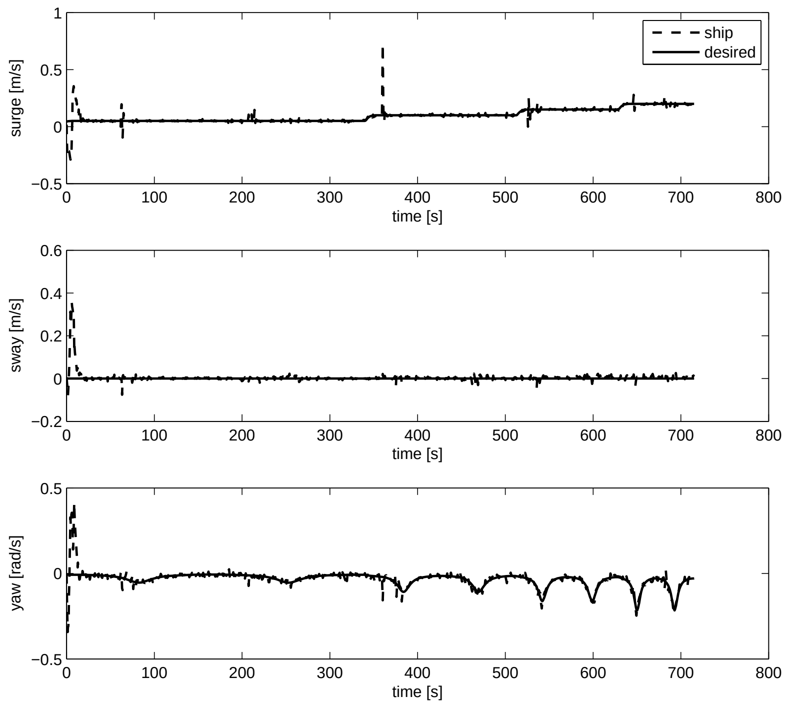

Figure 5: Desired velocity u_{ref}, v_{ref} and r_{ref} and ship velocity u, v and r estimated by the numerical differentiator.