# Optimal Flow Control of a Manufacturing System: an Introduction

M.A.R. Stoets (Michiel)

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Master's Thesis

Supervisor: Prof.dr.ir. J.E. Rooda
Coach: Dr.ir. A.A.J. Lefeber

Eindhoven University of Technology
Faculty of Mechanical Engineering
Systems Engineering Group

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# Preface

This master's thesis is the final part of a final assignment at the Systems Engineering Group of the Eindhoven University of Technology, the Department of Mechanical Engineering. The final assignment is the closing part of a five years curriculum of becoming a Master of Science in Mechanical Engineering, with a specialization in Systems Engineering. The Systems Engineering group aims to develop methods, techniques and tools for the design of advanced industrial systems. It focuses on production systems comprising multiple communicating subsystems that work in parallel. The final assignment is performed in the Optimization and Control research theme.

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# Summary

One of the important aspects in the design and control of manufacturing systems is the presence of random machine failures. This temporarily reduction of production capacity may cause the system not to meet its demand. Safety stocks are commonly used to prevent such a backlog. This may incorporate unnecessary high and costly inventories and work-in-process (WIP). Tradeoffs between backlog, inventory and WIP need to be made. View the manufacturing system as a dynamical system in which some variables are free to choose. Optimal control theory provides methods to derive a controller that achieves the desired balance of costs. The objectives of this research are to introduce the basic optimal control techniques, to provide an overview of the application of those techniques related to failure prone manufacturing systems, and to apply the optimal control techniques in a case. Often, a gap lies between complex, discrete-event models and standard techniques adapted to continuous-time models. This gap can be bridged by a concept of approximation. In the context of this research, Kimemia and Gershwin [Kim83] have proposed a flow model. The flow of products is modeled here as a continuous process, that enables the use of optimal control theory. The problem of optimal control is to find an admissible control function that minimizes the performance index subject to the system dynamics and constraints. The Minimum Principle provides insight in the optimal system behavior for deterministic problems. Dynamic programming provides means for deriving a feedback control law that achieves the optimal behavior also for stochastic problems. Literature is investigated that concerns the application of optimal control theory as a method to design an optimal flow controller to a failure prone manufacturing system. The focus lies on a two-machine flow shop. A linear performance index leads to an optimal control policy defined by hedging points. The control policy is then to produce at maximum rate, to demand, or not at all. In a deterministic single machine case, a quadratic performance index is considered. The control problem considers control constraints. Optimal control techniques are employed to find an optimal feedback control law. The resulting behavior shows graduate changes between the maximum, demand, and zero production rates. Remaining difficulties are the integration of control constraints and derivation of an explicit solution to the Hamilton-Jacobi-Bellman (HJB) equation. The flow model has been validated by means of simulation. A separation of time scales for planning and operations has been obtained that justifies the application of the flow model. Further research to the integration of state constraints, to the application of performance indices to describe the desired behavior, and to better approximate, continuous-time models is suggested.

# Samenvatting (in Dutch)

Eén van de belangrijke aspecten bij het ontwerpen en regelen van fabricagesystemen is de aanwezigheid van willekeurige machinestoringen. Een mogelijk gevolg van deze tijdelijk reductie van productiecapaciteit is dat het systeem niet aan de vraag kan voldoen. Vaak wordt een veiligheidsvoorraad aangehouden om productieachterstand te voorkomen. Echter, dit kan leiden tot onnodig hoge en dure voorraden in hoeveelheden onderhanden werk (WIP). Dit roept om een afweging tussen productieachterstand, voorraden en WIP. Het fabricagesysteem kan worden gezien als een dynamisch systeem waarin enkele variabelen vrij mogen worden gekozen. Met behulp van optimale besturingstheorie is het mogelijk om een regelaar te vinden die de gewenste balans tussen kosten realiseert. De doelen van dit onderzoek zijn als volgt gesteld. Introduceer de fundamentele optimale regeltechnieken. Geef een overzicht hoe optimale regeltechnieken toegepast kunnen worden in relatie tot storingsgevoelige fabricagesystemen. Pas deze technieken toe in een casus. Modellen van fabricagesystemen zijn veelal complex van aard en bevatten discrete gebeurtenissen. De standaard technieken op het vlak van optimalisatie en besturing zijn meestal toegespitst op modellen met een continu karakter. Middels een benaderingsconcept kan deze kloof worden overbrugd. Kimemia en Gershwin [Kim83] hebben een stroommodel voorgesteld dat als benaderingsconcept in dit onderzoek kan dienen. De modellering van de materiaalstroom als een continu process maakt het mogelijk om optimale besturingstheorie toe te passen. Het beschouwde probleem in optimale besturingstheorie is om een toelaatbare regelfunctie te vinden die een kostencriterium minimaliseert waarbij aan de systeemdynamica en beperkingen voldaan wordt. Het Minimum Principe geeft inzicht in het optimale systeemgedrag voor deterministische problemen. Dynamisch programmeren maakt het mogelijk een teruggekoppelde regelaar af te leiden voor algemene problemen. De toepassing van optimale besturingstheorie is onderzocht in de literatuur met betrekking tot het ontwerpen van een optimale regeling voor de materiaalstroom in storingsgevoelige fabricagesystemen. De nadruk ligt hierbij op een lijnproductiesysteem met twee machines. Een lineair kostencriterium leidt tot een regeling die gedefinieerd wordt door normvoorraden. De regeling resulteert in het instandhouden van deze voorraadniveau's door maximale productie, productie naar vraag of geen productie. Een kwadratisch kostencriterium is toegepast in een casus betreffende een deterministisch, enkel machine systeem. Hierbij zijn beperkingen op de regelvariabelen meegenomen. Optimale regeltechnieken zijn toegepast bij het vinden van een optimale regelwet met toestandsterugkoppeling. Het resulterende systeemgedrag kenmerkt zich door de geleidelijke overgangen tussen maximale productie,

productie naar vraag en geen productie. Het integreren van de beperkingen op de regelvariabelen en het vinden van een oplossing voor de Hamilton-Jacobi-Bellman (HJB) vergelijking blijven over als kenmerkende moeilijkheden. Door middel van simulatie is het stroommodel gevalideerd. Een scheiding van tijdschalen voor productieplanning en productie is bepaald die het gebruik van het stroommodel rechtvaardigt. Verder onderzoek is voorgesteld naar de integratie van beperkingen op de toestandsvariabelen, het gebruik van een kostencriterium om een gewenst systeemgedrag te beschrijven en naar beter benaderende continue modellen.

# Contents

**Preface** **i**

**Summary** **iii**

**Samenvatting (in Dutch)** **v**

**Symbols** **ix**

**1 Introduction** **1**

**2 Control design** **5**
2.1 Design process . . . . . 5
2.2 Control problem . . . . . 7
2.3 Concept of approximation . . . . . 11
2.4 Synopsis . . . . . 15

**3 Optimal control** **17**
3.1 Optimal control problem . . . . . 17
3.2 Minimum Principle . . . . . 18
3.3 Dynamic programming . . . . . 20

**4 Optimal control of a two-machine flow shop** **23**
4.1 Manufacturing system . . . . . 23
4.2 Optimal control problem . . . . . 25
4.3 Solutions . . . . . 28

**5 Single machine case** **33**
5.1 Manufacturing system . . . . . 33
5.2 Control problem . . . . . 35
5.3 Solutions . . . . . 36
5.4 Simulation . . . . . 39

**6 Conclusions** **51**

**7 Suggestions for further research** **53**

**A Free control** **59**
A.1 Open loop control . . . . . 60
A.2 Feedback control . . . . . 65
A.3 Synopsis . . . . . 70

**B Limited control** **71**
B.1 Numerical case . . . . . 72
B.2 Open loop control . . . . . 78
B.3 Feedback control . . . . . 83

**C Simulation model** **85**
C.1 Tools . . . . . 85
C.2 Deterministic model . . . . . 86
C.3 Stochastic model . . . . . 94

**D Simulation results** **97**
D.1 Parameter influence . . . . . 97
D.2 Flow model validation . . . . . 105
D.3 Open loop and feedback control . . . . . 111

# Symbols

| | |
|---|---|
| $\alpha$ | machine capacity |
| $\beta$ | buffer and upper depot size |
| $\Gamma$ | Gamma function |
| $\lambda$ | co-state variable |
| $\mu$ | constraint function |
| $\sigma$ | configuration state |
| $\Sigma$ | dynamical system |
| $\varphi$ | costs at the end of a period |
| | |
| $b$ | buffer level |
| $\mathfrak{B}$ | behaviour |
| $c$ | function of the control variable |
| $c^2$ | squared coefficient of variation |
| $\mathfrak{C}$ | set of admissible controllers |
| $d$ | demand rate |
| $e$ | relative performance error |
| $f$ | function describing the system dynamics |
| $H$ | Hamiltonian |
| $H^*$ | optimal Hamiltonian |
| $J$ | performance index |
| $J^*$ | optimal performance |
| $L$ | costs during a period |
| $q$ | state costs parameter |
| $r$ | control costs parameter |
| $\mathcal{S}$ | state space |
| $t$ | time |
| $\mathbb{T}$ | time axis |
| $u$ | production rate; control variable |
| $u^*$ | optimal control law |
| $\mathcal{U}$ | control space |
| $\mathbb{U}$ | universum |
| $V$ | cost-to-go |
| $\mathbb{W}$ | signal space |

| | |
|---|---|
| $x$ | surplus; state variable |
| $x^*$ | optimal state path |
| $z^*$ | hedging point |
| | |
| CSS | control switching sets |
| FMS | flexible manufacturing system |
| HJB | Hamilton-Jacobi-Bellman |
| WIP | work-in-process |

# Chapter 1

# Introduction

One of the important aspects in the design and control of manufacturing systems is the presence of random machine failures. This temporarily reduction of production capacity may cause the system not to meet its demand. Safety stocks are commonly used to prevent such a backlog. This may incorporate unnecessary high and costly inventories and work-in-process (WIP) levels. Tradeoffs between backlog, inventory, and WIP need to be made.

The introduction of safety stocks in the form of buffers also leads to new difficulties. As an example, consider a two-machine flow shop with a finite buffer as visualized in Figure 1.1. The manufacturing system produces parts at some production rate. Both

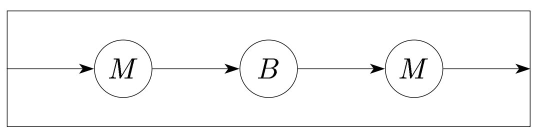



Figure 1.1: A two-machine flow shop.

machines are subjected to random machine failures. The buffer level is kept at a certain safety level to prevent backlog. In the case that the first machine is down, the supply of parts in the buffer enables the second machine to continue production at the same rate. After some time, when the first machine is still down, the supply of parts is diminished and the second machine is starved. This could be prevented by setting the second machine to a lower production rate when the first machine is down. Of course, a higher safety level provides the same effect. But then a side effect is encountered when the second machine is down and the first machine continues production at the same rate. After some time, when the second machine is still down, the capacity of the buffer is reached and the first machine is blocked. Again, this could be prevented by reducing the rate of production of the first machine. Therefore, the random machine failures cause the need for some kind of control of the production rates.

The manufacturing system makes part of a larger system called the industrial system. An industrial system is defined by Brandts [Bra93] as a system that produces products, for which in return it receives compensation. The industrial system is divided into a products system and a production system. The products define the industrial system's relationship with its customers. The objective of the industrial system is to produce products in the right quantity, at the right place, in the right moment, in the right quality, and for the right price. These five attributes form the conditions under which the production system must produce its products. The production system consist of the following subsystems, as modeled by Rooda [Roo00]:

- the manufacturing system,
- the information system, and
- the financial system.

The financial system is not considered in this research. Renders [Ren99] defines the manufacturing system, also called the primary system, as a set of resources, i.e., persons, machines, materials, and tools, that maintains the flow of products. Similar, the information system, also called the secondary system, is a set of resources that maintains the flow of materials. Part of the information system is the manufacturing control system, shorthand control system, that controls the flow of products.

The problem for the control system is to ensure that the manufacturing system achieves the desired tradeoff between inventory, backlog, and production. Costs can be introduced to specify the desired tradeoff. A balance of costs is obtained when the sum of costs is minimized. The control problem is then to find a control system that minimizes the costs. Such an optimal control problem shows similarities with that of an optimization problem. However, a solution to the optimization problem can be considered as static, optimal *values* for design variables are sought that are constant. The optimal control problem focuses on an optimal *function* that minimizes the costs. To find such a control function, knowledge of the dynamics of the manufacturing system is required. The concept of state equations is employed to deal with these dynamics. Several methods from optimal control theory are provided to solve the control problem. The use of state equations is complicated in practice due to the discrete nature of most manufacturing systems. Kimemia and Gershwin [Kim83] propose a flow model that models the flow of products as a continuous process. This simplifies the use of state equations and enables the use of optimal control theory to find a flow control that minimizes the costs, such that set objectives are satisfied.

The objectives of this research are:

- provide an overview of the application of optimal control theory in the flow control of failure prone manufacturing systems,
- introduce the basic techniques of optimal control theory, such as the Minimum Principle for the deterministic problem and dynamic programming for the general problem,
- apply these techniques in a case to provide insight in the application of optimal control theory, to understand the key difficulties, and to consider possible results, and

- validate by means of simulation the applied concept of approximation.

The remainder of this report is organized as follows. In Chapter 2 is explained what is considered as control design in this research. Concepts such as design, dynamical system, interconnection, and control problem are reviewed. Also a concept of approximation is presented that may be of use in the control design process. Chapter 3 deals with the basics of optimal control theory. A performance index is introduced to specify the desired tradeoff between costs. The optimal control problem is formulated as to find a function that minimizes the performance index subject to the system dynamics and constraints. Methods to derive a solution to the optimal control problem are presented. A literature investigation is reviewed in Chapter 4. Literature is investigated that concerns the application of optimal control theory as a method to design an optimal flow controller to a failure prone manufacturing system. The focus lies on a two-machine flow shop. Then, a case concerning the optimal flow control of a single machine manufacturing system is dealt with in Chapter 5. The relative simple problem provides the opportunity to understand the basics of optimal control techniques, to recognize key difficulties in the application of the techniques, and to consider the possible results that can be obtained. Finally, conclusions are made in Chapter 6 with respect to the objectives for this research. Suggestions for further research are presented in Chapter 7.

# Chapter 2

# Control design

The previous chapter expresses the need for a controller to the manufacturing system. In control design, an acceptable design is sought that fulfills the control function such that set objectives of the industrial system are achieved. First, the design process is explained. Then, a control problem is formulated. The problem is formulated in the context of control theory to enable standard techniques for the derivation of a controller. The design process takes place in a representative model of reality. Unfortunately, modeling industrial systems often leads to complex models that are not suited for the application of the favored standard techniques. Therefore, a concept of approximation is introduced.

## 2.1 Design process

Design is a creative process. However, a designer may wish to optimize this process. Formalizing the design process may provide means to do so. The designer has to deal with the problem of finding a design that fulfills some function. Often, solving such a problem is a repetitive cycle of gaining and applying knowledge obtained from previous results. Such a cycle is referred to in the behavioral sciences as an empirical cycle. In the special case that the subject is aware of the situation in which the cycle takes place, De Groot [deG72] refers to the empirical cycle with reflection:

observe — guess — expect — check — evaluate.

First, the subject becomes aware of the current situation by observations. To achieve a desired effect, the subject makes a guess about some actions. This guess is based upon previous results in a similar, though different situation. These actions are expected to have certain results. Then, the expectations can be checked with the desired effect. Finally, the subject evaluates the check: are the guessed actions expected to achieve the desired effect? No real actions are performed in the empirical cycle with reflection.

Obviously, consciously applying the cycle above lies close to what is referred to as problem solving. Roozenburg and Eekels [Roo98] recognize this, as they formalize the design process in five phases:

analysis — synthesis — simulation — evaluation — decision.

In these five phases, the problem of finding a design that fulfills a desired function is solved. In the analysis phase, a concrete design problem is formulated together with criteria that the design must satisfy. The synthesis phase generates a design. By means of simulation the expected behavior of the design is determined. Simulation is not done

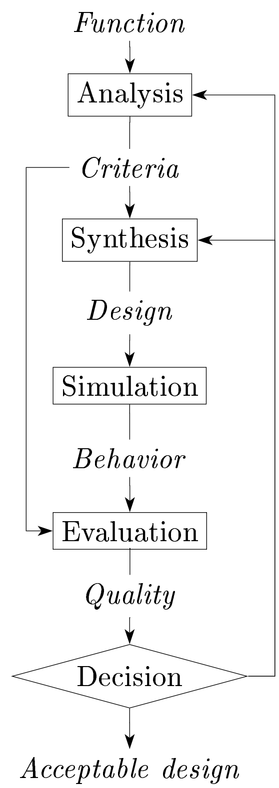



Figure 2.1: Design process.

in reality, but done in a representative model of reality. Formulating the problem and criteria in a model enables to solve the design problem in a more general setting. In the evaluation phase, the quality of the design is determined by checking if the expected behavior satisfies the criteria. Based upon this quality, a decision is made whether or not the design is acceptable enough to fulfill its function. Here, 'evaluation' refers to the 'check' in the empirical cycle with reflection. The formalized design process can be visualized as a flow chart, see Figure 2.1. The flow chart shows the design process as an iterative process. In the case of an unacceptable design, more knowledge is obtained by going through the cycle again, leading to a re-design or even re-formulating the problem and criteria.

Figure 2.1 provides a framework to the designer. Its purpose is not to specify in detail all the steps in a design process. It shows how general phases in the design process relate to each other. These relations provide an objective to each action, choice, or decision in the design process. For example, a model is made to formulate the problem and criteria (analysis) such that the behavior can be described (simulation) in some context that is relevant to the problem. In this research, the structure of Figure 2.1 is used to relate models, problems, techniques, and solutions with each other in the context of designing an optimal controller to a manufacturing system.

## 2.2 Control problem

The problem of control is formulated in this research in the context of the behavioral approach as presented by Willems [Wil91]. By means of this behavioral approach, Willems attempts to provide a mathematical framework for discussing dynamics on a general level [Wil89]. This framework centralizes around the notion that a model must describe the behavior of a dynamical system. The application of the behavioral approach in the context of control is explained by Willems in [Wil97]. An overview of the behavioral approach and its relation to control theory is presented in the form of a textbook by Polderman and Willems [Pol98].

First, the basic three ingredients behavior, behavioral equations, and latent variables are introduced. From these ingredients a model for a dynamical system is presented. This model is applied to formulate the problem of control in a general setting.

### Mathematical model

A mathematical model is viewed in [Wil91] as an exclusion law. It states that certain things can happen, and other things not. For a formal definition of a mathematical model, consider a certain phenomenon to be modeled. Assume that the phenomenon produces outcomes in a set $\mathbb{U}$, called the universum. A mathematical model for the phenomenon claims that certain outcomes are possible, while others are not. So the model defines a certain subset $\mathfrak{B}$, called the behavior, of the universum. Formally, [Wil91] defines a mathematical model as:

**Definition 2.2.1.** A mathematical model is a pair $(\mathbb{U}, \mathfrak{B})$ with $\mathbb{U}$ a set, called the universum — its elements are called outcomes — and $\mathfrak{B}$ a subset of $\mathbb{U}$, called the behavior.

So the proposed modeling approach consists of two main aspects:

- identify the outcomes of the phenomenon (specify the universum $\mathbb{U}$), and
- identify the behavior (specify $\mathfrak{B} \subseteq \mathbb{U}$).

In the identification of the outcomes, the attributes in $\mathbb{U}$ are divided in [Wil91] into two types of variables:

- manifest variables, and
- latent variables.

Manifest variables are the variables whose behavior the model aims at describing. Latent variables are auxiliary variables introduced in the first principles modeling process. The terminology first principles modeling refers to the fact that the laws that play a role in the system under consideration are the elementary laws from physics, chemistry, economics, etc. Manifest variables can be thought of as external to the system, while latent variables can be thought of as internal.

Besides identifying manifest and latent variables, the behavior must also be identified. Most mathematical models consist of a set of equations. These equations may serve as a law to exclude certain outcomes, namely those combinations of variables for which the equations are not satisfied. From this point of view, such a set of equations is referred to in [Wil91] as behavioral equations. These behavioral equations serve to specify the behavior as a set of solutions of a system of equations. Note that the behavioral equations specify the behavior uniquely. However, the inverse is not true. Another set of equations may specify the same behavior. As mentioned in [Wil91]: it is the behavior, the solution set of the behavioral equations, not the behavioral equations themselves, that is the essential result of a modeling procedure.

### Dynamical system

In Definition 2.2.1, no time aspects are taken into account. Such a mathematical model can be viewed as a static system. A dynamical system is viewed in [Wil91] as a mathematical model in which the elements of the universum are functions of time. Formally, a dynamical system is defined in [Wil91] as:

**Definition 2.2.2.** A dynamical system $\Sigma$ is defined as a triple

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B}),$$

with $\mathbb{T}$ a set called the time axis, $\mathbb{W}$ a set called the signal space, and $\mathfrak{B}$ a subset of $\mathbb{W}^{\mathbb{T}}$ called the behavior ($\mathbb{W}^{\mathbb{T}}$ is standard mathematical notation for the collection of all maps from $\mathbb{T}$ to $\mathbb{W}$).

The time axis $\mathbb{T}$ represents all possible time instances. It may be a subset of $\mathbb{R}$ (in continuous-time systems), or a subset of $\mathbb{Z}$ (in discrete-time systems). The signal space $\mathbb{W}$ represents all possible values of the variables. It can be seen that a discrete-event system can be represented by a finite set $\mathbb{W}$, because only a limited number of values are possible for a variable. The set $\mathbb{W}^{\mathbb{T}}$ represents all possible functions of time.

Some manifest variables can be considered free, their values are not determined by their past. These free variables can be imposed upon the dynamical system by the environment. The remaining manifest variables are completely specified if these free variables are chosen. It is sometimes possible to consider free variables as inputs to the system and the remaining manifest variables as outputs. This expresses a signal flow from input to output. However, the partition into inputs and output is in general not unique. It is the choice of the designer to consider a certain signal flow. Not every dynamical system has a recognizable signal flow. Therefore, this research focuses only on choices for the free variables. With these variables chosen, the remaining variables are specified by the behavioral equations and so does the behavior of the system.

A special class of latent variables are state variables. Think of the state that it should contain sufficient information about the past to determine the future behavior. This interpretation is expressed in the following axiom of state [Wil91]:

**Axiom of state.** Any trajectory from $\mathfrak{B}$ arriving in a particular state can be concatenated with any trajectory from $\mathfrak{B}$ proceeding from that same state.

So state variables can be seen as a summary of the preceding. As the memory of the system, they form the connection between the free variables and the remaining manifest variables.

The behavioral equations describe the relations between the variables in the system and their relations with the environment. Consider a dynamical system in which some manifest variables are specified as inputs, the remaining manifest variables as outputs, and latent variables as state variables. Such a dynamical system is called an input/state/output system. An input/state/output system can be represented by the following behavioral equations:

$$\dot{x} = f[x, u, t]; \quad y = g[x, u, t].$$

Here $u$ denotes the free variables that are imposed upon the system by its environment; the inputs. The state is denoted by $x$. The first equation then determines how the state trajectory evolves. The second equation, in which $y$ denotes the output, determines how the output trajectory evolves. It can be seen that for a linear time-invariant system the behavior is described by the following common behavioral equations:

$$\dot{x} = Ax + Bu; \quad y = Cx + Du.$$

Here $x$ is an $n$-vector, $u$ is an $m$-vector, and $y$ a $p$-vector. Consequently, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times m}$. These matrices are respectively called the system term, the input term, the output term, and the feedthrough term. The above mentioned representation of the behavioral equations are referred to as state equations. With the behavioral equations, the behavior of the dynamical system is specified as the set of solutions that satisfy these equations.

### Open loop and feedback control

The problem of control is to ensure that the considered dynamical system shows the desired behavior. The system to be controlled is usually called the plant. A control design is viewed in [Wil97] as a dynamical system, called the controller, that is interconnected to the plant such that the desired system behavior is achieved. This vision of control in the context of the behavioral approach is based upon the interconnection of dynamical systems.

Consider two dynamical systems $\Sigma_1 = (\mathbb{T}, \mathbb{W}, \mathfrak{B}_1)$ and $\Sigma_2 = (\mathbb{T}, \mathbb{W}, \mathfrak{B}_2)$ with the same time axis $\mathbb{T}$ and signal space $\mathbb{W}$. The interconnection of $\Sigma_1$ and $\Sigma_2$, denoted as $\Sigma_1 \wedge \Sigma_2$, is formally defined in [Wil97] as $\Sigma_1 \wedge \Sigma_2 := (\mathbb{T}, \mathbb{W}, \mathfrak{B}_1 \cap \mathfrak{B}_2)$. Thus in the interconnected system, variables must be acceptable to both $\mathfrak{B}_1$ and $\mathfrak{B}_2$. An illustration

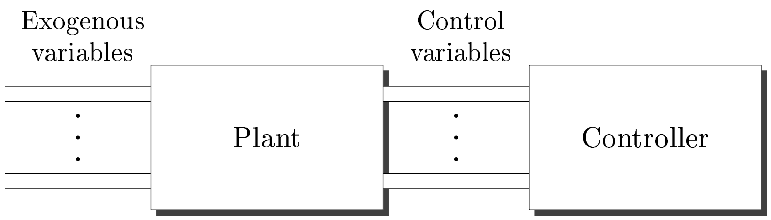



Figure 2.2: Controller interconnection [Pol98].

of the idea of controller interconnection is shown in Figure 2.2. With this definition of interconnection, a more formal description of the problem of control is given in [Wil97] as follows. Consider a plant, a dynamical system $\Sigma_p = (\mathbb{T}, \mathbb{W}, \mathfrak{B}_p)$, that is given. Let $\mathfrak{C}$ be a family of dynamical systems, called the set of admissible controllers, all with a common time axis $\mathbb{T}$ and common signal space $\mathbb{W}$. An element $\Sigma_c \in \mathfrak{C}$, $\Sigma_c = (\mathbb{T}, \mathbb{W}, \mathfrak{B}_c)$ is called an admissible controller, and the interconnection $\Sigma_p \wedge \Sigma_c$ is called the controlled system. The controller $\Sigma_c$ must be chosen such that the controlled system $\Sigma_p \wedge \Sigma_c$ shows a desired behavior. The problems of control are:

- first, to describe the set of admissible controllers,
- second, to describe the desired behavior that the controlled system should have, and
- third, to find an admissible controller such that the controlled system shows the desired behavior.

In an input/state/output system, the controller typically chooses the values for the input variables, the control variables, to achieve desired trajectories for the output variables. The way the controller chooses the values is determined by a control law. Two types of control can be distinguished:

- open loop control, and
- feedback control.

In open loop control, the controller chooses the input as an explicit function of time. The control law announces what control action must be taken at what time. In feedback control, the input is chosen based upon observed outputs or states. Here, the control law functions as a map from an observed output or state trajectory to a chosen control value. Unexpected events, small disturbances, or miscalculations due to uncertain parameters can be taken into consideration by the feedback controller thanks to the observations. In these cases of variability, feedback control leads to a better performance than open loop control.

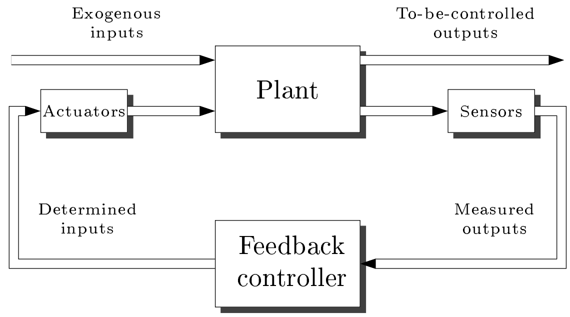



Figure 2.3: Intelligent control [Pol98].

Usually, a feedback controller for an input/state/output system is visualized by the signal flow graph shown in Figure 2.3. The plant has four terminals, two input terminals and two output terminals. Actuators generate the determined inputs and sensors measure the outputs. The dynamics of the actuators and sensors are usually considered as part of the plant. In [Wil97], this signal flow graph is referred to as the intelligent control paradigm. Note that the interconnection of the plant and the feedback controller is a similar one as that in Figure 2.2. However, a signal flow is specified in the picture of intelligent control, while in the controller interconnection in Figure 2.2 this signal flow is left unspecified. As mentioned in [Wil97], the intelligent control can be considered as a special class of controller interconnections; that in which a signal flow is specified.

To describe the desired behavior of the controlled system, consider the two main roots of control theory: regulation and trajectory optimization. In regulation, a controller is designed that steers to and keeps the to-be-controlled variables at a prescribed trajectory, e.g., a constant value, in the presence of external disturbances. In trajectory optimization, the dynamical system is transferred from a given initial state to a prescribed terminal state. Often paths are sought that are optimal in some sense, hence the term trajectory optimization. This research focuses on the latter case. What is considered to be optimal can be specified by means of a performance index. The performance index introduces costs for values of the variables. An optimal control law is a function that minimizes these costs. This approach is called optimal control and is dealt with in Chapter 3.

## 2.3 Concept of approximation

Assume that the set of admissible controllers and the desired behavior have been described. The control problem that remains, is to design an admissible controller that achieves the desired behavior of the controlled system. This can be a hard problem, especially when complex models are considered. In control theory, several standard techniques are available that aid the designer in the synthesis process of the desired controller. Unfortunately, the nature of the model is not always suitable for applying the favored or available standard techniques. This mismatch creates a gap between the complex model and the standard techniques. Therefore, a concept of approximation is introduced.

### Concept

Describing the desired behavior of the modeled phenomenon as well as possible often leads to complex models. The behavioral equations of such models might not be as amenable to solution. A familiar example of this difficulty is the analysis of nonlinear systems. Most real physical systems show nonlinear behavior resulting in complex models with nonlinear equations. Solutions to these equations are difficult to find. In the problem of regulation, the nonlinear models can be approximated by linearizing such that the behavioral equations are more amenable to solution. In a more general setting, Thompson [Tho99] describes three basic methods for finding solutions to the equations describing the model:

- obtain an exact solution,
- obtain an exact solution to an approximate problem, and
- obtain an approximate solution to the exact problem.

Finding a solution to a linearized problem can be seen as an example of the second method, where the original nonlinear problem is replaced by an approximate linear problem. An example of the third method is the application of numerical methods to find an approximate solution to the original problem. The question remains what method, approximating standard techniques or approximating the complex model, leads to a better solution. The answer lies in a comparison of solutions of both methods. The solution that describes the phenomenon being modeled best is the better solution. Thus, in this concept of approximation a closed loop is essential. After an approximate solution has been obtained, its behavior should be compared with the real world behavior. Differences should lead to an improvement or reformulation of the approximate problem or solution.

This closed loop is also recognized by Etman and Lefeber [Etm01] in the optimization and control of industrial systems. Modeling industrial systems often leads to complex, discrete-event models. Standard techniques in the field of optimization and control are mostly adapted to continuous-time models. This gap between models and techniques is also bridged by a concept of approximation. This concept is visualized in [Etm01] by Figure 2.4, clearly showing that the approximation bridges the gap between the complex model and the standard techniques available.

As an example, consider a complex model of a to-be-controlled plant. An approximation of this model can be made. Based upon this approximate model, standard techniques lead to a design for a controller to the plant. However, the interconnection of the controller to the plant may not be possible. Due to the approximation, controller

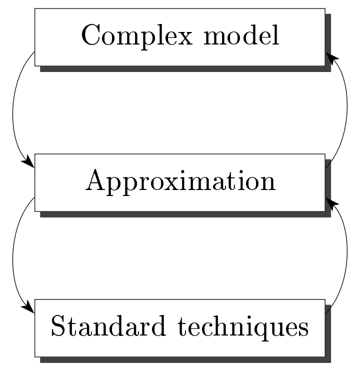



Figure 2.4: Concept of approximation.

and plant may have different time axes or signal spaces. Here comes the closed loop in Figure 2.4 in effect. Making another approximation or adjustment, this time of the designed controller, enables to interconnect the approximate controller to the complex model of the plant. Whether or not the resulting controlled system shows the desired behavior, may lead to adjustments of the approximate model or the approximate design. Making use of approximate techniques results in a loop in opposite direction.

### Flow model

In modeling industrial systems, a manufacturing system can be modeled as a set of resources upon which activities take place. An activity is a pair of discrete-events associated with a resource. The first event corresponds to the start of the activity, and the second is the end of the activity. Only one activity can occur at a resource at any time, e.g., operations, machine failures, inspection, and training sessions. The configuration of a resource determines what activities a resource may be able to perform at a given time. A flexible manufacturing system (FMS) can produce different part-types from a part-family. For each part-type the resources need to be configured (setups), which may take some time.

This model of a manufacturing system with discrete-event based activities can be viewed as a dynamical system in the context of Definition 2.2.2. As the variable that models the material flow is of discrete-event nature, its signal space is a finite set. The control design problem is to design a control system that is interconnected to the manufacturing system such that the desired behavior of the industrial system is achieved. Remains the difficulty that favored techniques from control theory are not suited to deal with the signal space of the discrete material flow. Therefore, introduce a concept of approximation; the flow model.

Kimemia and Gershwin [Kim83] have proposed a flow model that models the material flow as a continuous process. The flow model originates from the notion of a separation of frequencies at which activities occur. This notion is expressed by Gershwin [Ger89] in the following assumption:

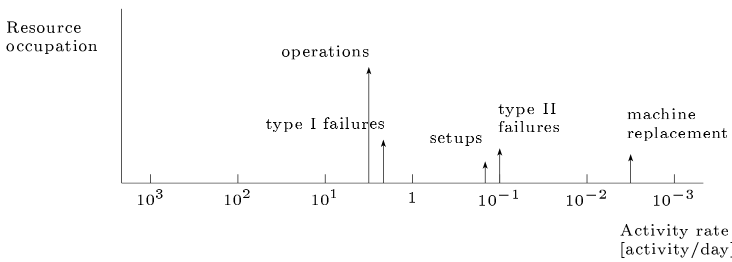
**Assumption 2.3.1.** Frequency separation: assume that activities can be grouped into sets $\mathbb{I}_1, \mathbb{I}_2, \ldots$ such that for each set $\mathbb{I}_k$, there is a characteristic frequency $f_k$ satisfying

$$0 \ll f_1 \ll f_2 \ll \cdots \ll f_k \ll f_{k+1}.$$

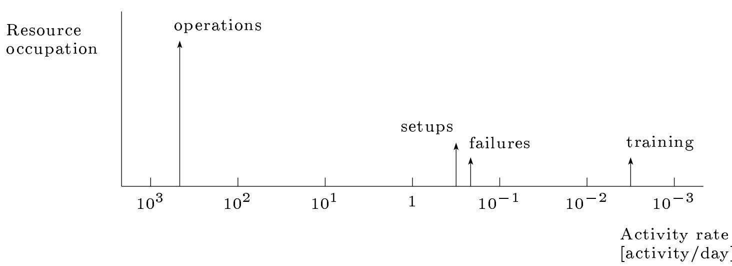
The activity rates satisfy

$$f_{k-1} \ll u_k \ll f_{k+1}.$$

A visualization of this separation of activity rates is shown in Figure 2.5. Here, two spectra of activity rates are shown for different kinds of production. Both spectra satisfy Assumption 2.3.1. Upon this frequency separation, [Ger89] bases its hierarchical




(a)




(b)

Figure 2.5: Frequency separation for two kinds of production.

decomposition of flow control for which the following central assumption holds [Ger89]:

**Assumption 2.3.2.** Hierarchical decomposition: when dealing with any dynamic quantity, treat quantities that vary much more slowly as static; and model quantities that vary much faster in a way that ignores the details of their variations.

Four activity sets are considered in [Kim83]. These are, in descending order of their characteristic frequency:

- operations,
- setups,

- failures and repairs, and
- planning.

In this research, manufacturing systems are assumed to be fully flexible, i.e., setups between configurations do not take any time. Thus, setups are not taken into account here. Consider a flow control of a failure prone manufacturing system that has to meet a certain demand for parts. A failure prone manufacturing system is a manufacturing system in which failures and repairs occur. This demand may be considered as static in the context of Assumption 2.3.2. Operations may be modelled such that their details are ignored. This validates the assumption made in [Kim83] that the material flow can be modelled as a continuous process, provided that the production rates are much higher than the rate of failure, repair, and planning.

The flow model enables now to apply the favoured standard techniques from control theory. A controller to the continuous-time model of the manufacturing system may be found. The controller must be adjusted somehow before it can be implemented in a control system that is interconnected to a discrete-event model of the manufacturing system. By means of simulation of the resulting interconnection, conclusions can be made for what activity rates the flow model is a valid approximation.

## 2.4 Synopsis

The problem of control is formulated in the context of the behavioural approach. By means of the flow model, standard techniques can be applied to find a solution to the problem. An acceptable control design must show the desired behaviour. In this research, an optimal behaviour is desired. Chapter 3 deals with techniques that enable to specify such an optimal behaviour and provide methods to derive a controller that achieves the optimal behaviour. Explicit optimal control designs are derived and tested in Chapter 5 where a single machine case is considered.

# Chapter 3

# Optimal control

In Chapter 1, the desire to control a manufacturing system in some optimal sense is expressed. Chapter 2 deals with the formulation of a general control problem. This chapter combines both aspects, optimization and control, into an optimal control problem. Two methods are presented to derive a design for an optimal controller. The first method, the Minimum Principle, only considers open loop control for a deterministic problem. The second, dynamic programming, is suited to derive a feedback control law for a general problem.

## 3.1 Optimal control problem

Consider a dynamical system of which the behavior is specified by the set of behavioral equations given by the following differential equation:

$$\dot{x}(t) = f[x(t), u(t), t], \qquad\qquad x(t_0) = x_0, \tag{3.1}$$

which is given for the interval $[t_0, t_f]$. Here, $x(t)$, an $n$-vector function, is determined by $u(t)$, an $m$-vector function. Introduce costs for values of $x(t)$ and $u(t)$ such that the desired behavior can be specified by a performance index (a scalar function) of the form

$$J = \varphi[x(t_f), t_f] + \int_{t_0}^{t_f} L[x(t), u(t), t]\mathrm{d}t. \tag{3.2}$$

Performance index $J$ can be seen as the sum of costs at the end of a period and costs during that period. The objective is to have minimal costs. The problem of optimal control is to find the functions $u(t)$ that minimize performance index $J$.

Not every function $u(t)$ is admissible. Let $\mathcal{U}$ denote the space of admissible functions $u(t)$. Hautus [Hau95] states that a satisfactory class of admissible functions is that of the piecewise continuous functions. Such functions are continuous except for a finite number of points, in which the right and left limit exist. Also has to be taken into account that constraints may be imposed on the functions $u(t)$ and $x(t)$. Clearly, constraints on $u(t)$ directly limit the space $\mathcal{U}$. Constraints on $x(t)$ have a more indirect effect on the space of admissible control functions. In this case, the behavior of the dynamical system must not only satisfy the system dynamics, but also has to satisfy the constraints on $x(t)$. This clearly limits the choice for $u(t)$.

The problem of optimal control can now be formulated in the context of a control problem as presented in Chapter 2. Let $u(t)$ describe the manifest variables that may be chosen freely, the control variables. Let $x(t)$ describe the latent variables that represent the state of the system, the state variables. The system dynamics functions as a set of behavioral equations of which the solution specifies the behavior of the dynamical system. The set of admissible controllers is specified by $\mathcal{U}$ and possible constraints on $u(t)$ and $x(t)$. The desired behavior is specified by performance index $J$. Then, the optimal control problem is to find an admissible control function $u(t)$ that minimizes performance index $J$ subject to the system dynamics and constraints.

## 3.2 Minimum Principle

A distinction is made in this research between constrained and unconstrained control problems. In the absence of constraints, the control function may be chosen freely. Therefore, such a control problem is referred to as a free control problem. In the case of constraints on the control variables, the choice for a control function is limited. Such a control problem is referred to as a limited control problem. The term limited is used instead of constrained to indicate the fact that only constraints on the control variables are taken into account in this research. First the Minimum Principle is presented for the free control problem. Then a method is presented to apply the Minimum Principle in a limited control problem.

### Free control problem

Consider the optimal control problem of finding an admissible control function that minimizes performance index (3.2) subject to system dynamics (3.1). Introduce a scalar function called the Hamiltonian by

$$H[x(t), u(t), \lambda(t), t] = L[x(t), u(t), t] + \lambda^T(t) f[x(t), u(t), t], \tag{3.3}$$

where $\lambda(t)$ is an $m$-vector function that denotes a new variable, the co-state. The co-state is defined as the solution of the following differential equation:

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x} = -\frac{\partial L}{\partial x} - \lambda^T(t)\frac{\partial f}{\partial x}, \tag{3.4}$$

which is referred to as the co-state equation. The boundary conditions for the co-state equation are:

$$\lambda^T(t_f) = \left.\frac{\partial \varphi}{\partial x}\right|_{t=t_f}. \tag{3.5}$$

Using the Hamiltonian and co-state equation, the Minimum Principle can be formulated. The Minimum Principle states that a control function that is optimal minimizes the Hamiltonian[1]. Or formally,

$$H^*[x^*(t), \lambda(t), u^*(t), t] = \min_{u(t)\in\mathcal{U}} H[x(t), \lambda(t), u(t), t], \tag{3.6}$$

for each $t$ at which $u^*(t)$ is continuous. Here, $u^*(t)$ denotes an optimal control function and $x^*(t)$ denotes the corresponding state function. The corresponding optimal value of the Hamiltonian is denoted by $H^*$.

Condition (3.6) is a necessary optimality condition. A necessary optimality condition is interpreted in [Hau95] as a property that is satisfied by a solution to the optimization problem. Such a condition is usually formulated as: *if a is an optimal point, then ...* . The most obvious way to apply (3.6) is to check whether a certain function $u^*(t)$ could be optimal or not. If $u^*(t)$, together with corresponding state $x^*(t)$ and co-state $\lambda(t)$, does not satisfy (3.6), then function $u^*(t)$ is not optimal. No other conclusions can be drawn. With the purpose of finding an optimal function, the Minimum Principle strongly limits the set of admissible functions to hopefully a unique solution. From necessary condition (3.6) an equation for $u^*(t)$ can be derived by determining the minimum of Hamiltonian (3.3):

$$\frac{\partial H}{\partial u} = 0 \quad \Rightarrow \quad u^*(t) = u^*[x^*(t), \lambda(t), t]. \tag{3.7}$$

Functions $x^*(t)$ and $\lambda(t)$ satisfy the following two-point boundary value problem:

$$\dot{x}(t) = f[x(t), u^*[x(t), \lambda(t), t], t], \qquad x(t_0) = x_0, \tag{3.8a}$$

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x}[x(t), u^*[x(t), \lambda(t), t], t], \qquad \lambda(t_f) = \left.\frac{\partial \varphi}{\partial x}\right|_{t=t_f}. \tag{3.8b}$$

Because the Minimum Principle is a strong necessary condition, it is expected that there exists a unique solution to (3.8). Substituting the solution to (3.8) into (3.7), the control function $u^*(t)$ is determined and is expected to be an optimal control law.

### Limited control problem

Again, consider the same optimal control problem as in the previous subsection, this time with constraints on the control variables included. Let the control variable constraints be represented by the following inequality constraints on functions of the control variable:

$$c[u(t), t] \leq 0, \tag{3.9}$$

[1] A justification of the Minimum Principle is given by Pontryagin [Pon62]. Due to a different sign convention, [Pon62] refers to the Maximum Principle.

where $c[u(t), t]$ is a vector function. Bryson and Ho [Bry75] present a method to implement these constraints into the Minimum Principle. The constraint functions $c[u(t), t]$ are adjoined to the Hamiltonian. The Hamiltonian is then defined as

$$H = L + \lambda^T f + \mu^T c, \tag{3.10}$$

where $\mu(t)$ is a vector function. The following constraints on $\mu(t)$ hold for all $t$:

$$\mu(t) = \begin{cases} \geq 0, & \text{if } c[u(t), t] = 0, \\ = 0, & \text{if } c[u(t), t] < 0. \end{cases} \tag{3.11}$$

Thus, whenever constraints are about to be violated, i.e., when $c[u(t), t] = 0$, functions $\mu(t)$ become active. From necessary condition (3.6) an equation for $u^*(t)$ can be derived by determining the minimum of Hamiltonian (3.10):

$$\frac{\partial H}{\partial u} = 0 \quad \Rightarrow \quad u^*(t) = u^*[x^*(t), \lambda(t), \mu(t), t]. \tag{3.12}$$

The optimal control law now is not only a function of $x^*(t)$ and $\lambda(t)$ as in (3.7), but also a function of $\mu(t)$. The solution of problem (3.8), $x^*(t)$ and $\lambda(t)$, may lead to values for $u^*(t)$ that violate constraints $c[u(t), t]$. Because of constraints (3.11), $\mu(t)$ takes values such that the effect of $x^*(t)$ and $\lambda(t)$ is diminished to admissible values for $u^*(t)$.

## 3.3 Dynamic programming

In the previous section, a method is presented to derive an optimal control law that minimizes performance index $J$. The control law maintains an optimal path starting from an initial point and proceeding optimally. Here, optimal implies that performance index $J$ is minimized. Every point $(x, t)$ directly on the optimal path is associated with this optimal control law. However, for starting from a point $(x, t)$ not on the optimal path and proceeding optimally, the solution to a new optimal control problem must be found. In other words, the Minimum Principle leads to an open loop type of control. As mentioned in Chapter 2, a feedback type of control is preferred to obtain better results in the case of any kind of variability. Somehow a control function must be found that associates each point $(x, t)$ with a proceeding optimal path that minimizes the performance index.

A unique optimal value of the performance index is associated with starting from an arbitrary point $(x, t)$ and proceeding optimally. These optimal values for all points $(x, t)$ can be captured in a function of $(x, t)$, the cost-to-go. Define $V[x, t]$ as the cost-to-go; the minimal costs associated with starting from a point $(x, t)$ and proceeding optimally. Or formally,

$$V[x, t] = \min_{u(t) \in \mathcal{U}} \left\{ \varphi[x(t_f), t_f] + \int_{t_0}^{t_f} L[x(t), u(t), t] \mathrm{d}t \right\}. \tag{3.13}$$

To derive the cost-to-go, introduce the Hamiltonian as (3.3). In the previous section, an optimal control law is determined by minimizing the Hamiltonian. Now, an optimal value of the Hamiltonian is derived:

$$H^*[x, \frac{\partial V}{\partial x}, t] = \min_{u \in \mathcal{U}} H[x, \frac{\partial V}{\partial x}, u, t]. \tag{3.14}$$

Using this optimal Hamiltonian $H^*$, the cost-to-go $V[x, t]$ can be determined by solving the Hamilton-Jacobi-Bellman (HJB) equation:

$$-\frac{\partial V}{\partial t} = H^*[x, \frac{\partial V}{\partial x}, t], \qquad V[x_f, t_f] = \varphi[x(t_f), t_f], \tag{3.15}$$

Next, the optimal feedback control law can be determined from (3.14). The optimal feedback control law determines for each point $(x, t)$ along the path an optimal program to be executed. An optimal open loop control law can be considered as a static optimal program to be followed by the controller. The feedback control law provides a dynamic program to the controller that may be changed along to way to deal with variability.

# Chapter 4

# Optimal control of a two-machine flow shop

This chapter deals with a literature investigation to the application of optimal control theory as a method to design an optimal flow controller to a failure prone manufacturing system. The focus lies here on a two-machine flow shop. First, a continuous-time model of the manufacturing system is presented. Then, an optimal control problem is formulated in the context of Chapter 3. Various forms of constraints and performance indices are discussed. The investigated literature provides several methods to solve the problem. Explicit solutions can be found for simplified problems. For more complex problems, heuristic and approximate methods are provided that may also lead to satisfactory results.

## 4.1 Manufacturing system

The literature investigation concentrates on a two-machine flow shop as visualized in Figure 4.1. The manufacturing system $WW$ produces different part-types from a part-family subject to a known demand. In Section 2.3, such a manufacturing system is modeled as a set of resources upon which activities take place. An activity is a pair of discrete-events associated with a resource. The first event corresponds to the start of the activity, and the second is the end of the activity. Here, the set of resources is

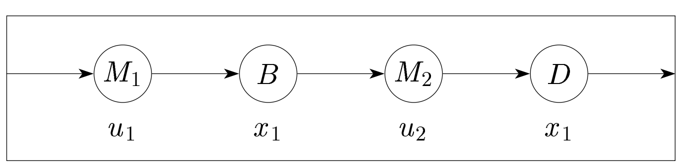



Figure 4.1: A two-machine flow shop $WW$.

given by two machines $M$, a buffer $B$, and a depot $D$. The depot differs from the buffer because it may also take negative values of depot levels to model any kind of backlog.

The objective is to control the flow of parts through the system. The environment provides an infinite supply of parts to the first machine. Parts are extracted from the depot by the environment at a known demand rate. Activities that can take place are operations, failures and repairs, and planning. When failure and repairs are considered, literature refers to failure prone manufacturing systems. The manufacturing system is assumed to be fully flexible, so setups between activities do not take any time and are thus not taken into account. By means of the flow model as presented in Section 2.3, an approximate model of system $WW$ is used. In this approximation, the flow of parts through the system is modeled as a continuous process. Let $u(t)$, a vector function, denote the production rates for the different part-types for every machine. These rates are considered to be free to choose and may function as control variables. Let $d(t)$, also a vector function, denote the known demand rates for the different part-types in the part-family. Consider these rates as exogenous variables.

Introduce a new variable, called the surplus, as the difference between the cumulative production of a system and the cumulative demand for the produced part-type. Formally, the surplus, denoted by $x(t)$, is defined by the following differential equation for a single machine system:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = u(t) - d(t), \qquad x(t_0) = x_0. \tag{4.1}$$

A positive surplus due to overproduction leads to an inventory and a negative surplus due to under production leads to backlog. The surplus can be used to describe the state of a manufacturing system.

The two-machine flow shop can now be modeled as a dynamical system. The system behavior can be described by the following set of differential equations:

$$\dot{x}_1(t) = u_1(t) - u_2(t), \qquad x_1(t_0) = x_{1,0}, \tag{4.2a}$$

$$\dot{x}_2(t) = u_2(t) - d(t), \qquad x_2(t_0) = x_{2,0}. \tag{4.2b}$$

Here, vector function $u_1(t)$ and $u_2(t)$ denote the production rates of respectively machine $M_1$ and machine $M_2$ for every part-type, see also Figure 4.1. Clearly, different values for $u_1(t)$ and $u_2(t)$ cause changes of the buffer levels, which are modeled by surplus $x_1(t)$. Likewise, the depot levels, modeled by surplus $x_2(t)$, change due to differences between production rates $u_2(t)$ and demand rates $d(t)$. The initial buffer and depot levels are denoted by $x_{1,0}$ and $x_{2,0}$ respectively. Dynamics (4.2) model a two-machine flow shop that produces multiple part-types; $u(t)$, $d(t)$, and $x(t)$ are compound vector functions here. Both machines may be extended to workstations that contain multiple parallel machines. This can be incorporated in the system dynamics by taking a sum over the machines in a workstation for each part-type. Also, re-entrant lines can be incorporated by some kind of permutation matrix. The interested reader may refer to Perkins and Kumar [Per95] that consider a multiple machine flow shop with re-entrant lines. The two-machine flow shop obviously forms the basis for a flow shop with more than two sequential machines. However, all these different variations on the system's architecture result in complex behavioral equations that are less amenable to solution.

Sometimes due to the complex structure of the behavioral equations, sometimes due to the increased number of variables.

The remainder of this literature investigation focuses mainly on a two-machine flow shop that produces a single part-type subject to a known, constant demand rate. Despite the relative simplicity of the manufacturing system, the corresponding optimal control problem may contain enough difficulties to let this be an interesting problem. Especially because the encountered difficulties are characteristic for the optimal flow control of discrete-event manufacturing systems.

## 4.2 Optimal control problem

Model the behavior of a two-machine flow shop that produces a single part-type subject to a known constant demand rate $d$ by the following system dynamics:

$$\dot{x}_1(t) = u_1(t) - u_2(t), \qquad x_1(t_0) = x_{1,0}, \tag{4.3a}$$

$$\dot{x}_2(t) = u_2(t) - d, \qquad x_2(t_0) = x_{2,0}. \tag{4.3b}$$

Where $u_1(t)$, $u_2(t)$ and $x_1(t)$, $x_2(t)$ are now scalar functions, representing the control and state variables respectively. Note that this is an approximate, continuous-time model of the discrete-event manufacturing system, obtained by making use of the flow model.

### Constraints

Constraints can be put on the control and state variables to incorporate the physical limitations of the manufacturing system in the control problem. Constrained control variables are caused by capacity limitations. As mentioned, several activities can occur on a resource, at most one at the time. An activity can only occur when the resource is configured for it, which is determined by the configuration state $\sigma(t)$. A setup is an activity that changes the configuration state. Now consider the failure prone flexible manufacturing system (FMS) mentioned by Kimemia and Gershwin [Kim83]. In [Kim83], only operations, failures, and repairs can occur on a machine. The latter two can be viewed as one activity. Then two machine states can be distinguished:

- up; operations can occur, and
- down; failure occurs and no operations can occur.

Two ways exist how the machine changes from one state to another; deterministic or stochastic. The latter can for example be represented by finite state Markov chains. It is assumed in this research that the controller has no influence upon the changes of configuration states. However, Hu, Vakili, and Yu [Hu94] study a single machine FMS producing a single part-type with operation dependent failure rates. They use a failure rate function of the form $au^\eta + c$ to denote tool wear in metal-removing processes.

Due to the two machine states, also two machine capacities have to be determined. So the machine capacity and thus the control space $\mathcal{U}$ becomes a (stochastic) set

$\mathcal{U}[x(t), \sigma(t)]$. Let $\alpha_i(\sigma(t))$ denote the production capacity for machine $i$ with $i = 1, 2$ and at time $t \geq 0$. Then, admissible control functions are given by:

$$0 \leq u_i(t) \leq \alpha_i(\sigma(t)), \qquad \text{for all } t \geq 0, \tag{4.4}$$

and with $i = 1, 2$ for the two-machine flow shop. Note that the control function $u(t)$ must be chosen such that the corresponding state function does not violate any state constraints. A deterministic process for machine state changes enables to approximate the model to that of one machine state. An approximate, read mean, machine capacity can then be used. Models with more than two machine states can also be found in the literature. These models incorporate slowly brake down of machines, i.e., reduced machine capacities. They have been also applied to model systems with identical parallel machines and a single part-type. Failure of one of the machines makes the system state go to another state that has decreased capacity.

The state constraints relate to bounds on surplus levels. Lower bounds address to nonnegative buffer levels, higher bounds to the maximum storage capacity of a buffer or depot. Note that internal buffer levels take only nonnegative values and external buffer levels, i.e., depot levels, may take negative values to model any backlog of the system. For a two-machine flow shop with a finite buffer and finite depot, let $\beta_i$ denote the size of buffer $i$ with $i = 1, 2$. The state constraints then become:

$$0 \leq x_1(t) \leq \beta_1, \qquad \text{for all } t \geq 0, \tag{4.5a}$$

$$x_2(t) \leq \beta_2, \qquad \text{for all } t \geq 0. \tag{4.5b}$$

Let $\mathcal{S} = [0, \beta_1] \times (-\infty, \beta_2] \subset \mathbb{R}^2$ denote the state space.

The stochastic transition from one machine state to another is the only case of variability considered in the literature investigating the optimal flow control of failure prone manufacturing systems. No variability in processing times is considered. However, process variability is an effect that is encountered in many, if not all, manufacturing systems. In the control theory that considers the control of mechanical systems, similar variability is encountered. Signal processing by actuators and sensors create uncertainties about respectively effects and information. These uncertainties, together with external disturbances, are modeled in the system dynamics by adding noise terms. This concept might also provide means to incorporate the process time variability in the optimal control of manufacturing systems.

## Costs

In the context of an optimal control problem, the introduction of costs enables to describe a certain desired optimal behavior. Two types of motives influence the choice for a desired behavior:

- financial motives; clearly these relate to real costs, e.g., costs for inventory, backlog, and production, and

- manufacturing motives; may relate to certain characteristics of the material flow, e.g., few and little or graduate disruptions, or the behavior of resources, e.g., the flexibility of machines.

The financial motives are a good example of *what* costs must be taken into account. This answers the question what variables must occur in the performance index. Manufacturing motives indicate *how* costs must be taken into account. The form of the performance index may be influenced by these motives. Costs for inventory and backlog can be related to the surplus, the state variables. Costs for production clearly relate to the control variables. Salama [Sal00] also addresses restarting costs in a slightly different setting as the dynamics specified above. The control variables are limited here to two options: produce at maximum rate or remain stand-by. The system is penalized by restarting costs when the machine goes from stand-by to production. However, the control variables in this research are assumed to vary piecewise continuously.

Most performance indices found in the investigated literature are chosen such that no extra difficulties in solving the optimal control problem are expected. Unfortunately, manufacturing motives are not considered in the choice for these indices. For example, an optimal control policy frequently found in the literature results in only three different production rates for the machines: maximum, to demand, and zero. Obviously, such a 'bang-bang' policy can create nervous machine behavior. No literature is found that investigates how criteria such as manufacturing motives can be taken into account in the choice for a performance index.

The structure of a performance index taking the sum of costs at the end of a period and costs during that period is frequently found in the literature. In the case of a stochastic capacity set, the expected value of the sum is taken. Then, the performance index for the stochastic and respectively the deterministic model is of the following general form:

$$J = \mathrm{E}\left\{\varphi(x(t_f)) + \int_{t_0}^{t_f} L[x(\tau), u(\tau), \tau]\mathrm{d}\tau \mid x(t_0) = x_0,\ \sigma(t_0) = \sigma_0\right\}, \tag{4.6a}$$

$$J = \varphi(x(t_f)) + \int_{t_0}^{t_f} L[x(\tau), u(\tau), \tau]\mathrm{d}\tau \mid x(t_0) = x_0. \tag{4.6b}$$

In the investigated literature, a linear function of positive (inventory) and negative (backlog) surplus values and of the production rates is taken for $L[x(t), u(t), t]$. Many performance indices in the literature take costs over an infinite horizon. For a deterministic model, average costs (over an increasing horizon) are specified by:

$$J = \lim_{t_f \to \infty} \frac{1}{t} \int_{t_0}^{t_f} L[x(\tau), u(\tau), \tau]\mathrm{d}\tau \mid x(t_0) = x_0, \tag{4.7a}$$

or discounted costs with a discount rate $\gamma > 0$:

$$J = \int_{t_0}^{\infty} \mathrm{e}^{-\gamma t} L[x(\tau), u(\tau), \tau]\mathrm{d}\tau \mid x(t_0) = x_0. \tag{4.7b}$$

Expected values, for $\sigma(t_0) = \sigma_0$, of the equations above are taken in the case of a stochastic problem.

Summarizing, the system dynamics for a two-machine flow shop subject to a constant demand rate is described by Equations (4.3). Control and state constraints are given by Equations (4.4) and (4.5) respectively. Examples of various forms of performance indices are given by Equations (4.6, 4.7). The optimal control problem is to find admissible control functions $u_1(t)$ and $u_2(t)$ that minimize performance index $J$ subject to the system dynamics and (stochastic) constraints.

## 4.3 Solutions

Most of the investigated literature considers failure prone manufacturing systems in which the configuration state changes are determined by a stochastic process. Therefore, the flow models of these systems contain stochastic terms. This excludes the application of the Minimum Principle as a method to derive a solution to the optimal control problem. A deterministic problem, or an approximation to a deterministic problem, includes this opportunity. Furthermore, feedback control is desired to deal with small disturbances, unexpected events, and so on. Therefore, the investigated literature focuses on the application of dynamic programming to find solutions to the optimal control problem.

Kimemia and Gershwin [Kim83] apply dynamic programming in their pioneering work to characterize an optimal feedback control policy to be defined by optimal surplus levels called hedging points. This hedging point policy maintains a nonnegative surplus level when over-capacity is available to hedge against future shortages brought by machine failures. General failure prone FMSs with total expected costs over a finite horizon are addressed in [Kim83]. It is assumed that the optimal feedback control law for a single machine manufacturing system producing a single part-type has the following form:

$$u^*[x(t), t] = \begin{cases} 0, & \text{if } x(t) > z^*, \\ d, & \text{if } x(t) = z^*, \\ \alpha, & \text{if } x(t) < z^*, \end{cases} \tag{4.8}$$

with $z^*$ the hedging point. Thus, if the present surplus $x(t)$ exceeds $z^*$, the system should produce nothing; if $x(t)$ is less than the optimal surplus level $z^*$, the system should produce at the maximum rate $\alpha$; if the surplus level exactly equals $z^*$, then the system should produce exactly enough to meet demand and thereby keep the inventory level at $z^*$. Of course this only holds for the up state. The optimal control law $u^*[x(t), t]$ remains zero when the system is down.

A verification theorem for the optimality of the hedging point policy is not provided in [Kim83]. However, Akella and Kumar [Ake86] rigorously prove that the optimal feedback control law is given by a single threshold inventory. They derive explicit values for the hedging point for a single machine, single part-type system with two machine states and discounted inventory and backlog costs over an infinite horizon.

Bielecki and Kumar [Bie88] also derive explicit values for the same system with average costs. They also show that a zero-inventory policy is optimal when the rate at which the system is down is small compared to the rate at which the system is up. In other words, the system is made efficient. Note that this implies not that it is necessary to have zero down states and infinite up states. Explicit values for the hedging points for a single machine, single part-type system with multiple machine states are derived by Sharifnia [Sha88]. These optimal surplus levels are a function of the machine state and hold for average inventory and backlog costs over an infinite horizon.

Zhang and Zhou [Zha94] extend [Ake86] to costs over a finite horizon. They obtain explicit values for the hedging points. These optimal surplus levels become now a function of time. This can be clarified by realizing that a backlog in the beginning of the horizon is more easy to correct by the controller than a backlog in the end of the horizon. For that reason, the optimum surplus level $z^*(t)$ increases for larger $t \in [t_0, t_f]$. So a failure near the end of the period does not create a backlog at the end of the period. The cumulative demand for that period is then satisfied.

No other solutions to the problem have been explicitly derived yet for stochastic manufacturing systems. The hedging point concept is adapted by many to form the basis of an approximate solution to more complex problems. The problem then converts to finding the optimal values of the hedging points. The question remains if the hedging point concept is an optimal control policy for a two-machine flow shop with finite buffers.

### Heuristic approach

Two difficulties remain when dynamic programming is applied to more complex problems:

- determine the cost-to-go, and
- derive an optimal feedback control law.

Various heuristic approaches have been applied to overcome these difficulties. Many use an approximation of the cost-to-go or apply a heuristic to derive the optimal control law. For example, Gershwin, Akella, and Choong [Ger85] use a quadratic approximation of the cost-to-go and apply linear programming to derive an optimal control law to the same problem as [Kim83]. Among others, Lou, Sethi, and Zhang [Lou94] research the structural properties of the cost-to-go for models with state constraints to come to good approximations. These properties may form the basis for other heuristic approaches. Presman, Sethi, and Zhang [Pre95] extend [Lou94] to an $N$-machine flow shop.

Also, many heuristic approaches take the hedging point concept as the basis of their optimal control law. The problem is then reduced to find the corresponding values of the optimal threshold levels. For this reason, many research has been done to the structural properties of the hedging point policy. Haurie and Van Delft [Hau91] and Sethi, Soner, Zhang, and Jiang [Set92] show that turnpike sets are a generalization of the hedging point concept and provide several structural properties. Turnpike sets originally come from economic growth theory. Liberopoulos and Caramanis [Lib95] show that the hedging points change into control switching sets (CSS) for multiple part-types. These sets determine what part-type to produce and at what rate, to stay close to the hedging points. Veatch and Caramanis [Vea99] make use of these CSS to extend the zero-inventory concept of [Bie88] to two part-types and multiple machine states. For problems with state constraints, the hedging points become functions which can be hard to derive explicitly. A heuristic is to approximate these functions. Yan, Yin, and Lou [Yan94] use this method to derive values for a two-machine flow shop with a nonnegative internal buffer.

Another heuristic method for this problem is provided by Van Ryzin, Lou, and Gershwin [Ryz93], called two-boundary control. They use a numerical solution to a discrete version of the HJB equation to derive optimal control laws to several cases. The characteristics of these numerical control laws are used to develop an approximate method. Alternative dynamics are used, in which both machines are subjected to the demand rate:

$$\dot{x}_i(t) = u_i(t) - d, \qquad x_i(t) = x_{i,0}, \tag{4.9a}$$

with $i = 1, 2$. An extra variable, the buffer level $b(t)$, is introduced. The rate of changes of the buffer level is defined as:

$$\dot{b}(t) = u_1(t) - u_2(t), \qquad b(t_0) = b_0, \tag{4.9b}$$

such that the buffer level is a function of the surplus:

$$b(t) = x_1(t) - x_2(t), \qquad b_0 = x_{1,0} - x_{2,0}. \tag{4.9c}$$

Two types of control are considered: surplus control, i.e., hedging points based on the depot level, and buffer control, i.e., hedging points based on the buffer level. The approximate, sub-optimal, method defines a surplus control for the second machine and a two-boundary control for the first machine. The two-boundary policy divides the state space into two regions, in one region surplus control is applied and in the other region buffer control.

Few other numerical solutions are applied due to the curse of dimensionality. This is because of the fact that the number of machine states, in contradiction to the number of control variables, increases exponentially for systems with more than one machine state as the number of machines increases. Samaratunga, Sethi, and Zhou [Sam97] provide some computational evaluations of control policies for stochastic manufacturing systems.

### Asymptotic approach

The setback of most heuristic methods is that no explicit verification is provided for the (sub)-optimality of the approximate control law. A heuristic approach that proves that the derived control law is asymptotically optimal under certain conditions is the asymptotic approach described below. As mentioned, the stochastic machine capacities make the problem of finding an optimal feedback control law more complex. Reduction to a deterministic model can be expected to resolve in a less complex problem that may be solved. Gershwin [Ger89] introduces the conjecture that a replacement of the large set of binary variables by a small set of real variables is a good approximation because of the large differences in frequencies among the events. Here, the set of binary variables indicates the precise time of events, i.e., the stochastic machine capacities. The small set of real, continuous variables represent the rates at which activities occur.

This conjecture is used by Lehoczky, Sethi, Soner, and Taksar [Leh91] to derive a limiting problem that replaces the stochastic machine capacities by mean capacities. In this way the problem becomes deterministic. The cost-to-go for the limiting problem can be determined and an optimal feedback control law in terms of hedging points is derived. The cost-to-go of the limiting problem converges to that of the original problem, as shown in [Leh91], when the rate of changes in machine states is much larger than the rate of fluctuation in demand. Recall that the previously used constant demand rates are only a result of central Assumption 2.3.2 to treat much more slowly varying quantities as static. Now, an asymptotic optimal feedback control law to the original problem can be derived from the optimal control law to the limiting problem.

The original problem of [Leh91] has no state constraints, it concerns an $M$-parallel machines system with multi part-types. Sethi, Yan, Zhang, and Zhou [Set93] complicate the problem by introducing state constraints. They address a single part-type, two-machine flow shop with a nonnegative internal buffer. The same asymptotic approach as [Leh91] is used to derive an optimal feedback control law to the limiting problem. Errors may occur when this control law would be used straightforwardly to derive an optimal control law to the original problem with state constraints. Because, due to failure, the asymptotic optimal control law may not be able to react on a possible violation of the state constraint. To prevent this, [Set93] introduces a method of 'lifting', i.e., a stronger state constraint in the limiting problem is used when the state comes near the lower bound.

Sethi, Zhang, and Zhou [Set97] extend this to a two-machine flow shop with lower and upper bounds on the internal buffer level. The method of 'lifting' is extended with a method of 'squeezing'. Fong and Zhou [Fon96b] extend the work of [Set93] to a two-machine flow shop with a finite buffer and depot, see Equation 4.5. They use a 'constraint domain approximation', comparable with the 'lifting' and 'squeezing' methods. Only open loop control laws are derived here, their following paper [Fon96a] extends the problem to feedback control laws.

With this asymptotic approach in mind, Sethi and Zhou [Set96] and Fong and Zhou [Fon00] address completely deterministic two-machine flow shops with a single part-type. [Set96] only takes a nonnegative internal buffer into account, where [Fon00] takes a finite buffer and depot, see Equation (4.5), into account.

# Chapter 5

# Single machine case

In Chapter 3, the basics of optimal control theory are presented. Chapter 4 deals with the application of optimal control theory as a method to derive an optimal flow control to failure prone manufacturing systems. Relative complex problems are reviewed in which the application of optimal control theory is less straightforward than presented in Chapter 3. This chapter deals with the optimal flow control of a single machine manufacturing system. A deterministic control problem is considered. The relative simplicity of the problem makes that the presented methods can be applied in a more straightforward manner. The opportunity is now given to understand the basics of the optimal control techniques. To recognize key difficulties in the application of the techniques. And to consider the possible results that can be achieved with the techniques. Also the applied concept of approximation, i.e., the flow model, can be investigated in the case.

The remainder of this chapter is organized as follows. First, a description of the manufacturing system is given. An approximate flow model of the system is made. Then, the optimal control problem is formulated. Constraints on the control variable are introduced, together with two performance indices. A performance index of a quadratic form is chosen. Open loop and feedback solutions to the problem are derived by means of the Minimum Principle and dynamic programming respectively. Finally, by means of simulation, the expected behavior of the controlled system is determined. Simulation results are presented that provide insight in the optimal system behavior and the validity of the flow model.

## 5.1 Manufacturing system

Consider a single machine manufacturing system $W$ as shown in Figure 5.1. System $W$ produces parts from a single part-type subject to a known, constant demand. The manufacturing system can be modeled as a set of resources upon which discrete activities take place. Here, the set of resources is given by a machine $M$ and a depot $D$. Parts enter the system at the machine where they are processed. After processing, the parts are temporarily stored in the depot. The parts leave the system from the depot. The environment provides an infinite supply of parts to the machine. Parts are extracted from the depot by the environment at a known, constant demand rate.

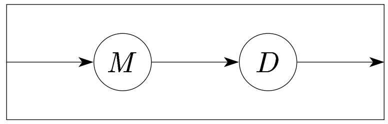



Figure 5.1: Single machine manufacturing system $W$.

Activities that can take place are operations and planning. A completely deterministic model of the system is considered. No process variability or stochastic machine failures is taken into account. The rate of operations may be set freely at any time to any nonnegative rate with maximum $\alpha$, the machine capacity. The depot level may take negative values to model any backlog. In that case, the depot temporarily stores orders instead of parts. A depot of infinite size is considered.

The objective is to control the flow of parts through the system in some optimal sense. Therefore, a controller $C$ is interconnected to manufacturing system $W$ as shown in Figure 5.2. The controller may choose the production rate of the machine and is aware of the depot level. The constant demand rate is also known to the controller.

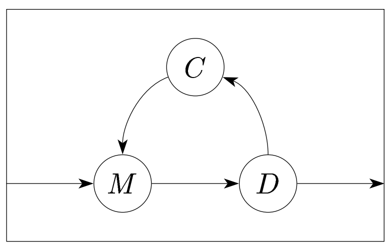



Figure 5.2: Controlled system.

Standard techniques can be applied to find an optimal controller. Unfortunately, the discrete-event model described above is not suited for the techniques presented in Chapter 3 that are adapted to continuous-time state equations. Assume that the characteristic frequency of operations is much higher than that of planning. According to Assumption 2.3.2, the complex discrete-event model can then be replaced by an approximate flow model. Here, the flow of parts is modeled as a continuous process.

Let $u(t)$, a scalar function, denote the production rate of the machine. As this rate can be chosen freely, it may function as the control variable. Let $d$, a scalar constant, denote the known, constant demand rate of the environment. Consider this rate as an exogenous variable. Introduce a new variable $x(t)$, called the surplus, as the difference between the cumulative production of the system and the cumulative demand for produced parts. A positive surplus due to overproduction leads to an inventory and a negative surplus due to under production leads to backlog. The surplus is used to describe the state of the system. With these variables, the behavior of the manufacturing system is modeled by the following system dynamics:

$$\dot{x}(t) = u(t) - d, \qquad\qquad x(t_0) = x_0, \tag{5.1}$$

where $x_0$ denotes the initial state. Differential equation (5.1) is given for the planning horizon $[t_0, t_f]$. Now, the optimal control problem can be formulated in the context of Chapter 3.

## 5.2 Control problem

The system dynamics is described by Equation (5.1). The production rate is limited to nonnegative rates with a maximum of $\alpha$, the machine capacity, or formally $0 \leq u(t) \leq \alpha$. In Chapter 3, control constraints are written as inequality constraints on functions of the control variable. Introduce $c[u(t), t]$ as a vector function of the control variable, then $0 \leq u(t) \leq \alpha$ can be written as:

$$c[u(t), t] \leq 0, \qquad\qquad c[u(t), t] = \begin{bmatrix} -u(t) \\ u(t) - \alpha \end{bmatrix}. \tag{5.2}$$

Performance indices that are found in the investigated literature are of a linear form. Such a form results in a 'bang-bang' policy causing nervous system behavior by chattering of the control variable. Therefore, a quadratic performance index is chosen to describe the desired behavior. Define a performance index of the form:

$$J_{(u)} = \int_{t_0}^{t_f} q \cdot x(t)^2 + r \cdot u(t)^2 \mathrm{d}t. \tag{5.3}$$

Parameters $q$ and $r$ are positive constants that can be chosen to specify the preference for minimal surplus costs or minimal production costs. Hence, these constants are referred to as cost parameters. The control problem is to find an admissible control function $u(t)$ that minimizes performance index (5.3) subject to system dynamics (5.1) and constraints (5.2). Such a control function is referred to as an optimal control law.

Also, define a performance index of the form:

$$J_{(u-d)} = \int_{t_0}^{t_f} q \cdot x(t)^2 + r \cdot (u(t) - d)^2 \mathrm{d}t, \tag{5.4}$$

where $q$ and $r$ are cost parameters. The term $(u(t) - d)$ in (5.4) expresses the desire to keep the production rate close to the demand rate. Here, the control problem is to find an admissible control function $u(t)$ that minimizes performance index (5.4) subject to system dynamics (5.1) and constraints (5.2). The following notation is used in the remainder to indicate which control problem is dealt with. A $(u)$, subscript or in text, indicates that a variable, function, model, system, or problem concerns the control problem with performance index (5.3). A $(u - d)$, subscript or in text, refers to the control problem with performance index (5.4).

## 5.3 Solutions

First, an equivalent problem is considered with the absence of constraints; the free control problem. Insight is gained in the optimal system behavior. Then the problem with constraints is considered; the limited control problem. Both problems are first solved by applying the Minimum Principle to derive open loop control laws. These open loop control laws are used to gain insight in the optimal control and state trajectories. The results are then used in the application of dynamic programming to derive feedback control laws for the problems.

### Free control

The free control problem is to find an admissible control function $u(t)$ that minimizes the performance index, (5.3) or (5.4), subject to system dynamics (5.1). For the free control problem, the Minimum Principle and dynamic programming can be applied relatively straightforward along the line presented in Chapter 3. A complete derivation of the optimal free control laws is given in Appendix A. For control problem $(u)$, the optimal feedback control law results into:

$$u^*[x,t] = d - d \cdot \operatorname{sech}\left(\sqrt{q/r}(t-t_f)\right) + \sqrt{q/r} \cdot \tanh\left(\sqrt{q/r}(t-t_f)\right) x, \tag{5.5a}$$

which results into

$$x^*(t) = K_1 \cdot \sinh\left(\sqrt{q/r}(t-t_f)\right) + K_2 \cdot \cosh\left(\sqrt{q/r}(t-t_f)\right), \tag{5.5b}$$

$$\begin{aligned} u^*(t) = d + K_1\sqrt{q/r} \cdot \cosh\left(\sqrt{q/r}(t-t_f)\right) \\ + K_2\sqrt{q/r} \cdot \cosh\left(\sqrt{q/r}(t-t_f)\right), \end{aligned} \tag{5.5c}$$

where

$$K_1 = -d/\sqrt{q/r}, \tag{5.5d}$$

$$K_2 = \operatorname{sech}\left(\sqrt{q/r}(t_0-t_f)\right)\left(x_0 + d\sinh\left(\sqrt{q/r}(t_0-t_f)\right)/\sqrt{q/r}\right). \tag{5.5e}$$

Clearly, feedback law (5.5a) maps every point $(x,t)$ to an optimal control action. For the deterministic system $(u)$, the optimal state path $x^*(t)$ is described by (5.5b). The optimal open loop control law $u^*(t)$, given by (5.5c), achieves that optimal state path. A visualization of the optimal system behavior for problem $(u)$ is shown in Figure 5.3. Here, the optimal control and state paths are plotted for a range of initial states $x_0$. The final time $t_f$ is 5, cost parameters $q$ and $r$ are both $\frac{1}{2}$, and the demand rate $d$ is 1. Note that for the range of state paths, Figure 5.3(b), the initial state $x_0$ increases from bottom to top. In contrary, the initial state for the range of control paths in Figure 5.3(a) increases in the opposite direction.

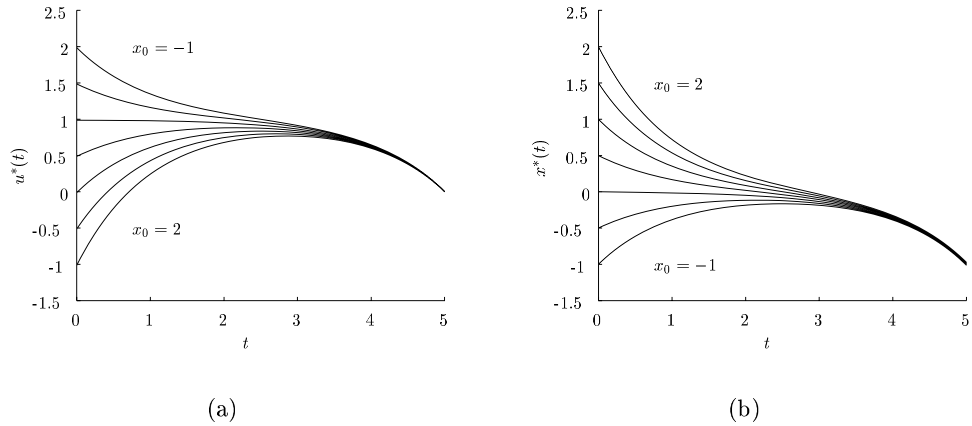



Figure 5.3: Numerical example of: (a) the optimal open loop control path $u^*(t)$ and (b) the optimal state path $x^*(t)$ for the free system $(u)$ for a range of initial states $x_0$ from -1 to 2 with step size $\frac{1}{2}$.

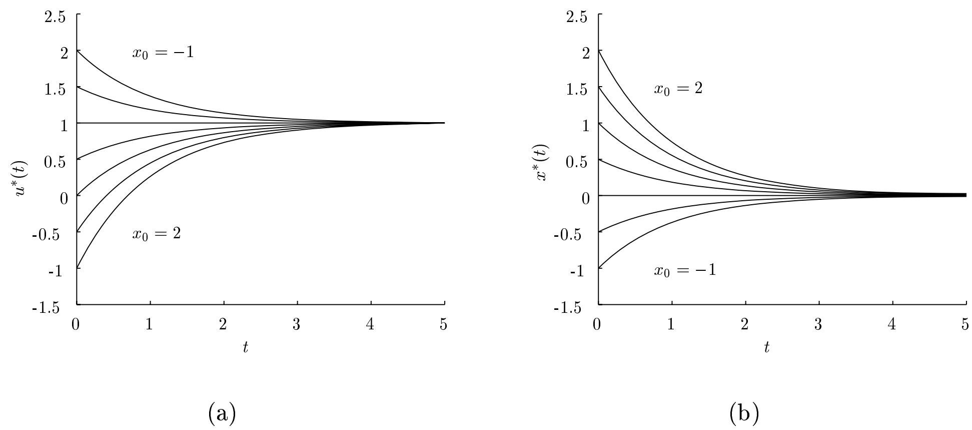



Figure 5.4: Numerical example of: (a) the optimal open loop control path $u^*(t)$ and (b) the optimal state path $x^*(t)$ for the free system $(u - d)$ for a range of initial states $x_0$ from -1 to 2 with step size $\frac{1}{2}$.

For control problem $(u - d)$, the optimal feedback control law results into:

$$u^*[x, t] = d + \sqrt{q/r} \cdot \tanh\left(\sqrt{q/r}(t - t_f)\right) x, \tag{5.6a}$$

which results into

$$x^*(t) = x_0 \operatorname{sech}\left(\sqrt{q/r}(t_0 - t_f)\right) \cdot \cosh\left(\sqrt{q/r}(t - tf)\right), \tag{5.6b}$$

$$u^*(t) = x_0 \sqrt{q/r} \operatorname{sech}\left(\sqrt{q/r}(t_0 - t_f)\right) \cdot \sinh\left(\sqrt{q/r}(t - t_f)\right). \tag{5.6c}$$

Again, feedback law (5.6a) is a map for every point $(x, t)$ to an optimal control action. The optimal state and control paths are described by respectively Equation (5.6b) and (5.6c). Figure 5.4 shows the optimal system behavior for problem $(u - d)$. The optimal control and state paths are plotted for the same range as that in Figure 5.3. The other parameters are also kept the same. Clearly, the term $(u(t) - d)$ in the performance index has its effects. After the offset in state has been reduced, the value of the control law approximately equals that of the demand rate. The combination of system dynamics (5.1) and performance index (5.3) makes that producing at demand rate with no surplus gives minimal costs. Also in the case of a quadratic performance index. Control problem $(u - d)$ can then also be considered as some kind of regulation control problem.

### Limited control

The limited control problem is to find an admissible control function $u(t)$ that minimizes the performance index, (5.3) or (5.4), subject to system dynamics (5.1) and constraints (5.2). For this problem, the methods presented in Chapter 3 cannot be applied that straightforward as done for the free control problem. To overcome this difficulty, insight in the limited system behavior has been obtained from a numerical case. The effect of control constraints on the free system behavior has been investigated. This resulted in a method for deriving an optimal open loop control law, see Section B.1 in Appendix B. Applying the Minimum Principle with control constraints stranded in a numerical problem. No explicit analytical solution for the open loop control problem could be found, see Section B.2 in Appendix B.

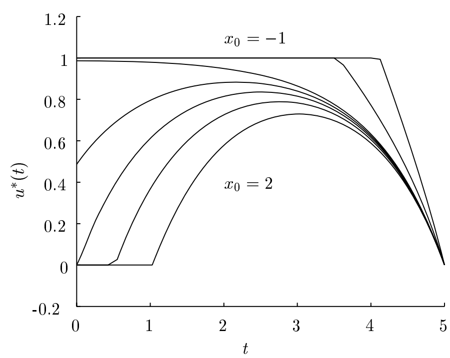
However, for feedback control, the limited control law turns out to be the saturation of the free feedback control law. In solving the limited control problem, constrained and unconstrained paths must be pieced together such that all necessary conditions are satisfied. The junction point of constrained and unconstrained paths is referred to as a corner, see Bryson and Ho [Bry75]. At a corner, the control path can be discontinuous.

From Figures 5.3 and 5.4 can be concluded that the control law is likely to violate a constraint at the beginning of the horizon. The value of the limited control law is thus set to that of the violated constraint. As a consequence of the control action, some different state than the initial state $x_0$ is reached. For that state and time, the optimal control problem must be solved again. If the solution still violates the constraint, the limited control law remains equal to the constraint. This loop continues until the solution of the optimal control problem does not violate the constraint anymore. The limited control law may then take the value of the derived solution. This moment occurs in the corner. Finding a solution every present state and time to the present optimal control problem is exactly what dynamic programming does.

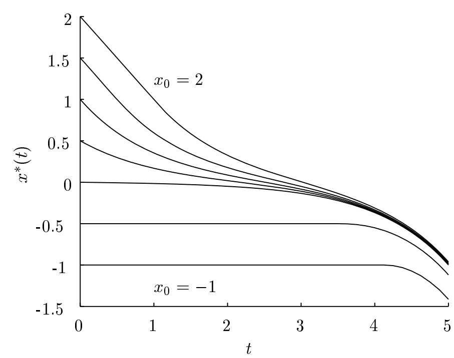
Then, the optimal limited feedback control law $u^*[x,t]$ is the saturation of the optimal free feedback control law $\bar{u}^*[x,t]$. For both control problems $(u)$ and $(u-d)$, the optimal limited feedback control law $u^*[x,t]$ results into:

$$u^*[x,t] = \begin{cases} \alpha, & \text{if} \quad \bar{u}^*[x,t] > \alpha, \\ \bar{u}^*[x,t], & \text{if} \quad 0 \leq \bar{u}^*[x,t] \leq \alpha, \\ 0, & \text{if} \quad \bar{u}^*[x,t] < 0. \end{cases} \tag{5.7}$$

Here, a bar indicates that a variable of function concerns the free equivalent of a limited control problem. For control problem $(u)$ the free feedback control law $\bar{u}^*[x,t]$ is given by (5.5a) and for control problem $(u-d)$ the free feedback control law is given by (5.6a). In every point $(x,t)$, the value of the free feedback control law is compared to that of the constraints. If the free feedback control law violates a constraint, the value of the limited feedback law is set to that of the violated constraint. The optimal system behavior for limited control problems $(u)$ and $(u-d)$ is shown by Figures 5.5 and 5.6. Control and state paths are plotted for a range of initial states $x_0$ for the same numerical parameter values as for Figures 5.3 and 5.4. The machine capacity $\alpha$ is set equal to the demand rate.




(a)




(b)

Figure 5.5: Numerical example of: (a) the optimal open loop control path $u^*(t)$ and (b) the optimal state path $x^*(t)$ for the limited system $(u)$ for a range of initial states $x_0$ from -1 to 2 with step size $\frac{1}{2}$.

## 5.4 Simulation

Several control designs have been derived in the previous section. By means of simulation the expected behavior of the controlled system is determined. A simulation model is created in which the derived control laws are implemented. Simulations are performed to investigate the influence of parameters, to validate the approximate flow

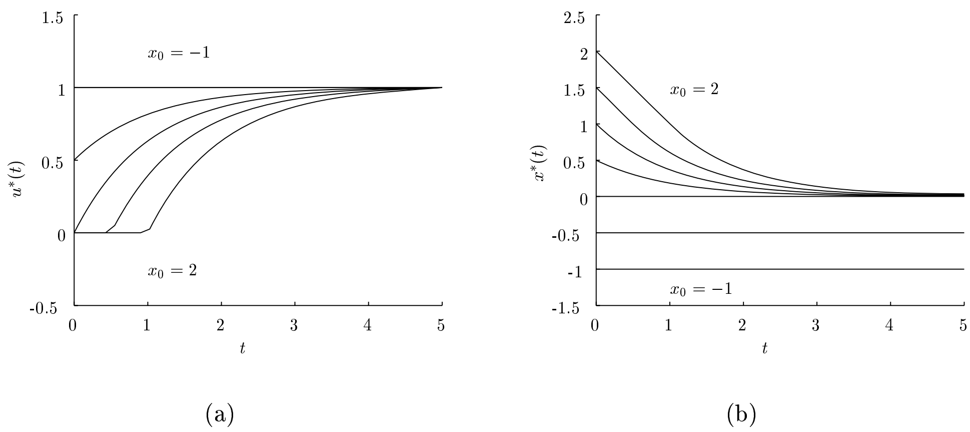



Figure 5.6: Numerical example of: (a) the optimal open loop control path $u^*(t)$ and (b) the optimal state path $x^*(t)$ for the limited system $(u - d)$ for a range of initial states $x_0$ from -1 to 2 with step size $\frac{1}{2}$.

model, and to compare open loop and feedback control. Visualization of the control and state paths are included to illustrate the results.

## Model

A discrete-event simulation model is created to simulate the behavior of the controlled system. The controlled system is the interconnection of controller $C$ to manufacturing system $W$, see Figure 5.2. Optimal controllers to an approximate (continuous-time) model of the manufacturing system are derived in the previous section. Due to the approximation, the time axes and signal spaces of the derived controller and the original model of the plant differ. The plant is a discrete-event system but the controllers are continuous-time systems. A signal conversion from controller to plant overcomes this difficulty.

The continuous-time controller is implemented in a control system that determines the control actions at discrete-time intervals. According to the Shannon [Sha49] sampling theorem, the discrete-intervals, i.e., the sample time, must be taken at least two times smaller than the duration of shortest activity in the manufacturing system to obtain a well sampled signal. In the model of the machine is some kind of preemption assumed. Therefore, a control action can be incorporated in the manufacturing system immediately, even when a part is being processed. Note that, in contradiction to what is considered as common preemption, the part may not be unloaded from the machine before it is completely processed.

Appendix C describes the discrete-event simulation model of the controlled system. The control system, represented by controller $C$, is partly discrete-event and partly discrete-time. The controller updates its depot information whenever a part enters or leaves the depot. This is a discrete-event process. A control action is determined and incorporated every sample time. This is a discrete-time process. Due to sampling of the control signal, feedback control laws are expected to return better results.

### Setup and results

The simulation focuses on three subjects:

- parameter influence,
- validation of flow model, and
- open loop and feedback control.

Because of the influence that the parameters have on the system behavior, difficulties may be encountered. An initial state not equal to zero may lead to violation of control constraints. The cost parameters provide means for adjusting the influence of control and state costs. The system behavior can then be manipulated to fulfill the designer's wishes and needs.

The control designs have been derived in an approximate flow model of the manufacturing system. The designs are implemented in the simulation model with a signal converter. To validate the approximations that are done during the synthesis phase, simulated system behavior of the approximated model and the discrete-event simulation model are compared. Finally, the benefits of feedback control are investigated when unexpected events and uncertain parameters are considered.

#### Parameter influence

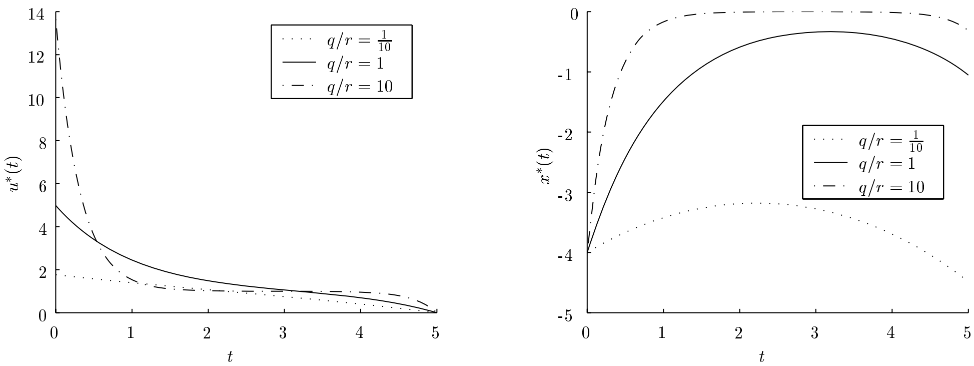
The influence of three parameter types is investigated:

- initial state $x_0$,
- planning horizon $t_f$, and
- cost parameters $q$ and $r$.

The influence of the initial state is shown in the figures in Section 5.3. The controller reduces the surplus cost created by the initial state in the first part of the horizon. This may cause relative extreme control actions that may violate the control constraints. Expansion of the planning horizon shows that the controller steers to an optimal control value where the state remains close to zero, see Appendix D. This corresponds with the hedging point concept as discussed in Chapter 4. Obviously, for a deterministic system without random machine failures and variability, the hedging point equals zero.

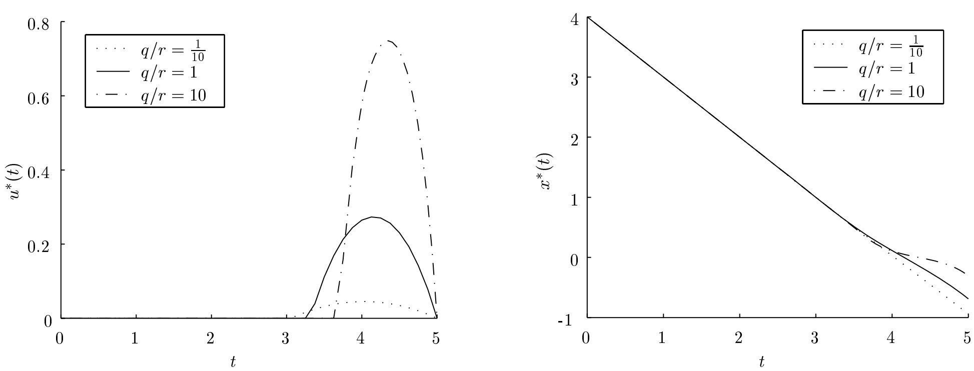
As mentioned, cost parameters $q$ and $r$ specify a preference for state and control costs respectively. Changing the ratio between these two parameters changes the system behavior substantially. Figures 5.7 and 5.8 show the optimal system behavior for the free and limited control problem ($u$) respectively. Three different ratios for $\frac{q}{r}$ are considered.

For a fair comparison, $q$ and $r$ are set such that for every ratio $\frac{q}{r}$ the same optimal performance is achieved, i.e., $J^*_{(u)} = 1$. For $\frac{q}{r} = \frac{1}{10}$, less costs for control actions are preferred than for control results. Consequently, an offset in state is accepted to let the control costs take smaller values. For $\frac{q}{r} = 10$, the opposite occurs. The offset in state is reduced much faster than for equal cost parameters. This results into relative extreme control actions at the beginning of the horizon.




(a) $x_0 < 0$ (b) $x_0 < 0$

Figure 5.7: Influence of cost parameters $q$ and $r$ with free control $(u)$ on (a) the optimal control path $u^*(t)$ and (b) the optimal state path $x^*(t)$.




(a) $x_0 > 0$ (b) $x_0 > 0$

Figure 5.8: Influence of cost parameters $q$ and $r$ with limited control $(u)$ on (a) the optimal control path $u^*(t)$ and (b) the optimal state path $x^*(t)$.

### Flow model validation

In Section 5.1 is assumed that the characteristic frequency of operations is much higher than that of planning. According to Assumption 2.3.2, the flow model is expected to be a valid approximation of the original, discrete-event model. The question that remains is *how much* higher the rate of operations must be. If the flow model is a valid approximation, then the performance of the approximate controlled system must be equal or close to that of the discrete-event controlled system. Introduce a relative performance error $e$ defined by:

$$e = \frac{J_{DE} - J_{CT}}{J_{CT}}, \tag{5.8}$$

where $J_{DE}$ denotes the performance of the discrete-event system and $J_{CT}$ denotes the performance of the continuous-time system. It is expected that for production rates not sufficiently high enough, the performance $J_{DE}$ is higher than performance $J_{CT}$. Recall that performance index $J$ is desired to be minimal.

Various simulations have been performed for a range of demand rates $d$. The final time $t_f$ is taken 10, cost parameters $q$ and $r$ are both $\frac{1}{2}$, initial state $x_0 = -4d$ or $x_0 = 4d$, and capacity $\alpha$ is set equal to $d$. For every setup the relative error $e$ has been calculated. Results are plotted in Figure D.10 in Appendix D on page 105. From Figure D.10, Tables 5.1 and 5.2 are created. In these tables the demand rates are given for which the flow model is expected invalid, critical, and valid.

Table 5.1: Invalid, critical, and valid demand rates $d$ for initial state $x_0 = -4d$.

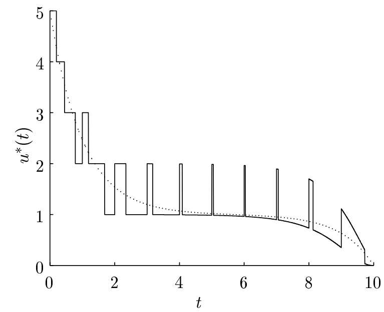
| $\mathbf{x_0 = -4d}$ | **invalid** $\mathbf{e > 0.005}$ | **critical** $\mathbf{e \approx 0.005}$ | **valid** $\mathbf{e < 0.005}$ |
|---|---|---|---|
| free $(u)$ | 1 | 10 | 100 |
| free $(u-d)$ | 1 | 10 | 100 |
| limited $(u)$ | 1 | 200 | 500 |
| limited $(u-d)$ | 1 | 200 | 500 |

Table 5.2: Invalid, critical, and valid demand rates $d$ for initial state $x_0 = 4d$.

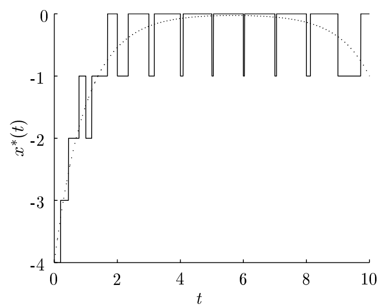
| $\mathbf{x_0 = 4d}$ | **invalid** $\mathbf{e > 0.005}$ | **critical** $\mathbf{e \approx 0.005}$ | **valid** $\mathbf{e < 0.005}$ |
|---|---|---|---|
| limited $(u)$ | 1 | 200 | 500 |
| limited $(u-d)$ | 1 | 200 | 500 |

Figure 5.9 shows the system behavior for free control problem $(u)$ for demand rates where the flow model is considered invalid, critical, and valid. This is also shown in Figure 5.10 for limited control problem $(u)$. In both figures, the dotted line functions as a reference to the optimal, continuous-time behavior. From Tables 5.1 and 5.2 can be concluded for what time scales the flow model seems a good approximation. Consider

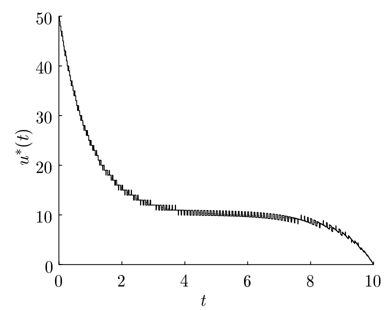
the case of the free control problem with negative initial states. For a horizon of length 10, the time scale of operations must be at least 100 times smaller than that of the planning horizon.



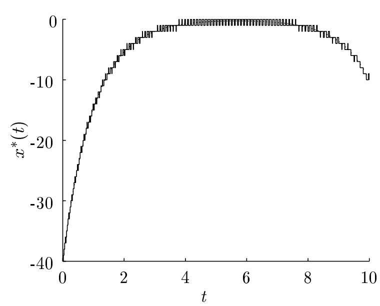

(a) invalid: $d = 1$



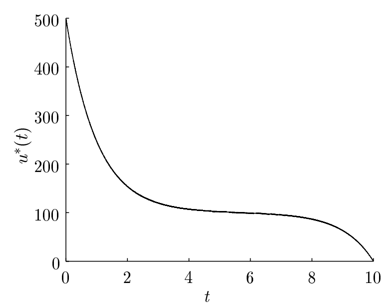

(b) invalid: $d = 1$



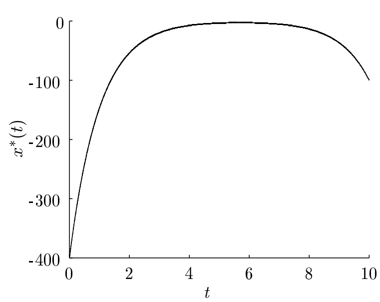

(c) critical: $d = 10$




(d) critical: $d = 10$

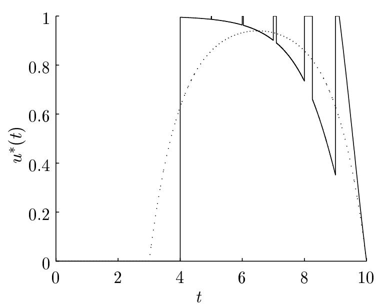



(e) valid: $d = 100$

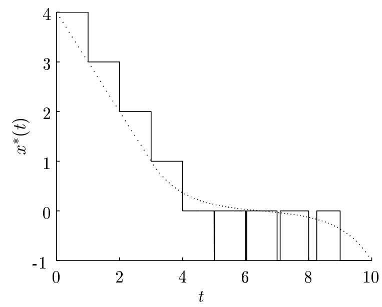



(f) valid: $d = 100$

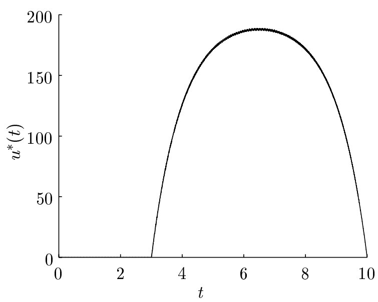
Figure 5.9: Optimal control paths $u^*(t)$ and optimal state paths $x^*(t)$ with free control ($u$) for demand rates $d$ where the flow model is (a, b) invalid, (c, d) critical, and (e, f) valid and with negative initial states $x_0$.



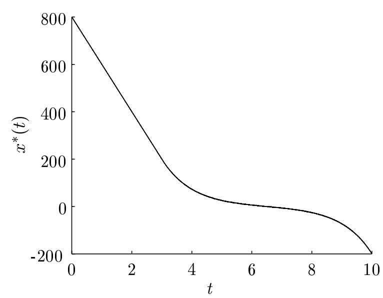

(a) invalid: $d = 1$



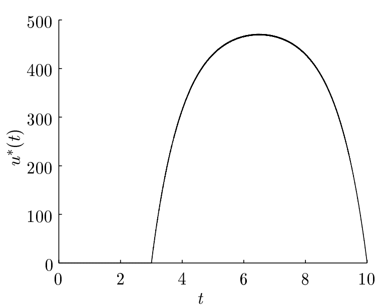

(b) invalid: $d = 1$



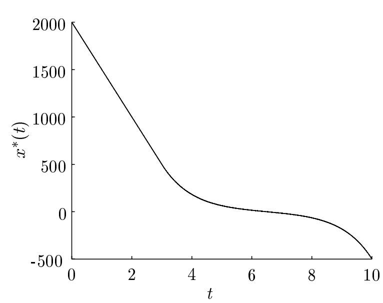

(c) critical: $d = 200$




(d) critical: $d = 200$




(e) valid: $d = 500$




(f) valid: $d = 500$

Figure 5.10: Optimal control paths $u^*(t)$ and optimal state paths $x^*(t)$ with limited control ($u$) for demand rates $d$ where the flow model is (a, b) invalid, (c, d) critical, and (e, f) valid and with positive initial states $x_0$.

### Open loop and feedback control

As mentioned in Chapter 2, feedback control takes unexpected events, small disturbances, or miscalculations due to uncertain parameters into consideration by means of the observed variables. The discrete-event flow of parts through the system leads to unexpected events that are not expected by the continuous-time controller. Therefore, only feedback control laws have been applied in the previous simulations. In this subsection, the effect of open loop and feedback control laws on the system behavior is investigated. In the deterministic simulation model, the control signal is sampled. This is done at such a high rate, that the open loop control laws also achieve sufficiently well results, see Section D.3 in Appendix D.

To exploit the true benefit of feedback control, the controlled system is simulated with uncertain exogenous variables. Here, the constant demand rate is considered to be a stochastic variable. This is modeled by introducing variability to the inter request time. The inter request time is the interval, denoted by $t_d$, for which the environment extracts parts from the depot. It is simply defined by:

$$t_d = \frac{1}{d}. \tag{5.9}$$

By means of a Gamma distribution, low, moderate and high variability distributions can be modeled. For a given mean, denoted by $\mu$, the shape of the distribution is determined by the squared coefficient of variation. The squared coefficient of variation, denoted by $c^2$, is defined by:

$$c^2 = \frac{\sigma^2}{\mu^2}, \tag{5.10}$$

where $\sigma^2$ denotes the variance. According to Hopp and Spearman [Hop00], a squared coefficient of variation substantially smaller than 1 indicates a lowly variable distribution, where highly variable distributions are indicated by a squared coefficient of variation substantially higher than 1. Distributions with a squared coefficient of variation near 1 are called moderately variable. Only the mean value of $t_d$ is known to the controller.

Figures 5.11 and 5.12 show the results for respectively the free and limited control problem ($u$). For the free control problem, the initial state is set $x_0 = -4d$, and $x_0 = 4d$ for the limited control problem. Cost parameters $q$ and $r$ both equal $\frac{1}{2}$ and the planning horizon $t_f$ is 10. The capacity $\alpha$ is set equal to the demand rate $d$. The system behavior is simulated for demand rates for which the validity of the flow model is considered to be critical. Three values of inter request time variability are considered:

- lowly variable; $c^2 = 0.1$,
- moderately variable; $c^2 = 1.0$, and
- highly variable; $c^2 = 10$.

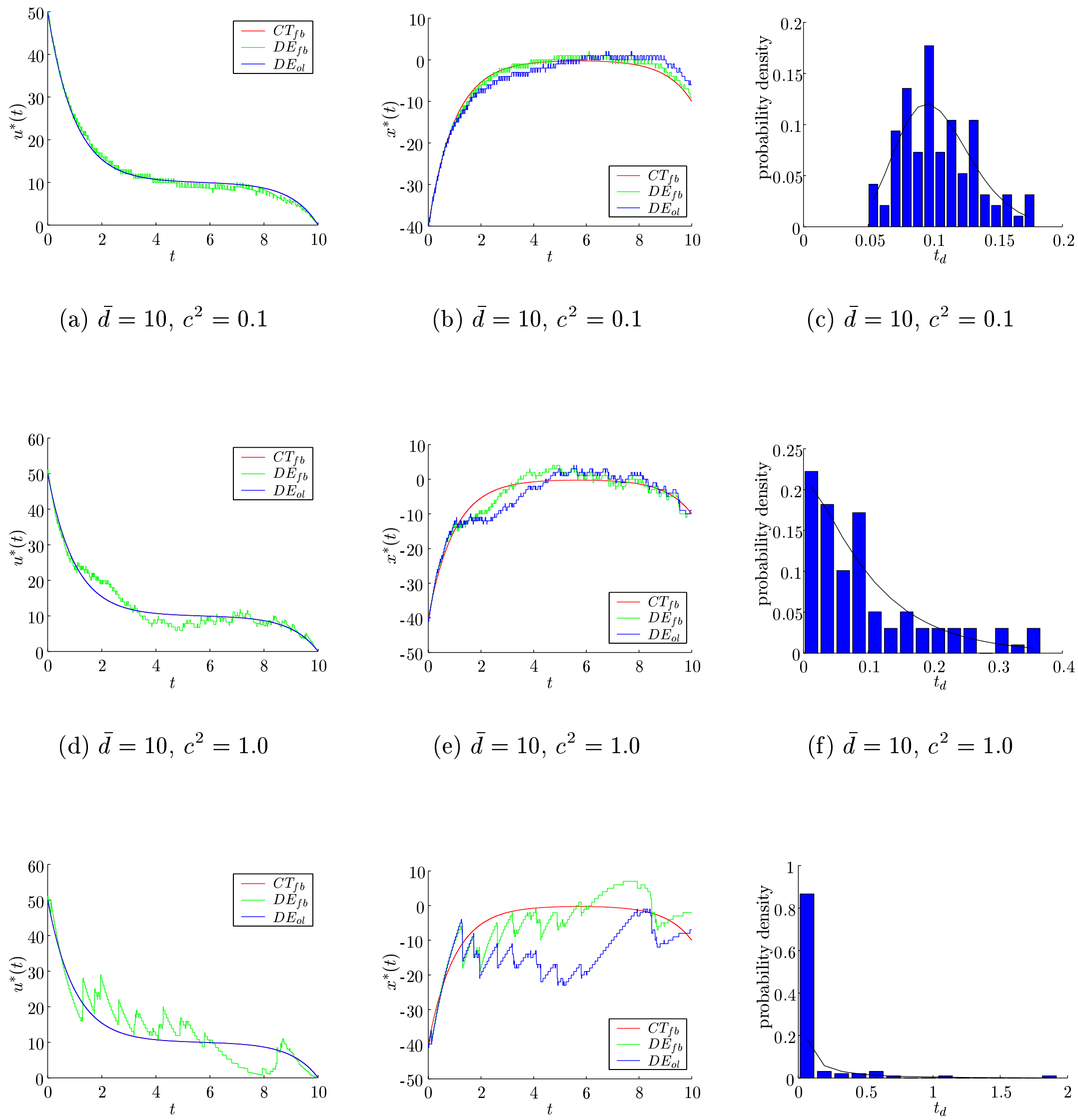
The resulting probability distributions of the inter request time are shown at the right side of the figure. Control and state paths are plotted for:

- the continuous-time model with feedback control; $CT_{fb}$,
- the discrete-time model with feedback control; $DE_{fb}$, and

- the discrete-time model with open loop control; $DE_{ol}$.

No variability is taken into account in the $CT_{fb}$-model. These control and state paths are only plotted as a reference.

Both figures show the corrupting influence of variability. Because the open loop controller does not anticipate on the changes in the state path, these changes are not compensated. Due to its observations, the feedback controller adjusts its actions to compensate the external changes to the state path. Other results in Section D.3 in Appendix D show that for higher demand rates the effect of inter request time variability on the system behavior decreases relatively.

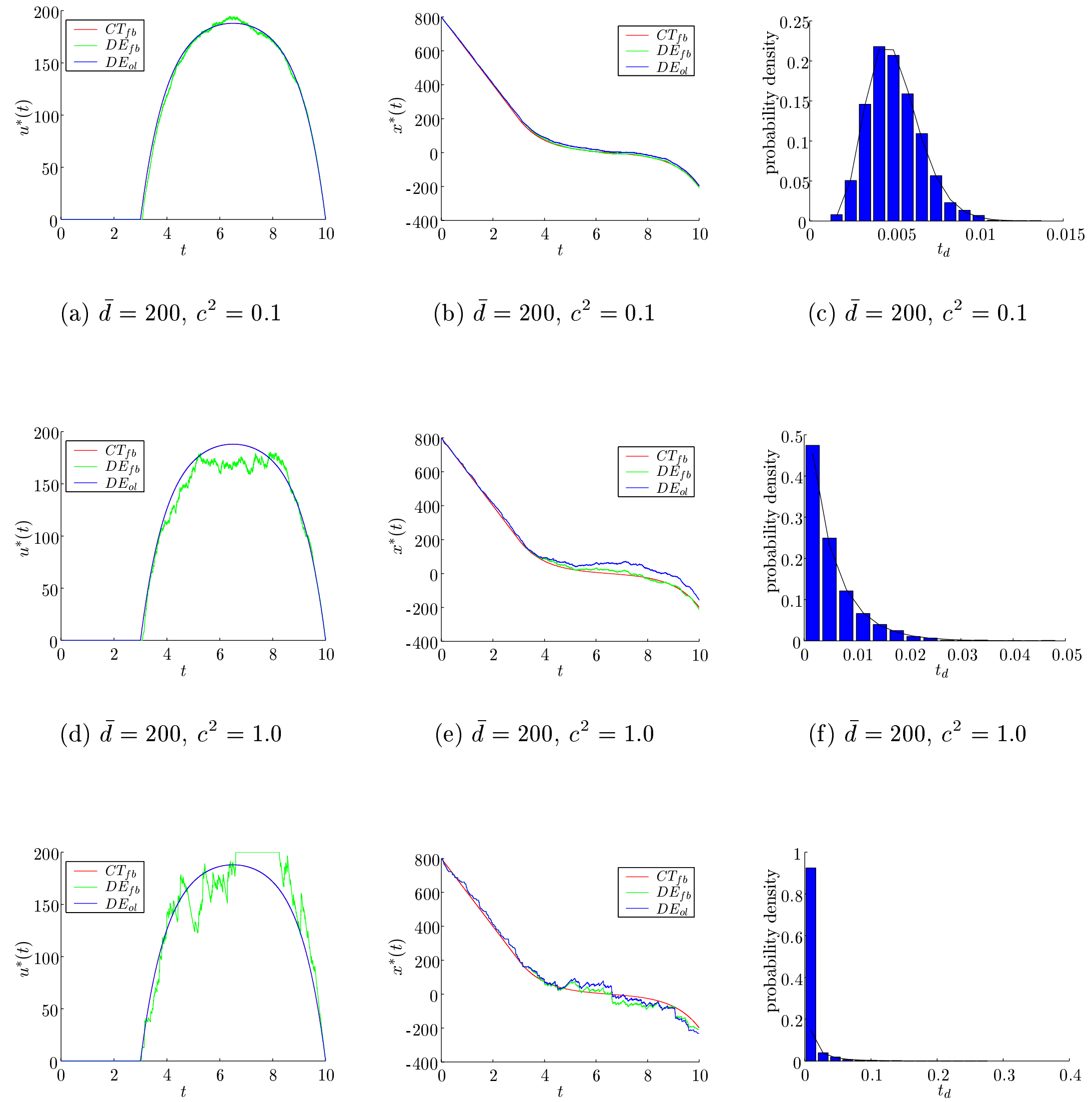



(a) $\bar{d} = 10,\ c^2 = 0.1$ (b) $\bar{d} = 10,\ c^2 = 0.1$ (c) $\bar{d} = 10,\ c^2 = 0.1$

(d) $\bar{d} = 10,\ c^2 = 1.0$ (e) $\bar{d} = 10,\ c^2 = 1.0$ (f) $\bar{d} = 10,\ c^2 = 1.0$

(g) $\bar{d} = 10,\ c^2 = 10.0$ (h) $\bar{d} = 10,\ c^2 = 10.0$ (i) $\bar{d} = 10,\ c^2 = 10.0$

Figure 5.11: Open loop and feedback control in the stochastic case with free control ($u$) and negative initial state $x_0$. Control and state paths are shown for several values of inter request time variability: (a, b, c) lowly variable, (d, e, f) moderately variable, and (g, h, i) highly variable. Validity of the flow model is critical with an average demand rate $\bar{d} = 10$.




(a) $\bar{d} = 200,\ c^2 = 0.1$ (b) $\bar{d} = 200,\ c^2 = 0.1$ (c) $\bar{d} = 200,\ c^2 = 0.1$

(d) $\bar{d} = 200,\ c^2 = 1.0$ (e) $\bar{d} = 200,\ c^2 = 1.0$ (f) $\bar{d} = 200,\ c^2 = 1.0$

(g) $\bar{d} = 200,\ c^2 = 10.0$ (h) $\bar{d} = 200,\ c^2 = 10.0$ (i) $\bar{d} = 200,\ c^2 = 10.0$

Figure 5.12: Open loop and feedback control in the stochastic case with limited control ($u$) and positive initial state $x_0$. Control and state paths are shown for several values of inter request time variability: (a, b, c) lowly variable, (d, e, f) moderately variable, and (g, h, i) highly variable. Validity of the flow model is critical with an average demand rate $\bar{d} = 200$.

# Chapter 6

# Conclusions

The previous chapters deal with the application of optimal control theory as a method to design a flow controller for failure prone manufacturing systems. The problem of optimal control is defined, optimal control techniques are introduced, several solutions are reviewed, and a case is considered that deals with the application itself. The following conclusions with respect to the objectives set in Chapter 1 can be drawn.

### Overview

The flow model proposed by Kimemia and Gershwin [Kim83] provides means to apply optimal control techniques for finding a flow controller to a failure prone manufacturing system. Linear cost functions let the optimal control policy to be defined by hedging points, i.e., optimal surplus levels. The control policy is then to produce at a maximum rate, to demand, or not at all. Only for the stochastic single machine problem explicit solutions have been obtained. For other problems, several other approximate methods are reviewed.

### Optimal control

An optimal control problem consists of the system dynamics, a performance index, and constraints. The problem is find an admissible control function that minimizes the performance index subject to the system dynamics and constraints. The Minimum Principle provides insight in the optimal system behavior for deterministic problems, with or without constraints. Dynamic programming provides means for deriving a feedback control law that achieves the optimal behavior also for stochastic problems

### Application

A deterministic, single machine problem with constraints on the control variable is considered. Quadratic performance indices are chosen. Two key difficulties remain:

- dealing with the constraints when applying the Minimum Principle, and
- finding an explicit solution to the HJB-equation.

In the considered case can the latter be overcome by making use of the open loop results. The quadratic performance index leads to a graduate change of production rates. Machine capacities are properly modeled by control constraints.

### Flow model

The flow model has been validated by means of simulation. A separation of time scales for planning and operations has been obtained that justifies the use of the flow model. In the simulation model, the continuous-time controller is converted to a discrete-time control system that is interconnected to the manufacturing system.

# Chapter 7

# Suggestions for further research

This research has resulted into several conclusions, as presented in the previous chapter. Some research objectives need further research. Also, several questions have evolved during the research. The remaining objectives and evolved questions result into the following suggestions for further research.

Optimal control methods have been presented that only take into account constraints on the control variables. To deal with problems that consider finite buffers, constraints on the state variables must be taken into account also. Investigate methods that take state constraints into account in the optimal control problem.

Deterministic problems have been dealt with only. The interesting problem of random machine failures remains. The stochastic capacity set results into a stochastic term in the HJB-equation. Investigate methods that deal with such stochastic, partial differential equations. From a control engineering point of view, investigate also the possibilities of noise terms modeling variability. This may provide means to also deal with the problem of process time variability.

In the context of control as expressed in Chapter 2, a performance index is seen as a way to describe the desired behavior of the controlled system. Here, the controlled system is the interconnection of manufacturing system and control system. Objectives are to maintain and control the flow of products. Key aspects are WIP, cycle time, and throughput. Find ways to incorporate these aspects in the performance index.

It is assumed that the production rates can be chosen freely. Consider also the case that this choice is limited to a finite set of production rates at discrete-events. Therefore, find other ways of describing the dynamics by state equations that take the discrete-event nature of the manufacturing system into account. The optimal control techniques can then also be used for lower production rates.

All suggested research items can be considered in a two-machine problem with a single part-type. Other architectures, e.g., multiple machine job shops with multiple part-types, may be also considered. However, it is expected that here other research and engineering problems are of interest than merely throughput control.

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# Appendix A

# Free control

Consider a single machine manufacturing system subject to a constant demand and with the absence of constraints. This system is referred to as the free system, as is described in Chapter 5. Let $u(t)$ denote the production rate of the machine, the control variable, and let the constant demand rate, denoted by $d$, be known for the interval $[t_0, t_f]$. The system dynamics are modeled by the following differential equation:

$$\dot{x}(t) = u(t) - d, \qquad x(t_0) = x_0. \tag{A.1}$$

The state variable $x(t)$ models the surplus, i.e., the difference between the cumulative production of the system and the cumulative demand for the produced part-type. The initial surplus level is modeled by the initial state $x_0$. Define a performance index $J_{(u)}$ of the form:

$$J_{(u)} = \int_{t_0}^{t_f} q \cdot x(t)^2 + r \cdot u(t)^2 \mathrm{d}t. \tag{A.2}$$

Parameters $q$ and $r$ are positive constants that can be chosen to specify the preference for minimal surplus costs or minimal production costs. Therefore, these parameters are referred to as cost parameters. The control problem is to find an admissible control function $u(t)$ that minimizes performance index (A.2) subject to the system dynamics (A.1). Such a control function is referred to as an optimal free control law.

Besides performance index (A.2), this research also deals with a slightly different variant of (A.2):

$$J_{(u-d)} = \int_{t_0}^{t_f} q \cdot x(t)^2 + r \cdot (u(t) - d)^2 \mathrm{d}t. \tag{A.3}$$

The term $(u(t) - d)$ in (A.3) expresses the desire to keep the production rate close to the demand rate. The control problem now is to find an admissible control function $u(t)$ that minimizes performance index (A.3) subject to the system dynamics (A.1).

This appendix deals with the derivation of both an open loop control law and a feedback control law for the free system for both control problems (A.1, A.2) and (A.1, A.3).

First an optimal open loop control law, denoted by $u^*(t)$, is derived by applying the Minimum Principle to the free system. Then by applying dynamic programming, an optimal feedback control law, denoted by $u^*[x(t), t]$, is derived. Several numerical examples are included to illustrate the obtained results.

## A.1 Open loop control

The following notation is used in this report to indicate which control problem is dealt with. A $(u)$, subscript or in text, indicates that a variable, function, model, system, or problem concerns control problem (A.1, A.2). A $(u-d)$, subscript or in text, indicates that a variable, function, model, system, or problem concerns control problem (A.1, A.3).

### Control problem $(u)$

Consider the following open loop control problem (A.1, A.2): find an admissible open loop control function $u(t)$ that minimizes performance index (A.2) subject to the system dynamics (A.1). According to Equation (3.3) on page 18, a Hamiltonian of the following form can be derived for the problem (A.1, A.2):

$$H = q \cdot x(t)^2 + r \cdot u(t)^2 + \lambda(t)(u(t) - d). \tag{A.4}$$

Here $\lambda(t)$ denotes the co-state variable that is defined as the solution of the following differential equation:

$$\dot{\lambda}(t) = -2q \cdot x(t), \qquad \lambda(t_f) = 0. \tag{A.5}$$

The Minimum Principle states that a control function that is optimal minimizes the Hamiltonian. This necessary condition implies that:

$$H^*[x^*(t), \lambda(t), u^*(t), t] = \min_{u(t)} H[x(t), \lambda(t), u(t), t]. \tag{A.6}$$

The corresponding optimal value of the Hamiltonian is denoted by $H^*$. Because the Hamiltonian is of a quadratic form, the necessary condition for $u(t)$ to be an optimal control law can be written as:

$$0 = \frac{\partial H}{\partial u} = 2r \cdot u(t) + \lambda(t). \tag{A.7}$$

The following equation for the optimal open loop control law $u(t)$ can be derived from (A.7):

$$u^*(t) = -\frac{1}{2r} \cdot \lambda(t). \tag{A.8}$$

By substituting (A.8) into the system dynamics, the following two differential equations that determine the state function are obtained:

$$\dot{x}(t) = -\frac{1}{2r} \cdot \lambda(t) - d, \qquad x(t_0) = x_0, \tag{A.9a}$$

$$\dot{\lambda}(t) = -2q \cdot x(t), \qquad \lambda(t_f) = 0. \tag{A.9b}$$

A second order differential equation of the state function can be determined by substituting (A.9b) into the time derivative of (A.9a):

$$\ddot{x}(t) - \frac{q}{r} \cdot x(t) = 0, \tag{A.10}$$

with boundary conditions:

$$x(t_0) = x_0, \tag{A.11a}$$

$$\dot{x}(t_f) = -\frac{1}{2r} \cdot \lambda(t_f) - d = -d. \tag{A.11b}$$

For a control law that is optimal, its corresponding state path must satisfy (A.10). Substituting the time derivative of the state function that satisfies (A.10) into system dynamics (A.1) yields the optimal open loop control law. A general solution of the second order differential equation is:

$$x(t) = C_1 \cdot \mathrm{e}^{t\sqrt{q/r}} + C_2 \cdot \mathrm{e}^{-t\sqrt{q/r}}. \tag{A.12}$$

It is convenient to write this solution in terms of hyperbolic functions such as sinh and cosh. Table A.1 shows the hyperbolic functions used in this report together with some convenient properties. More properties can be found in literature such as [Råd90]. A

Table A.1: Hyperbolic functions.

| **sinh** | **cosh** | **tanh** | **sech** |
|---|---|---|---|
| $y = \sinh x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}$ | $y = \cosh x = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}$ | $y = \tanh x = \frac{\sinh x}{\cosh x}$ | $y = \operatorname{sech} x = \frac{1}{\cosh x}$ |
| $\dot{y} = \cosh x$ | $\dot{y} = \sinh x$ | $\dot{y} = \frac{1}{\cosh^2 x}$ | $\dot{y} = -\operatorname{sech} x \tanh x$ |
| $\cosh(-x) = \cosh x$ | $\sinh(-x) = -\sinh x$ | $\tanh(-x) = -\tanh x$ | $\operatorname{sech}(-x) = \operatorname{sech} x$ |

time-to-go $t - t_f$ is also introduced to simplify the analysis of the system behavior. It is then possible to describe the behavior of the system from an arbitrary state $x(t)$ at time $t$ to the final state $x(t_f)$ at final time $t_f$. Substituting the time-to-go into (A.12) results into:

$$x(t) = C_1 \mathrm{e}^{t_f} \cdot \mathrm{e}^{\sqrt{q/r}(t-t_f)} + C_2 \mathrm{e}^{-t_f} \cdot \mathrm{e}^{-\sqrt{q/r}(t-t_f)}. \tag{A.13}$$

Writing (A.13) in terms of hyperbolic functions yields:

$$x(t) = \left(C_1 \mathrm{e}^{t_f} - C_2 \mathrm{e}^{-t_f}\right) \cdot \sinh\left(\sqrt{q/r}(t - t_f)\right) \\ + \left(C_1 \mathrm{e}^{t_f} + C_2 \mathrm{e}^{-t_f}\right) \cdot \cosh\left(\sqrt{q/r}(t - t_f)\right), \quad \text{(A.14)}$$

or with new constants $K_1$ and $K_2$:

$$x(t) = K_1 \cdot \sinh\left(\sqrt{q/r}(t - t_f)\right) + K_2 \cdot \cosh\left(\sqrt{q/r}(t - t_f)\right). \quad \text{(A.15)}$$

The constants can be derived by substituting the boundary conditions (A.11) into the general solution (A.15). Condition (A.11b) results in:

$$\dot{x}(t_f) = \sqrt{q/r} K_1 = -d \quad \Rightarrow \quad K_1 = -d/\sqrt{q/r}. \quad \text{(A.16a)}$$

Using this, condition (A.11a) yields:

$$x(t_0) = -d \sinh\left(\sqrt{q/r}(t_0 - t_f)\right) / \sqrt{q/r} + K_2 \cdot \cosh\left(\sqrt{q/r}(t_0 - t_f)\right) = x_0 \\ \Rightarrow \quad K_2 = \operatorname{sech}\left(\sqrt{q/r}(t_0 - t_f)\right) \left(x_0 + d \sinh\left(\sqrt{q/r}(t_0 - t_f)\right) / \sqrt{q/r}\right). \quad \text{(A.16b)}$$

Substituting the time derivative of (A.15) into system dynamics (A.1) yields the following optimal open loop control law $u^*(t)$:

$$u^*(t) = d + K_1 \sqrt{q/r} \cdot \cosh\left(\sqrt{q/r}(t - t_f)\right) + K_2 \sqrt{q/r} \cdot \sinh\left(\sqrt{q/r}(t - t_f)\right). \quad \text{(A.17)}$$

Summarizing, the optimal state function $x^*(t)$ and corresponding optimal open loop control law $u^*(t)$ for the free open loop control problem $(u)$ are:

$$x^*(t) = K_1 \cdot \sinh\left(\sqrt{q/r}(t - t_f)\right) + K_2 \cdot \cosh\left(\sqrt{q/r}(t - t_f)\right), \quad \text{(A.18a)}$$

$$u^*(t) = d + K_1 \sqrt{q/r} \cdot \cosh\left(\sqrt{q/r}(t - t_f)\right) \\ + K_2 \sqrt{q/r} \cdot \sinh\left(\sqrt{q/r}(t - t_f)\right), \quad \text{(A.18b)}$$

where

$$K_1 = -d/\sqrt{q/r}, \quad \text{(A.18c)}$$

$$K_2 = \operatorname{sech}\left(\sqrt{q/r}(t_0 - t_f)\right) \left(x_0 + d \sinh\left(\sqrt{q/r}(t_0 - t_f)\right) / \sqrt{q/r}\right). \quad \text{(A.18d)}$$

### Control problem $(u - d)$

Consider the following open loop control problem (A.1, A.3): find an admissible open loop control function $u(t)$ that minimizes performance index (A.3) subject to the system dynamics (A.1). The occurrence of $(u(t) - d)$ in both system dynamics and performance index enables to introduce a dummy control variable $\tilde{u}(t) = u(t) - d$. Substituting $\tilde{u}(t)$ into (A.1) and (A.3) yields the following system dynamics and performance index for the free system with dummy control variable $\tilde{u}(t)$:

$$\dot{x}(t) = \tilde{u}(t), \qquad x(t_0) = x_0, \tag{A.19}$$

$$J_{(\tilde{u})} = \int_{t_0}^{t_f} q \cdot x(t)^2 + r \cdot \tilde{u}(t)^2 \mathrm{d}t. \tag{A.20}$$

The open loop control problem is now to find an admissible open loop dummy control function $\tilde{u}(t)$ that minimizes performance index (A.20) subject to the system dynamics (A.19). Substituting the optimal dummy control law back into $\tilde{u}(t) = u(t) - d$ yields the optimal open loop control law $u^*(t)$ for the free open loop control problem $(u - d)$.

Deriving the dummy control law goes along the same lines as shown in the previous subsection. From Equations (A.19, A.20), the following Hamiltonian and co-state equation are obtained:

$$H = q \cdot x(t)^2 + r \cdot \tilde{u}(t)^2 + \lambda(t)\tilde{u}(t), \tag{A.21}$$

$$\dot{\lambda}(t) = -2q \cdot x(t), \qquad \lambda(t_f) = 0. \tag{A.22}$$

Again, the Minimum Principle states that a control function that is optimal minimizes the Hamiltonian. From $\frac{\partial H}{\partial u} = 0$ results that such a function is described by the following equation:

$$\tilde{u}^*(t) = -\frac{1}{2r} \cdot \lambda(t). \tag{A.23}$$

By substituting (A.23) into system dynamics (A.19), the same set of differential equations as Equations (A.9) is obtained. An equal second order differential equation of the state function as (A.10) can also be obtained. However, different boundary conditions hold for the dummy control problem (A.19, A.20):

$$x(t_0) = x_0, \tag{A.24a}$$

$$\dot{x}(t_f) = -\frac{1}{2r} \cdot \lambda(t_f) = 0. \tag{A.24b}$$

The following solution of the second order differential equation is obtained:

$$x(t) = K_1 \cdot \sinh\left(\sqrt{q/r}(t - t_f)\right) + K_2 \cdot \cosh\left(\sqrt{q/r}(t - t_f)\right). \tag{A.25}$$

Where applying boundary conditions (A.24) yields:

$$\dot{x}(t_f) = \sqrt{q/r}K_1 = 0 \quad \Rightarrow \quad K_1 = 0, \tag{A.26a}$$

$$\begin{aligned} x(t_0) &= K_2 \cdot \cosh\left(\sqrt{q/r}(t_0 - t_f)\right) = x_0 \\ &\Rightarrow \quad K_2 = x_0 \operatorname{sech}\left(\sqrt{q/r}(t_0 - t_f)\right). \end{aligned} \tag{A.26b}$$

The resulting optimal state function for the problem with dummy control variable $\tilde{u}(t)$ is then:

$$x^*(t) = K_2 \cdot \cosh\left(\sqrt{q/r}(t - t_f)\right), \tag{A.27a}$$

Substituting the time derivative of (A.27a) into system dynamics (A.19) yields the following optimal open loop dummy control law $\tilde{u}^*(t)$:

$$\tilde{u}^*(t) = K_2\sqrt{q/r} \cdot \sinh\left(\sqrt{q/r}(t - t_f)\right). \tag{A.27b}$$

The optimal state function $x^*(t)$ and corresponding optimal open loop control law $u^*(t)$ for the free open loop control problem $(u - d)$ can be obtained by simply substituting $\tilde{u} = u(t) - d(t)$ into (A.27b):

$$x^*(t) = K_2 \cdot \cosh\left(\sqrt{q/r}(t - t_f)\right), \tag{A.28a}$$

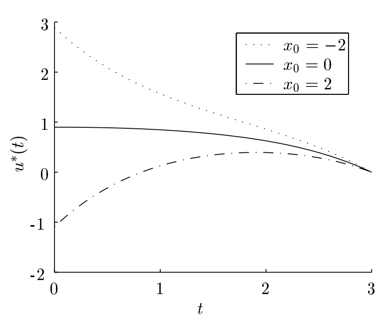
$$u^*(t) = d + K_2\sqrt{q/r} \cdot \sinh\left(\sqrt{q/r}(t - t_f)\right), \tag{A.28b}$$

with

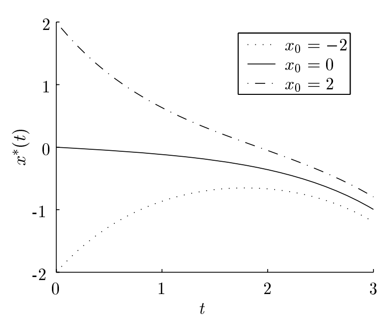
$$K_2 = x_0 \operatorname{sech}\left(\sqrt{q/r}(t_0 - t_f)\right). \tag{A.28c}$$

## Numerical example

Figures A.1 and A.2 show numerical examples of the free system behavior for respectively problem $(u)$ and $(u - d)$. In these examples, both cost parameters $q$ and $r$ equal $\frac{1}{2}$, initial time $t_0$ is 0, final time $t_f$ is 3, and the demand rate $d$ is set to 1. The left side of both figures shows the control path, the right side shows the state path. Paths are shown for initial states $x0 = -2, 0 - 2$.

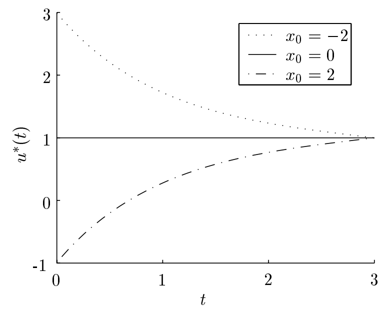



(a)

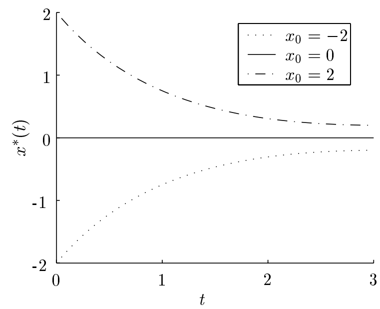



(b)

Figure A.1: Numerical example of the free system behavior $(u)$: (a) the optimal open loop control law $u^*(t)$ and (b) the optimal state function $x^*(t)$ for several values of the initial state $x_0$.




(a)




(b)

Figure A.2: Numerical example of the free system behavior $(u - d)$: (a) the optimal open loop control law $u^*(t)$ and (b) the optimal state function $x^*(t)$ for several values of the initial state $x_0$.

Consider the control and state paths for problem $(u)$, Figure A.1. In the case $x_0 = 0$, initially no costs for the state are taken into account in performance index (A.2). An option for the controller could be to keep the control variable equal to $d$ for the remainder of the period. In the case of linear, absolute costs, this would be an optimal control policy. However, quadratic costs are taken into account here. A straight line from $u(t_0) = 1$ to $u(t_f) = 0$ for the control variable and a corresponding straight line from $x(t_0) = 0$ to $x(t_f) = -1$ for the state variable lead to less costs than the function $u(t) = d$ does. So due to the quadratic costs, some backlog is created to achieve an optimal performance. This clearly illustrates the effect on the system behavior *how* costs are taken into account.

In the case of problem $(u - d)$, performance index (A.3) is applied. Here, the term $(u(t) - d)$ indicates that no control costs are taken into account if the control variable equals $d$. So a control function $u(t) = d$ is an optimal control function in the case of $x_0 = 0$. The paths for $x_0 = -2, 2$ tend to go the optimal path for $x_0 = 0$. This can be explained by the fact that all control paths should share the same final control value, defined by boundary condition $\lambda(t_f) = 0$.

## A.2 Feedback control

Optimal feedback control laws $u^*[x, t]$ are derived in this section for both control problems $(u)$ and $(u - d)$ by applying dynamic programming. First, define $V[x, t]$ as the cost-to-go; the minimal costs associated with starting from a point $(x, t)$ and proceeding optimally. The cost-to-go can be determined by solving the Hamilton-Jacobi-Bellman (HJB) equation. When this solution is determined, a feedback control law can be found that is associated with minimal costs defined by the cost-to-go.

### Control problem $(u)$

Consider the following feedback control problem (A.1, A.2): find an admissible feedback control function $u^*[x,t]$ that minimizes performance index (A.2) subject to the system dynamics (A.1). A Hamiltonian $H$ equal to (A.4) can be defined in which co-state $\lambda$ satisfies co-state equation (A.5). The HJB-equation for this problem is formally written as the following partial differential equation:

$$-\frac{\partial V}{\partial t} = H^*[x, \frac{\partial V}{\partial x}, t], \qquad V[x, t_f] = 0, \tag{A.29}$$

where,

$$H^*[x, \frac{\partial V}{\partial x}, t] = \min_u H[x, \frac{\partial V}{\partial x}, u, t], \tag{A.30}$$

and

$$H[x, \lambda, u, t] = q \cdot x^2 + r \cdot u^2 + \lambda(u - d). \tag{A.31}$$

Though (A.30) and necessary condition (A.6) from the Minimum Principle seem similar, both equations suit a different purpose. Necessary condition (A.6) is applied to find an optimal control law $u(t)$ that minimizes the Hamiltonian. The purpose of (A.30) is that it defines the optimal Hamiltonian from which, by substituting into (A.29), the cost-to-go can be derived. The value of the optimal Hamiltonian can be determined by substituting a control function that minimizes the Hamiltonian into (A.31). Such a control function is derived in Section A.1 from (A.6):

$$u^* = -\frac{1}{2r} \cdot \lambda. \tag{A.32}$$

Substituting (A.32) into Hamiltonian (A.31) yields:

$$H^*[x, \lambda, t] = q \cdot x^2 - \frac{1}{4r} \cdot \lambda^2 - d \cdot \lambda, \tag{A.33}$$

which is in terms of the co-state $\lambda$. The Hamiltonian $H^*$ in (A.30) is specified in terms of $\frac{\partial V}{\partial x}$. This form can simply be obtained by substituting $\frac{\partial V}{\partial x}$ for $\lambda$ into (A.33), resulting in:

$$H^*[x, \frac{\partial V}{\partial x}, t] = q \cdot x^2 - \frac{1}{4r} \cdot \left(\frac{\partial V}{\partial x}\right)^2 - d \cdot \frac{\partial V}{\partial x}. \tag{A.34}$$

Then, the HJB-equation for the free system with dummy control variable is given by the following partial differential equation:

$$-\frac{\partial V}{\partial t} = q \cdot x^2 - \frac{1}{4r} \cdot \left(\frac{\partial V}{\partial x}\right)^2 - d \cdot \frac{\partial V}{\partial x}, \qquad V[x, t_f] = 0. \tag{A.35}$$

Unfortunately, an explicit solution to HJB-equation (A.35) cannot be determined by Mathematica. However, the results obtained from the derivation of the open loop control law can be applied to overcome this difficulty. Optimal state function (A.18a) and optimal open loop control law (A.18b) result in an optimal performance, denoted by $J^*_{(u)}$. This optimal performance describes the minimal costs for the interval $[t_0, t_f]$. Substituting (A.18) into performance index (A.2) yields (with the aid of Mathematica):

$$J^*_{(u)} = \frac{r}{q}\Bigg( - d^2 q(t_0 - t_f) + 2dqx_0\left(\operatorname{sech}\left(\sqrt{q/r}(t_0 - t_f)\right) - 1\right) \\ + \sqrt{q/r}(d^2 r - qx_0^2)\tanh\left(\sqrt{q/r}(t_0 - tf)\right)\Bigg). \tag{A.36}$$

These are the minimal costs associated with starting from a point $(x_0, t_0)$ and proceeding optimally. Hence the similarity between the optimal performance $J^*_{(u)}[x_0, t_0]$ and the cost-to-go $V[x, t]$. Then, substituting $(x, t)$ for $(x_0, t_0)$ into optimal performance (A.36) yields the cost-to-go $V[x, t]$:

$$V[x,t] = \frac{r}{q}\Bigg( - d^2 q\cdot(t - t_f) + 2dq\cdot\left(\operatorname{sech}\left(\sqrt{q/r}(t - t_f)\right) - 1\right)x \\ + \sqrt{q/r}\cdot(d^2 r - q\cdot x^2)\tanh\left(\sqrt{q/r}(t - t_f)\right)\Bigg). \tag{A.37}$$

Now that the cost-to-go has been determined, the problem is to find a feedback control law that is associated with these minimal costs. Similar to the substitution of $\lambda$ by $\frac{\partial V}{\partial x}$ in (A.33), $\lambda$ can also be substituted by $\frac{\partial V}{\partial x}$ in (A.32). The equation for a control function that is optimal can then be written as:

$$u^* = -\frac{1}{2r}\cdot\frac{\partial V}{\partial x}. \tag{A.38}$$

With $V[x, t]$ specified by (A.37), the optimal feedback control law $u^*[x, t]$ for the free feedback control problem $(u)$ results in:

$$u^*[x,t] = d - d\cdot\operatorname{sech}\left(\sqrt{q/r}(t - t_f)\right) + \sqrt{q/r}\cdot\tanh\left(\sqrt{q/r}(t - t_f)\right)x \tag{A.39}$$

Because the system is completely deterministic, the optimal state function that goes with the optimal feedback control law can also be determined. Therefore, feedback control law (A.39) can be substituted in system dynamics (A.1) that can then be solved. With the aid of Mathematica, the following optimal state function is obtained:

$$x^*(t) = x_0\operatorname{sech}\left(\sqrt{q/r}(t_0 - t_f)\right)\cdot\cosh\left(\sqrt{q/r}(t - t_f)\right) \\ - d/\sqrt{q/r}\cdot\sinh\left(\sqrt{q/r}(t - tf)\right) \\ + d/\sqrt{q/r}\tanh\left(\sqrt{q/r}(t_0 - t_f)\right)\cdot\cosh\left(\sqrt{q/r}(t - t_f)\right). \tag{A.40}$$

Due to the deterministic nature, substituting state function (A.40) into feedback control law (A.39) leads to the same equation as for the optimal open loop control law $u^*(t)$ derived in the previous section.

### Control problem $(u - d)$

Consider the following feedback control problem (A.1, A.3): find an admissible feedback control function $u^*[x,t]$ that minimizes performance index (A.3) subject to the system dynamics (A.1). Again, introduce a dummy control variable $\tilde{u} = u - d$. Then, the HJB-equation for the dummy control problem (A.19, A.20) is formally written as the following partial differential equation:

$$-\frac{\partial V}{\partial t} = H^*[x, \frac{\partial V}{\partial x}, t], \qquad V[x, t_f] = 0, \tag{A.41}$$

where,

$$H^*[x, \frac{\partial V}{\partial x}, t] = \min_{\tilde{u}} H[x, \frac{\partial V}{\partial x}, \tilde{u}, t], \tag{A.42}$$

and

$$H[x, \lambda, \tilde{u}, t] = q \cdot x^2 + r \cdot \tilde{u}^2 + \lambda \tilde{u}. \tag{A.43}$$

In Section A.1, Equation (A.23) is presented as a control function that minimizes Hamiltonian (A.43). Substituting (A.23) into (A.43) and rewriting into terms of $\frac{\partial V}{\partial x}$ yields the optimal Hamiltonian $H^*$:

$$H^*[x, \frac{\partial V}{\partial x}, t] = q \cdot x^2 - \frac{1}{4r} \cdot \left(\frac{\partial V}{\partial x}\right)^2. \tag{A.44}$$

Then, the HJB-equation for the free control problem with dummy control variable is given by the following partial differential equation:

$$-\frac{\partial V}{\partial t} = q \cdot x^2 - \frac{1}{4r} \cdot \left(\frac{\partial V}{\partial x}\right)^2, \qquad V[x, t_f] = 0. \tag{A.45}$$

Unlike Equation (A.35), it is possible to find a solution to Equation (A.45). A solution is characterized by the form:

$$V[x, t] = c(t) x^2, \tag{A.46}$$

with $c(t)$ a certain function of the time to be determined. From (A.46) it follows that:

$$\frac{\partial V}{\partial x} = 2 \cdot c(t) x, \text{ and} \tag{A.47a}$$

$$\frac{\partial V}{\partial t} = \dot{c}(t) x^2. \tag{A.47b}$$

Substituting (A.47) into HJB-equation (A.45) yields the following nonlinear differential equation:

$$-\dot{c}(t) = q - \frac{1}{r} \cdot c(t)^2, \qquad c(t_f) = 0. \tag{A.48}$$

With the aid of Mathematica, the following solution can be obtained for this equation:

$$c(t) = -\sqrt{qr} \cdot \tanh\left(\sqrt{q/r}(t - t_f)\right). \tag{A.49}$$

Substituting (A.49) back into (A.46) yields an explicit solution to the HJB-equation:

$$V[x,t] = -\sqrt{qr} \cdot \tanh\left(\sqrt{q/r}(t - t_f)\right) x^2. \tag{A.50}$$

Now that the cost-to-go has been determined, the problem is to find a feedback control law that is associated with these minimal costs. Substitute $\lambda(t)$ by $\frac{\partial V}{\partial x}$ in (A.23), such that the equation for a feedback control function that is optimal can then be re-written as:

$$\tilde{u}^*[x,t] = -\frac{1}{2r} \cdot \frac{\partial V}{\partial x}. \tag{A.51}$$

With $V[x,t]$ specified by (A.50), the optimal feedback control law for the free control problem with dummy control variable is specified by:

$$\tilde{u}^*[x,t] = \sqrt{q/r} \cdot \tanh\left(\sqrt{q/r}(t - t_f)\right) x. \tag{A.52}$$

The optimal feedback control law $u^*[x,t]$ for the free feedback control problem $(u-d)$ can be obtained by simply substituting $\tilde{u} = u - d$ into (A.52):

$$u^*[x,t] = d + \sqrt{q/r} \cdot \tanh\left(\sqrt{q/r}(t - t_f)\right) x. \tag{A.53}$$

This result could also be obtained via the approach as applied for problem $(u)$ in the previous subsection. Therefore, consider the optimal performance $J^*_{(u-d)}$ for problem $(u-d)$ determined with the aid of Mathematica:

$$J^*_{(u-d)} = -\sqrt{qr} \tanh\left(\sqrt{q/r}(t_0 - t_f)\right) x_0^2. \tag{A.54}$$

With $(x,t)$ substituted for $(x_0, t_0)$, optimal performance (A.54) yields the same cost-to-go as (A.50). Again, the deterministic nature of the system enables to derive an optimal state function that goes with feedback control law (A.53). This optimal state function $x^*(t)$ results from solving differential equation (A.1) with $u^*[x,t]$ substituted. With the aid of Mathematica, the following result is obtained:

$$x^*(t) = x_0 \operatorname{sech}\left(\sqrt{q/r}(t_0 - t_f)\right) \cdot \cosh\left(\sqrt{q/r}(t - t_f)\right). \tag{A.55}$$

Again, the same equation as for the optimal open loop control law $u^*(t)$ is obtained by substituting state function (A.55) into feedback control law (A.53).

## A.3 Synopsis

In the previous sections, both an optimal open loop control law and an optimal feedback control law are derived for the free control problem $(u)$ as well for free control problem $(u-d)$. The Minimum Principle resulted in open loop control laws and corresponding state functions, specified by Equation (A.18) for control problem $(u)$ and specified by Equation (A.28) for control problem $(u-d)$. Dynamic programming resulted in feedback control laws, specified by Equation (A.39) for control problem $(u)$ and specified by Equation (A.53) for control problem $(u-d)$. For each control problem, the open loop and feedback control laws are equivalent, i.e., they show identical system behavior for the specified deterministic continuous-time system. However, open loop control laws (A.18b, A.28b) are only optimal if state functions (A.18a, A.28a) are obtained.

The obtained control laws are derived with the purpose of implementation into a high speed production line. Production rates may change continuously, but the surplus level is subject to the discrete changes, see also Chapter 5. So it is not possible that the state paths in this implementation obtain paths as specified by Equations (A.18a, A.28a). This limits the practical use of open loop control laws as optimal control laws. Feedback control laws are less sensitive to such changes in the state path. It can be concluded from Equations (A.18b, A.28b) that their value is not only a function of time, but also a function of the current state. So whenever a state is encountered that is not on the optimal path, the control action is adjusted to remain optimal.

In the remaining part of this report, open loop control laws are derived to obtain insight in the system behavior and to perform as a possible basis for feedback control laws. Feedback control laws are derived to implement an optimal control law into a discrete-event model of a manufacturing system.

# Appendix B

# Limited control

Consider a single machine manufacturing system subject to the a constant demand and with constraints on functions of the control variable. This system is referred to as the limited system, as is described in Chapter 5. Let $u(t)$ denote the control variable, and let the constant demand rate, denoted by $d$, be known for the interval $[t_0, t_f]$. The system dynamics are modeled by the following differential equation:

$$\dot{x}(t) = u(t) - d, \qquad x(t_0) = x_0, \tag{B.1}$$

where the state variable, denoted by $x(t)$, models the surplus. The initial surplus level is modeled by the initial state $x_0$. Define a performance index $J_{(u)}$ of the form:

$$J_{(u)} = \int_{t_0}^{t_f} q \cdot x(t)^2 + r \cdot u(t)^2 \mathrm{d}t, \tag{B.2}$$

where $q$ and $r$ are positive constants. The production rate is limited to nonnegative rates with a maximum of $\alpha$, the machine capacity, or formally $0 \leq u(t) \leq \alpha$. The theory of the Minimum Principle with control constraints, see Chapter 3, desires that control constraints are written as constraints on functions of the control variable. Introduce $c[u(t), t]$ as a vector function of the control variable, then $0 \leq u(t) \leq \alpha$ can be written as:

$$c[u(t), t] \leq 0, \qquad c[u(t), t] = \begin{bmatrix} -u(t) \\ u(t) - \alpha \end{bmatrix}. \tag{B.3}$$

The limited control problem is to find an admissible control function $u(t)$ that minimizes performance index (B.2) subject to the system dynamics (B.1) and constraints (B.3). Such a control function is referred to as an optimal limited control law.

Similar as in Appendix A, define also a performance index $J_{(u-d)}$ of the form:

$$J_{(u-d)} = \int_{t_0}^{t_f} q \cdot x(t)^2 + r \cdot (u(t) - d)^2 \mathrm{d}t. \tag{B.4}$$

The limited control problem now is to find an admissible control function $u(t)$ that minimizes performance index (B.4) subject to the system dynamics (B.1) and constraints (B.3). The same notation as introduced in Appendix A is also used to indicate what control problem is dealt with. A $(u)$ refers to the control problem with performance index (B.2), and a $(u-d)$ refers to the control problem with performance index (B.4).

This appendix deals with the derivation of both an open loop control law and a feedback control law for the limited system for both control problems (B.1, B.2, B.3) and (B.1, B.3, B.4). First a numerical case of a corresponding free system, i.e., the limited system with the absence of constraints, is studied. Analysis of the behavior of the controlled free system leads to an optimal open loop control law for the limited system. The gained insight from the numerical case is then employed to derive an optimal open loop control law $u^*(t)$, by applying the Minimum Principle with control constraints to the limited system.

The open loop system behavior also leads to the insight that an optimal limited feedback control law $u^*[x(t), t]$ is in fact a saturation of the corresponding optimal free feedback control law. Several numerical examples are included to illustrate the obtained results.

## B.1 Numerical case

To gain more insight in the difficulties and strategies of deriving an optimal control law for the limited system, a simplification of the limited control problem is studied first. Only control problem $(u)$ is studied in this section. Eliminating the control constraints from control problem (B.1, B.2, B.3) simplifies the limited control problem to a free control problem. The free control problem $(u)$ is to find an admissible control function $u(t)$ that minimizes performance index (B.2) subject to the system dynamics (B.1). In the remainder of this appendix, a bar indicates that a variable or function concerns the free equivalent of a limited control problem.

From Appendix A, the optimal state function $\bar{x}^*(t)$ and corresponding optimal open loop control law $\bar{u}^*(t)$ for the free control problem $(u)$ are:

$$\bar{x}^*(t) = K_1 \cdot \sinh\left(\sqrt{q/r}(t-t_f)\right) + K_2 \cdot \cosh\left(\sqrt{q/r}(t-t_f)\right), \tag{B.5a}$$

$$\begin{aligned}\bar{u}^*(t) = d + K_1\sqrt{q/r} \cdot \cosh\left(\sqrt{q/r}(t-t_f)\right)& \\ + K_2\sqrt{q/r} \cdot \sinh\left(\sqrt{q/r}(t-t_f)\right)&,\end{aligned} \tag{B.5b}$$

where

$$K_1 = -d/\sqrt{q/r}, \tag{B.5c}$$

$$K_2 = \operatorname{sech}\left(\sqrt{q/r}(t_0-t_f)\right)\left(x_0 + d\sinh\left(\sqrt{q/r}(t_0-t_f)\right)/\sqrt{q/r}\right). \tag{B.5d}$$

### Numerical example

A numerical example of the free system is shown in Figure B.1. Here the optimal open loop control law $\bar{u}^*(t)$ and optimal state function $\bar{x}^*(t)$ for the free system are plotted for a range of initial states $x_0$. The final time $t_f$ is 3, cost parameters $q$ and $r$ are both $\frac{1}{2}$, and the demand rate $d$ is 1. Note that for the range of state paths, Figure B.1(b), the initial state $x_0$ increases from bottom to top. In contrary, the initial state for the range of control paths in Figure B.1(a) increases in the opposite direction. Introduce $\alpha = 1$ as the upper control constraint and 0 as the lower control constraint (nonnegative production rates). It is obvious, according to Figure B.1, that the free control law violates both lower and upper constraints for certain values of $x_0$. However, for some values of $x_0$, the free control law does not violate the constraints.

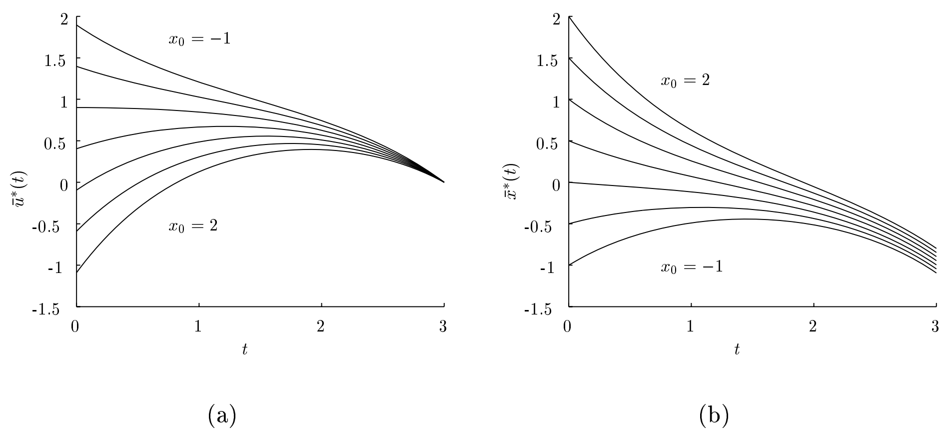



Figure B.1: Numerical example of: (a) the optimal open loop control law $\bar{u}^*(t)$ and (b) the optimal state function $\bar{x}^*(t)$ for the free system for a range of initial states $x_0$ from -1 to 2 with step size $\frac{1}{2}$.

Consider the case with an initial state $x_0 = -1$. The free control law starts at approximately $u(t_0) = 1.9$. From Figure B.1 can be concluded that the free control law violates the upper constraint from $t = 0$ to approximately $t = \frac{3}{2}$. In this case, an optimal control strategy for the limited system would be to remain at 1 until a certain time $t_1$. From then on follow a path given by the optimal free control law for the interval $[t_1, t_f]$. See Figure B.2(a) for a visualization of this strategy.

Then consider the case with an initial state $x_0 = 2$. According to Figure B.1, this causes the free control law to take negative control actions from $t = 0$ to approximately $t = 1$. An optimal strategy for the limited control law in this case, would be to remain zero until a certain time $t_1$. From then on follow a path given by the optimal free control law for the interval $[t_1, t_f]$. This is a similar strategy as that when the upper constraint is violated. A visualization of this strategy is shown in Figure B.2(b).

Finally, consider the case with an initial state $x_0 = 0$. The free control law does

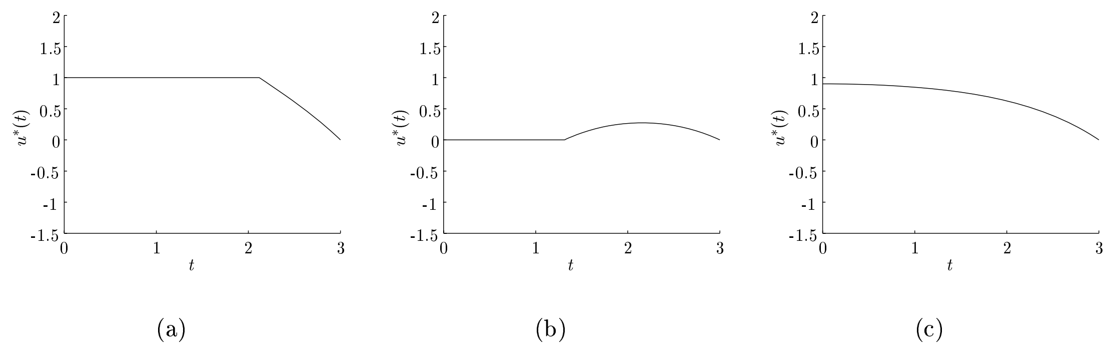



Figure B.2: Limited control strategies with initial states (a) $x_0 = -1$, (b) $x_0 = 2$, and (c) $x_0 = 0$.

not violate any constraints, as can be seen in Figure B.1. Then it is not necessary to constrain the free control law at all. The optimal control strategy for the limited control law is then to follow the path given by the optimal free control law for the interval $[t_0, t_f]$, see Figure B.2(c).

From these cases can be concluded that the limited control strategy depends on the initial state. The initial state $x_0$ determines whether the free control law

- must be limited to $\alpha$ on the interval $[t_0, t_1]$,
- must be limited to 0 on the interval $[t_0, t_1]$, or
- may remain free for the complete interval $[t_0, t_f]$.

Though the initial state determines the limited control strategy, it is the initial value of the free control law that marks the violation of constraints. So conditions on the initial free control law can be specified such that the limited control strategy

- limits the free control law on $[t_0, t_1]$, if $\bar{u}^*(t_0) > \alpha$,
- limits the free control law on $[t_0, t_1]$, if $\bar{u}^*(t_0) < 0$, and
- lets the free control law remain free on $[t_0, t_f]$, if $0 \leq \bar{u}^*(t_0) \leq \alpha$.

The conditions above can be written as functions of the initial state because the initial free control law $\bar{u}^*(t)$ is known from (B.5b). A superscript $u$ denotes an initial state that results in an initial free control law equal to the upper constraint $\alpha$. An initial state that results in an initial free control law equal to the lower constraint 0 is denoted by the superscript $l$. Then, the limited control strategy

- limits the free control law to $\alpha$ on $[t_0, t_1]$, if $x_0 < x_0^u$,
- limits the free control law to 0 on $[t_0, t_1]$, if $x_0 > x_0^l$, and

- lets the free control law remain free on $[t_0, t_f]$, if $x_0^u \leq x_0 \leq x_0^l$.

As a consequence, the optimal control path is divided into two intervals, a constrained path on $[t_0, t_1]$ and an unconstrained path on $[t_1, t_f]$. Each interval has its specific system dynamics and system behavior. An optimal control law can be derived for each interval. Note that the latter strategy, let the free control remain free on $[t_0, t_f]$ can be considered as a special case of the previous two. In this case, the time $t_1$ equals $t_0$ such that the length of the interval $[t_0, t_f]$ equals zero. In solving the limited control problem, the constrained and unconstrained paths must be pieced together to satisfy all necessary conditions.

### Limited control on $[t_0, t_1]$

Let $u_{01}(t)$ denote the limited control function for the interval $[t_0, t_1]$. The limited control strategy deduced from the numerical case results in the following formal description of the limited control law:

$$u_{01}(t) = \begin{cases} \alpha, & \text{if} \quad x_0 < x_0^u, \\ \bar{u}^*(t), & \text{if} \quad x_0^u \leq x_0 \leq x_0^l, \\ 0, & \text{if} \quad x_0 > x_0^l. \end{cases} \tag{B.6}$$

The critical initial states $x_0^u$ and $x_0^l$ are derived by setting the initial free control law equal to respectively the upper constraint $\alpha$ and the lower constraint 0. This results in:

$$\bar{u}^*(t_0) = \alpha \qquad \Rightarrow \qquad x_0^u = \frac{d + (\alpha - d)\cosh\left(\sqrt{q/r}(t_0 - t_f)\right)}{\sqrt{q/r}\sinh\left(\sqrt{q/r}(t_0 - t_f)\right)}, \tag{B.7a}$$

$$\bar{u}^*(t_0) = 0 \qquad \Rightarrow \qquad x_0^l = \frac{d - d\cosh\left(\sqrt{q/r}(t_0 - t_f)\right)}{\sqrt{q/r}\sinh\left(\sqrt{q/r}(t_0 - t_f)\right)}. \tag{B.7b}$$

In the case that $x_0$ lies between $x_0^u$ and $x_0^l$, the limited control function equals $\bar{u}(t)$ according to (B.6). However, $t_1$ equals $t_0$ in this case such that the length of the interval $[t_0, t_1]$ is zero. Clearly, any admissible control law on an interval of length zero is an optimal control law. So $u_{01}(t)$ may take any value between 0 and $\alpha$ in this case. Take $\bar{u}^*(t_0)$ to satisfy that the final control of the first interval equals the initial control of the second interval. As a result, the limited control function on the interval $[t_0, t_1]$ is a constant, denoted by $u_{01}$:

$$u_{01} = \begin{cases} \alpha, & \text{if} \quad x_0 < x_0^u, \\ \bar{u}^*(t_0), & \text{if} \quad x_0^u \leq x_0 \leq x_0^l, \\ 0, & \text{if} \quad x_0 > x_0^l. \end{cases} \tag{B.8}$$

With the limited control function known on the interval $[t_0, t_1]$, the system dynamics on the first interval can be written as the following differential equation:

$$\dot{x}_{01}(t) = u_{01} - d, \qquad\qquad x_{01}(t_0) = x_0, \tag{B.9}$$

where $x_{01}(t)$ denotes the state function for the interval $[t_0, t_1]$. Because $u_{01}$ is a constant, a solution of the differential equation can easily be obtained by integrating, resulting in the state function for the first interval:

$$x_{01}(t) = -(u_{01} - d) \cdot (t_0 - t) + x_0. \tag{B.10}$$

The system behavior of the limited system for the interval $[t_0, t_1]$ is then described by the limited control function $u_{01}$ and state function $x_{01}(t)$, respectively Equations (B.8) and (B.10).

### Limited control on $[t_1, t_f]$

Let $u_{1f}(t)$ denote the limited control function for the interval $[t_1, t_f]$. According to the limited control strategy deduced from the numerical case, no constraints are active in this interval. The system dynamics can be modeled by the following differential equation:

$$\dot{x}_{1f}(t) = u_{1f}(t) - d, \qquad x_{1f}(t_1) = x_{01}(t_1) = x_1, \tag{B.11}$$

where $x_{1f}(t)$ denotes the state function for the interval $[t_1, t_f]$. The initial state $x_1$ for the second interval equals the final state of the first interval. Because of the absence of control constraints, (B.11) is similar to (B.1), but now holds for the interval $[t_1, t_f]$ and with initial state $x_1$. Substituting the correct boundaries $t_1$ and $t_f$ and the initial state $x_1$ into (B.5) results into the following state function $x_{1f}(t)$ and corresponding limited control function $u_{1f}(t)$ for the interval $[t_1, t_f]$:

$$x_{1f}(t) = K_1 \cdot \sinh\left(\sqrt{q/r}(t - t_f)\right) + K_2(t_1) \cdot \cosh\left(\sqrt{q/r}(t - t_f)\right), \tag{B.12a}$$

$$\begin{aligned} u_{1f}(t) = d + K_1\sqrt{q/r} \cdot \cosh\left(\sqrt{q/r}(t - t_f)\right) \\ + K_2(t_1)\sqrt{q/r} \cdot \sinh\left(\sqrt{q/r}(t - t_f)\right), \end{aligned} \tag{B.12b}$$

where

$$K_1 = -d/\sqrt{q/r}, \tag{B.12c}$$

$$K_2(t_1) = \operatorname{sech}\left(\sqrt{q/r}(t_1 - t_f)\right)\left(x_{01}(t_1) + d\sinh\left(\sqrt{q/r}(t_1 - t_f)\right)/\sqrt{q/r}\right), \tag{B.12d}$$

and with the time $t_1$ to be determined later. Note that constant $K_2(t_1)$ is a function of time $t_1$ as the initial state $x_1 = x_{01}(t_1)$ for the interval $[t_1, t_f]$ occurs in $K_2(t_1)$.

### Limited control on $[t_0, t_f]$

In the previous two subsections, limited control functions have been derived for the intervals $[t_0, t_1]$ and $[t_1, t_f]$. These individual control functions, Equations (B.8) and

(B.12b), can be composed to a global limited control function $u(t)$ for the interval $[t_0, t_f]$, resulting in:

$$u(t) = \begin{cases} u_{01}, & \text{if } t < t_1, \\ u_{1f}(t), & \text{if } t \geq t_1. \end{cases} \tag{B.13}$$

A value for $t_1$ can be determined by stating that the initial control on the second interval must be equal to the final control on the first interval. In this way, a limited control function that is continuous is obtained, which corresponds with the limited control strategy deduced from the numerical case. The following formal condition can be given from which $t_1$ can be determined:

$$u_{1f}(t_1) = u_{01}. \tag{B.14}$$

Because the limited control function $u_{1f}(t)$ is in fact an optimal control law for the interval $[t_1, t_f]$, the deduced value for $t_1$ from (B.14) is an optimal value, denoted by $t_1^*$. A value for $t_1^*$ is given by the root of $u_{1f}(t_1) - u_{01}$. Unfortunately, this root cannot be determined in an analytical way, but must be determined numerically.

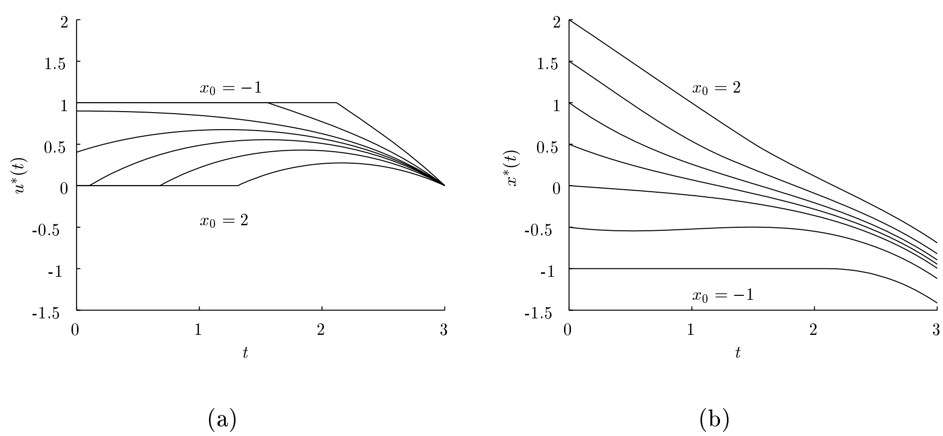



Figure B.3: Numerical example of: (a) the optimal open loop control law $u^*(t)$ and (b) the optimal state function $x^*(t)$ for the limited system for a range of initial states $x_0$ from -1 to 2 with step size $\frac{1}{2}$.

With $t_1^*$ determined, the optimal limited control law $u^*(t)$ and the corresponding optimal state function $x^*(t)$ for the interval $[t_0, t_f]$ are:

$$u^*(t) = \begin{cases} u_{01}, & \text{if } t < t_1^*, \\ u_{1f}(t), & \text{if } t \geq t_1^*, \end{cases} \tag{B.15a}$$

$$x^*(t) = \begin{cases} x_{01}(t), & \text{if } t < t_1^*, \\ x_{1f}(t), & \text{if } t \geq t_1^*. \end{cases} \tag{B.15b}$$

A numerical example of the limited system is shown in Figure B.3 as a limited equivalent of Figure B.1. Here the optimal open loop control law and the optimal state function for the limited system are plotted for a range of initial states $x_0$. Note that for $x_0 = -1$ the optimal limited control law leaves the upper constraint at approximately $t = 2.2$, instead of $t = \frac{3}{2}$ as expected in Section B.1.

## B.2 Open loop control

An optimal control law for the limited system $(u)$ is obtained in the previous section. A numerical example of the free system is studied from which insight in the system behavior is gained. From this insight, a formal description of the control law as a function of the optimal value for $t_1$ is deduced. However, in Chapter 3 a theory is discussed that deals with constraints on functions of the control. This theory can be applied to derive a formal description of an optimal open loop control law for a single machine manufacturing system with control constraints, i.e., the limited system. First control problem $(u)$ is considered, then control problem $(u-d)$ is considered.

### Control problem $(u)$

Consider the control problem $(u)$ that is specified by the following dynamics, performance index, and constraints on functions of the control variable:

$$\dot{x}(t) = u(t) - d, \qquad x(t_0) = x_0, \tag{B.16}$$

$$J_{(u)} = \int_{t_0}^{t_f} q \cdot x(t)^2 + r \cdot u(t)^2 \mathrm{d}t, \tag{B.17}$$

$$c[u(t), t] \leq 0, \qquad c[u(t), t] = \begin{bmatrix} -u(t) \\ u(t) - \alpha \end{bmatrix}. \tag{B.18}$$

The open loop control problem is to find an admissible open loop control function $u(t)$ that minimizes performance index (B.17) subject to the system dynamics (B.16) and constraints (B.18). For problem (B.16, B.17, B.18) the following Hamiltonian can be derived:

$$H = q \cdot x(t)^2 + r \cdot u(t)^2 + \lambda(t)(u(t) - d) + \mu_1(t)(-u(t)) + \mu_2(t)(u(t) - \alpha). \tag{B.19}$$

The general additional constraints on $\mu(t)$, Equation (3.11) on page 20, can also be written as:

$$\mu(t)c[u(t), t] = 0, \qquad \text{and } \mu(t) \geq 0 \text{ for all } t. \tag{B.20}$$

Then, the following constraints on the functions $\mu_1(t)$ and $\mu_2(t)$ respectively must hold:

$$\mu_1(t)(-u(t)) = 0, \qquad \mu_1(t) \geq 0, \text{ and} \tag{B.21a}$$

$$\mu_2(t)(u(t) - \alpha) = 0, \qquad \mu_2(t) \geq 0. \tag{B.21b}$$

Define the co-state variable $\lambda(t)$ as the solution of the co-state equation:

$$\dot{\lambda}(t) = -2q \cdot x(t), \qquad \lambda(t_f) = 0. \tag{B.22}$$

The necessary condition for $u(t)$ to be an optimal control law then yields:

$$0 = \frac{\partial H}{\partial u} = 2r \cdot u(t) + \lambda(t) - \mu_1(t) + \mu_2(t). \tag{B.23}$$

This results in the following equation for the optimal open loop control law for the limited system:

$$u^*(t) = \frac{1}{2r} \cdot (-\lambda(t) + \mu_1(t) - \mu_2(t)), \tag{B.24}$$

with $\lambda(t)$, $\mu_1(t)$, and $\mu_2(t)$ to be determined later. This equation is an extended version of its equivalent for the free system ($u$), see Equation (A.8), Appendix A on page 60. In both cases, the co-state $\lambda(t)$ determines the value of the control law. Its effect is limited by the functions $\mu_1(t)$ and $\mu_2(t)$ whenever a control constraint is violated. As the co-state reaches a value such that a constraint is violated, its value is added or subtracted with the value of $\mu_1(t)$ or $\mu_2(t)$ respectively, such that the limited control law has a value equal to the violated constraint. As a consequence, the functions $\mu_1(t)$ and $\mu_2(t)$ also depend on the value of the co-state $\lambda(t)$. In general, three cases can be considered:

- no constraints are violated,
- the lower constraints are violated, and
- the upper constraints are violated.

It is useful to describe these cases as functions of the co-state $\lambda(t)$, due to the influence of $\lambda(t)$ on $u^*(t)$, $\mu_1(t)$, and $\mu_2(t)$.

First, consider the case where no constraints are violated. Then the control law satisfies:

$$0 \leq u^*(t) \leq \alpha. \tag{B.25}$$

If no constraints are violated, both functions $\mu_1(t)$ and $\mu_2(t)$ do not have any effect on the value of $\lambda(t)$, i.e., they remain zero. The control law for this case then results in:

$$u^*(t) = -\frac{1}{2r} \cdot \lambda(t). \tag{B.26}$$

Substituting (B.26) into (B.25) yields

$$-2r\alpha \leq \lambda(t) \leq 0 \tag{B.27}$$

as the condition for $\lambda(t)$ in the case that no constraints are violated. Second, consider the case where the lower constraint is violated, i.e., $\lambda(t) > 0$. Function $\mu_1(t)$ is then active, such that the control law results in:

$$u^*(t) = \frac{1}{2r} \cdot (-\lambda(t) + \mu_1(t)). \tag{B.28}$$

Substituting (B.28) into constraint (B.21a) yields for function $\mu_1(t)$:

$$\mu_1(t) = \lambda(t), \qquad \text{if } \lambda(t) > 0. \tag{B.29}$$

Finally consider the case where the upper constraint is violated, i.e., $\lambda(t) < -2r\alpha$. Then function $\mu_2(t)$ is active, such that the control law results in:

$$u^*(t) = \frac{1}{2r} \cdot (-\lambda(t) - \mu_2(t)). \tag{B.30}$$

Substituting (B.30) into constraint (B.21b) yields for the function $\mu_2(t)$:

$$\mu_2(t) = -\lambda(t) - 2r\alpha, \qquad \text{if } \lambda(t) < -2r\alpha. \tag{B.31}$$

Summarizing the results from above, can be concluded that the optimal open loop control law $u^*(t)$ for the limited control problem $(u)$ is described by:

$$u^*(t) = \frac{1}{2r} \cdot (-\lambda(t) + \mu_1(t) - \mu_2(t)), \tag{B.32}$$

where

$$\mu_1(t) = \begin{cases} 0, & \text{if } \lambda(t) \leq 0, \\ \lambda(t), & \text{if } \lambda(t) > 0, \end{cases} \tag{B.33a}$$

$$\mu_2(t) = \begin{cases} 0, & \text{if } \lambda(t) \geq -2r\alpha, \\ -\lambda(t) - 2r\alpha, & \text{if } \lambda(t) < -2r\alpha, \end{cases} \tag{B.33b}$$

and with $\lambda(t)$ to be determined. The problem that has to be solved to determine $\lambda(t)$ is given by:

$$\dot{x}(t) = u^*(t) - d, \qquad x(t_0) = x_0, \tag{B.34a}$$

$$\dot{\lambda}(t) = -2q \cdot x(t), \qquad \lambda(t_f) = 0, \tag{B.34b}$$

with $u^*(t)$ specified by (B.32). It is possible to derive a second order differential equation in $\lambda(t)$ from problem (B.34). However, note that functions $\mu_1(t)$ and $\mu_2(t)$ are not continuously differentiable, i.e., their time derivative is not a continuous function. Due to this fact, an explicit solution for $\lambda(t)$ cannot be obtained. A solution to the problem can be obtained by transforming problem (B.34) into a numerical problem. The co-state $\lambda(t)$ can then be determined numerically, such that the optimal open loop limited control law (B.32) is completely specified. For visualization of and simulation with the open loop control law is made use of the numerical problem that is finished with in Section B.1.

### Control problem $(u - d)$

Consider the control problem $(u - d)$ that is specified by the following dynamics, performance index, and constraints on functions of the control variable:

$$\dot{x}(t) = u(t) - d, \qquad x(t_0) = x_0, \tag{B.35}$$

$$J_{(u-d)} = \int_{t_0}^{t_f} q \cdot x(t)^2 + r \cdot (u(t) - d)^2 \mathrm{d}t, \tag{B.36}$$

$$c[u(t), t] \leq 0, \qquad c[u(t), t] = \begin{bmatrix} -u(t) \\ u(t) - \alpha \end{bmatrix}. \tag{B.37}$$

The open loop control problem is to find an admissible control function $u(t)$ that minimizes performance index (B.36) subject to the system dynamics (B.35) and constraints (B.37). Similar as in Appendix A, introduce a dummy control variable $\tilde{u}(t) = u(t) - d$. Substituting $\tilde{u}(t)$ into (B.35), (B.36), and (B.37) yields the following dynamics, performance index, and constraints on functions of the dummy control variable for the limited system $(u - d)$ with dummy control variable $\tilde{u}(t)$:

$$\dot{x}(t) = \tilde{u}(t), \qquad x(t_0) = x_0, \tag{B.38}$$

$$J_{(\tilde{u})} = \int_{t_0}^{t_f} q \cdot x(t)^2 + r \cdot \tilde{u}(t)^2 \mathrm{d}t, \tag{B.39}$$

$$c[\tilde{u}(t), t] \leq 0, \qquad c[\tilde{u}(t), t] = \begin{bmatrix} -\tilde{u}(t) - d \\ \tilde{u}(t) - (\alpha - d) \end{bmatrix}. \tag{B.40}$$

The open loop control problem is now to find an admissible dummy control function $\tilde{u}(t)$ that minimizes performance index (B.39) subject to the system dynamics (B.38) and constraints (B.40).

The following Hamiltonian can be derived for the problem (B.38, B.39, B.40):

$$H = q \cdot x(t)^2 + r \cdot \tilde{u}(t)^2 + \mu_1(t)(-\tilde{u} - d) + \mu_2(t)(\tilde{u}(t) - (\alpha - d)), \tag{B.41}$$

with the following constraints on the functions $\mu_1(t)$ and $\mu_2(t)$ respectively:

$$\mu_1(t)(-\tilde{u}(t) - d) = 0, \qquad \mu_1(t) \geq 0, \text{ and} \tag{B.42a}$$

$$\mu_2(t)(\tilde{u}(t) - (\alpha - d)) = 0, \qquad \mu_2(t) \geq 0. \tag{B.42b}$$

The co-state variable $\lambda(t)$ is defined as the solution of co-state equation:

$$\dot{\lambda}(t) = -2q \cdot x(t), \qquad \lambda(t_f) = 0. \tag{B.43}$$

The Minimum Principle states that an optimal control function minimizes the Hamiltonian. From $\frac{\partial H}{\partial u}$ results that such an optimal function is described by the following equation:

$$\tilde{u}^*(t) = \frac{1}{2r} \cdot (-\lambda(t) + \mu_1(t) - \mu_2(t)), \tag{B.44}$$

with $\lambda(t)$, $\mu_1(t)$, and $\mu_2(t)$ to be determined later.

Similar as in Equation (B.24), the functions $\mu_1(t)$ and $\mu_2(t)$ limit the effect of $\lambda(t)$, thus preventing $\tilde{u}^*(t)$ from violating constraints (B.40). As a consequence, functions $\mu_1(t)$ and $\mu_2(t)$ also depend on the value of $\lambda(t)$. Again, three cases can be considered, which conditions can be written as functions of the co-state $\lambda(t)$:

- no constraints are violated if $-2r(\alpha - d) \leq \lambda(t) \leq 2rd$,
- the lower constraint is violated if $\lambda(t) > 2rd$, and
- the upper constraint is violated if $\lambda(t) < -2r(\alpha - d)$.

Function $\mu_1(t)$ becomes active when the lower constraint is violated. Substituting (B.44) with $\mu_2(t) = 0$ into constraint (B.42a) yields the value for $\mu_1(t)$. In the case that the upper constraints is violated, function $\mu_2(t)$ must become active. Its value results from substituting (B.44) with $\mu_1(t) = 0$ into (B.42b).

Summarizing the results from above, can be concluded that the optimal open loop control law $u^*(t)$ for the limited control problem $(u - d)$ is described by:

$$u^*(t) = d + \frac{1}{2r} \cdot (-\lambda(t) + \mu_1(t) - \mu_2(t)), \tag{B.45}$$

where

$$\mu_1(t) = \begin{cases} 0, & \text{if } \lambda(t) \leq 2rd, \\ \lambda(t) - 2rd, & \text{if } \lambda(t) > 2rd, \end{cases} \tag{B.46a}$$

$$\mu_2(t) = \begin{cases} 0, & \text{if } \lambda(t) \geq -2r(\alpha - d), \\ -\lambda(t) - 2r(\alpha - d), & \text{if } \lambda(t) < -2r(\alpha - d), \end{cases} \tag{B.46b}$$

and with $\lambda(t)$ to be determined. The problem that has to be solved to determine $\lambda(t)$ is given by:

$$\dot{x}(t) = u^*(t) - d, \qquad x(t_0) = x_0, \tag{B.47a}$$

$$\dot{\lambda}(t) = -2q \cdot x(t), \qquad \lambda(t_f) = 0, \tag{B.47b}$$

where $u^*(t)$ is specified by (B.45). This is a similar problem as that for limited control problem $(u)$, see Equation (B.34).

## B.3 Feedback control

In solving the limited control problem, constrained and unconstrained paths must be pieced together such that all necessary conditions are satisfied. The junction point of constrained and unconstrained paths is referred to as a corner, see Bryson [Bry75]. At a corner, the control path can be discontinuous. In Section B.1, when discussing the numerical example, the assumption has been made that the line $u(t) = \alpha$ and the free control path $\bar{u}^*(t)$ may be pieced together at the point where they intersect. However, results from Figure B.3 show that this is not the case. This can be explained as follows.

Consider the start of the horizon where the free control law violates a constraint. The value of the limited control law is thus set to that of the violated constraint. As a consequence of the control action some different state than the initial state $x_0$ is reached. For that state and time, the optimal control problem must be solved again. The solution still violates the constraint, thus the limited control law remains equal to constraint. This loop continues until the solution of the optimal control problem does not violate the constraint anymore. The limited control law may then take the value of the derived solution. This moment occurs in the corner. Finding a solution every present state and time to the present optimal control problem is exactly what dynamic programming does. Then, the optimal limited feedback control law $u^*[x,t]$ is the saturation of the optimal free feedback control law $\bar{u}^*[x,t]$. For both control problems $(u)$ and $(u-d)$, the optimal limited feedback control law $u^*[x,t]$ results into:

$$u^*[x,t] = \begin{cases} \alpha, & \text{if} \quad \bar{u}^*[x,t] > \alpha, \\ \bar{u}^*[x,t], & \text{if} \quad 0 \leq \bar{u}^*[x,t] \leq \alpha, \\ 0, & \text{if} \quad \bar{u}^*[x,t] < 0, \end{cases} \tag{B.48}$$

where for control problem $(u)$ $\bar{u}^*[x,t]$ is given by Equation (A.39) and for control problem $(u-d)$ $\bar{u}^*[x,t]$ is given by Equation (A.53). In every point $(x,t)$, the value of the free feedback control law is compared to that of the constraints. If the free feedback control law violates a constraint, the value of the limited feedback law is set to that of the violated constraint.

# Appendix C

# Simulation model

This appendix deals with a discrete-event model of a single machine manufacturing system suited for simulation. A control system that applies optimal control laws is interconnected to the manufacturing system. The control laws are designed in a continuous-time flow model of the manufacturing system. A discrete-time converter is applied to enable the interconnection. First, applied software tools are shortly described. Then, a deterministic and stochastic discrete-event simulation model is presented.

## C.1 Tools

The following software tools are used to perform the simulations.

$\chi$: A formalism based on communicating sequential processes. Rooda [Roo00] uses $\chi$ 0.3 to describe the (dynamic) behavior of industrial systems. In this research $\chi$ 0.7.5c is used, in which Python access is added. A manual for $\chi$ 0.7 is provided by Kleijn and Rooda [Kle01].

MATLAB: A flexible environment for technical computing. Calculations in MATLAB are based upon arrays. The optimal continuous-time control laws derived in Appendices A and B are implemented in MATLAB function-files. Also, post-processing of the obtained experiment data is performed in MATLAB.

Python: An object-oriented programming language. It can be used for writing stand alone programs, quick scripts, and prototypes of complex applications. A good introduction to the basics of the Python language is given by Lutz and Ascher [Lut99]. Python is used in this research to function as an interface between $\chi$ and MATLAB.

The $\chi \rightarrow$ Python-interface provided by Hofkamp [Hof01] provides a way to let $\chi$ interact with Python. The *pymat* module provided by Sterian [Ste99] provides a way to use the functionality of MATLAB in a Python environment.

## C.2 Deterministic model

A discrete-event model of system $W$ with controller $C$ as described in Chapter 5 is presented. The model is specified in the formalism $\chi$. The model calls several external functions specified in the Python language. One of those functions starts calculations in MATLAB. A visualization of the model is shown in Figure C.1. In the following

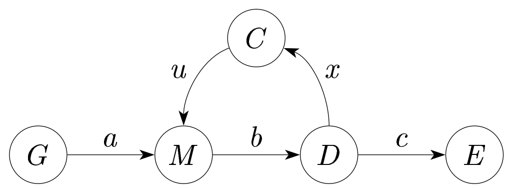



Figure C.1: Discrete-event model.

$\chi$-specifications, the parts are modeled by natural numbers. The production rate and surplus level, the state, may take real values, as expressed in the following type-mapping:

$$
\begin{aligned}
&\mathsf{type}\ \text{part} &= \text{nat}\\
&,\quad \text{rate} &= \text{real}\\
&,\quad \text{state} &= \text{real.}
\end{aligned}
\tag{C.1}
$$

Below follow the descriptions and $\chi$-specifications of system $W$ and its processes and functions. Note that boldface symbols stand for variables of the string type. A description of a system, process, or function starts with an function part in which is explained what the subject does. Then, a purpose part explains how the subject's variables relate to those of other processes. Finally, an effect part explains how the subject's variables achieve the expressed function.

**System** $W$

Function: To produce parts such that an optimal system behavior is achieved.

Purpose: The control system, represented by controller $C$, controls the flow of parts through the manufacturing system, represented by machine $M$ and depot $D$. Generator $G$ and end process $E$ represent the environment of system $W$, see also Figure C.1.

Effect: The generator models the supply of parts by the environment, the end process models the demand for parts, i.e., orders, by the environment. The difference of the cumulative production of the machine and the cumulative demand of the environment is modeled in the depot by a surplus of parts and orders. Based upon the present surplus and present time, the controllers decides at what rate the machine should produce. These optimal production rates are optimal in a sense that they are derived from a continuous-time control law that minimizes a defined performance index, see also Appendices A and B.

The interactions of the processes as modeled in Figure C.1 are implemented in the $\chi$-specification of system $W$ [C.2].

$$\begin{array}{l} \mathsf{syst}\ W() = \\ [\![\ a,\ b,\ c : -\mathrm{part},\ u : -\mathrm{rate},\ x : -\mathrm{state} \\ \mid G(a) \parallel M(a,\ b,\ u) \parallel D(b,\ c,\ x) \parallel E(c) \parallel C(x,\ u) \\ ]\!] \end{array} \tag{C.2}$$

### Generator $G$

Function: To model the supply of parts by the environment.

Purpose: Supply parts to machine $M$, see also Figure C.1.

Effect: Send a part, represented by a natural 1, to the machine when possible. This effect is implemented in the $\chi$-specification of process $G$ [C.3].

$$\begin{array}{l} \mathsf{proc}\ G(a : !\,\mathrm{part}) = \\ [\![\ *[\ \mathrm{true} \longrightarrow a\,!\,1\ ]\ ]\!] \end{array} \tag{C.3}$$

### End process $E$

Function: To model the constant demand for parts by the environment.

Purpose: Take parts from depot $D$ at a specified demand rate, see also Figure C.1.

Effect: First, the inter request time $t_d$ is calculated from the demand rate $d$. The value for parameter $d$ is returned by external function *GetReal*. Then, every $t_d$ period of time, a single part is taken from the depot. This effect is implemented in the $\chi$-specification of process $E$ [C.4].

$$\begin{array}{l} \mathsf{proc}\ E(a : ?\,\mathrm{part}) = \\ [\![\ p : \mathrm{part},\ t_d : \mathrm{real} \\ \mid t_d := 1.0/\mathit{GetReal}(\mathbf{d}) \\ ;\ *[\ \mathrm{true};\ \Delta\, t_d \longrightarrow a\,?\,p\ ] \\ ]\!] \end{array} \tag{C.4}$$

### Machine $M$

Function: To produce parts at an optimal rate.

Purpose: When required, a part is received from generator $G$. A part is processed for an optimal period of time and is then send to depot $D$. The optimal production time is determined by controller $C$, see also Figure C.1.

Effect: The machine has two states: on and off. Only if the machine is in the on state, a part can be loaded from the generator. If a part is loaded, the machine is busy. Initially, the machine is in the off state and waits to be switched on.

This is done when a new optimal production rate $u_n \neq 0.0$ is received from the controller. The present optimal production rate $u$ is updated and a part can be loaded. After an optimal period of time, i.e., at finishing time $t_p$, the part is send to the depot and the machine becomes idle. Now, a new part may be loaded and processed at the present rate $u$. While processing a part, the controller may send an new optimal rate $u_n \neq 0.0$. Because some form of preemption is allowed, this new rate can directly be incorporated in the processing of the loaded part. Therefore, a new optimal finishing time $t_p$ is returned by function *DelayLeft*. When the controller sends a new optimal rate $u_n = 0.0$, the machine is switched off. Any loaded part is kept on the machine. A switching off time $t_o$ is stored, such that when the machine is switched on again, the part is finished at the new optimal rate. This effect is implemented in the $\chi$-specification of process $M$ [C.5].

$$
\begin{array}{l}
\textsf{proc}\ M(a : ?\,\mathrm{part},\ b : !\,\mathrm{part},\ c : ?\,\mathrm{rate}) = \\
[\![\ p : \mathrm{part},\ u,\ u_n : \mathrm{rate},\ \mathit{on},\ \mathit{busy} : \mathrm{bool},\ t_o,\ t_p : \mathrm{real} \\
\mid\ \mathit{on} := \mathrm{false};\ \mathit{busy} := \mathrm{false};\ t_o := 0.0 \\
;\ *[\quad \mathit{on} \wedge \neg \mathit{busy};\ a\,?\,p \\
\qquad\quad \longrightarrow t_p := \tau + 1.0/u;\ \mathit{busy} := \mathrm{true} \\
\quad\ [\!]\ \ \mathit{on} \wedge \mathit{busy};\ \Delta\, t_p - \tau \\
\qquad\quad \longrightarrow b\,!\,1;\ \mathit{busy} := \mathrm{false} \\
\quad\ [\!]\ \ \mathrm{true};\ c\,?\,u_n \\
\qquad\quad \longrightarrow [\ u_n \neq 0.0 \wedge \mathit{busy} \\
\qquad\qquad\qquad \longrightarrow t_p := \tau + u/u_n \mathit{DelayLeft}(\tau,\ t_o,\ t_p) \\
\qquad\qquad\qquad\quad ;\, t_o := 0.0;\ u := u_n \\
\qquad\qquad\quad [\!]\ u_n \neq 0.0 \wedge \neg \mathit{busy} \\
\qquad\qquad\qquad \longrightarrow \mathit{on} := \mathrm{true};\ u := u_n \\
\qquad\qquad\quad [\!]\ u_n = 0.0 \\
\qquad\qquad\qquad \longrightarrow \mathit{on} := \mathrm{false};\ t_o := \tau \\
\quad\ ]\,] \\
]\!]
\end{array}
\qquad [\mathrm{C.5}]
$$

**Function** *DelayLeft*

Function: To calculate the remaining production delay.

Purpose: Based upon the present time, finishing time, and if necessary the switching off time, the remaining production time for a part being processed is calculated. All times are based upon the present production rate.

Effect: If the machine has not been switched off while the part was loaded, the remaining production time is simply the difference between the finishing time $t_p$ and the present time $t$. Else, the finishing time $t_p$ has to be increased with the time that the machine has been switched off, $t_o$. This effect is implemented in the $\chi$-specification of function *DelayLeft* [C.6].

$$
\begin{array}{l}
\mathsf{func}\ \mathit{DelayLeft}(t,\ t_o,\ t_p : \mathrm{real}) \rightarrow \mathrm{real} = \\
[\![\ [\ t_o = 0.0 \longrightarrow \uparrow t_p - t \\
\quad [\!]\ t_o > 0.0 \longrightarrow \uparrow t_p + (t - t_o) - t \\
\quad ] \\
]\!]
\end{array}
\qquad [\mathrm{C.6}]
$$

**Depot** $D$

Function: To temporarily store parts and orders.

Purpose: Parts from machine $M$ are stored in depot $D$ and send to end process $E$. Here, the depot level models the surplus, the difference between the cumulative production and the cumulative demand. The changed depot level is send to controller $C$, see also Figure C.1.

Effect: When a part is received from the machine, the depot level $x$ is increased by 1. The depot level is decreased by 1 when the end process takes a part from the depot. After every change of depot level, the value of $x$ is send to the controller. A depot of infinite size is modeled, thus $x$ may take any value in $\mathbb{Z}$. This effect is implemented in the $\chi$-specification of process $D$ [C.7].

$$
\begin{array}{l}
\mathsf{proc}\ D(a : ?\,\mathrm{part},\ b : !\,\mathrm{part},\ c : !\,\mathrm{state}) = \\
[\![\ p : \mathrm{part},\ x : \mathrm{state} \\
|\ x := \mathit{GetReal}(\mathbf{x0})\,;\ c\,!\,x \\
;\ *[\ \ [\ \mathrm{true};\ a\,?\,p \longrightarrow x := x + 1.0 \\
\qquad [\!]\ \mathrm{true};\ b\,!\,1\ \ \longrightarrow x := x - 1.0 \\
\qquad ] \\
\quad ;\ c\,!\,x \\
\quad ] \\
]\!]
\end{array}
\qquad [\mathrm{C.7}]
$$

**Controller** $C$

Function: To control the flow of parts such that an optimal performance is achieved. Also, store state and control path data and terminate the experiment.

Purpose: Based upon the present depot level and present time, an optimal production rate is calculated and send to machine $M$. The present depot level is received from depot $D$, see also Figure C.1.

Effect: Here, the production rate $u$ is the control variable, the depot level $x$ is the state variable. Every $t_s$ period of time, the controller goes through the following sequence. First, calculate the present optimal production rate $u$. This value is returned by the external function $\mathit{Control}(x, \tau)$ for given present state $x$ and present time $\tau$. The present state is received from the depot. Then, send the optimal production rate to the machine. Last, store the present state and control path data. This is done by returning list $[\tau, x, u]$ to external function *StoreReals*. At final time $t_f$, the final path data are stored and dumped to a file by returning list [**path**] to external function *DumpStores*. Finally, the experiment is terminated. This effect is implemented in the $\chi$-specification of process $C$ [C.8].

$$
\begin{array}{l}
\mathsf{proc}\ C(a : ?\,\mathrm{state},\ b : !\,\mathrm{rate},\ ) = \\
[\![\ u : \mathrm{rate},\ x : \mathrm{state},\ t_f,\ t_s,\ t_u : \mathrm{real}, \\
\mid t_f := \mathit{GetReal}(\mathbf{tf})\,;\ t_s := \mathit{GetReal}(\mathbf{ts}) \\
;\ a\,?\,x\,;\ u := \mathit{Control}(x,\ \tau)\,;\ b\,!\,u \\
;\ [\ \mathit{StoreReals}([\tau,\ x,\ u],\ \mathbf{path}) \longrightarrow \mathrm{skip}\ ] \\
;\ t_u := t_s \\
;\ *[\ \mathrm{true}\,;\ a\,?\,x \longrightarrow \mathrm{skip} \\
\quad [\!]\ \mathrm{true}\,;\ \Delta\, t_u - \tau \\
\qquad \longrightarrow u := \mathit{Control}(x,\ \tau)\,;\ b\,!\,u \\
\qquad\quad ;\ [\ \mathit{StoreReals}([\tau,\ x,\ u],\ \mathbf{path}) \longrightarrow \mathrm{skip}\ ] \\
\qquad\quad ;\ t_u := t_u + t_s \\
\quad [\!]\ \mathrm{true}\,;\ \Delta\, t_f - \tau \\
\qquad \longrightarrow [\ \mathit{StoreReals}([\tau,\ x,\ u],\ \mathbf{path}) \longrightarrow \mathrm{skip}\ ] \\
\qquad\quad ;\ [\ \mathit{DumpStores}([\mathbf{path}]) \longrightarrow \mathrm{skip}\ ] \\
\qquad\quad ;\ \mathrm{terminate} \\
\quad ] \\
]\!]
\end{array}
\tag{C.8}
$$

System $W$ is instantiated in the following experiment environment:

$$\mathsf{xper}\ = [\![\ W()\ ]\!]\ . \tag{C.9}$$

The described $\chi$-specifications [C.1] up to and including [C.9] are united into a single $\chi$-model *exp_DE.chi*. Several external functions are applied in the $\chi$-specifications above. By means of the $\chi \rightarrow$ Python-interface, these external functions are included in the experiment. The external functions are specified in a Python module named *func.py*. The $\chi \rightarrow$ Python-interface tells the $\chi$-compiler to use this module by specifying a list of external specifications. This list is specified in a script *func.ext* [C.10].

```
// func.ext
//
language "python"
file "func"
// Python -> Chi
ext GetReal(p: string) -> real = "Get"
ext Control(x, t: real) -> real
// Chi -> Python
ext StoreReals(xs: real*, n: string) -> bool = "StoreData"
ext DumpStores(ns: string*) -> bool
```

[C.10]

Here, *GetReal*, and *Control* are functions in Python that return values back to $\chi$. Functions *StoreReals* and *DumpStores* store data from $\chi$ to Python. In Python, functions *GetReal* and *StoreReals* call respectively functions *Get* and *StoreData*. The Python module *func.py* that specifies the external functions starts with the import of several other Python modules [C.11].

```
# func.py
#
import os
from Numeric import *
import pymat
import pickle
global G
G = None
```
[C.11]

Then, a set of parameters for the experiment is loaded. This is done by making use of the *pickle* module [Lut99]. Pickling converts objects to serialized byte streams, which may be stored in files or sent across a network. In some kind of simulation module, a dictionary object, see [C.12], has been created and pickled to a file *setup*.

```
pard = { 'q': 0.5, 'r': 0.5, 't0': 0.0, 'tf': 10.0
       , 'ts': 0.001, 'd': 10.0, 'alpha': 10.0, 'x0': -40.0
       , 'c2': 1.0, 'seed': 1, 'law': 'fbfreecontrolu'
       }
```
[C.12]

The set of parameters is then loaded by [C.13] from *setup* as dictionary *pard*. While performing multiple experiments, this pickling construct enables a some kind of simulation module to easily change the parameter setup for each experiment.

```
# Load parameters from file setup
input = open('setup', 'r')
pard = pickle.load(input)
input.close()
```
[C.13]

Below follow the descriptions and Python specifications of the external functions as applied above.

### Function *Get*

Function: To get the experiment setup.

Purpose: Based upon the parameter name, represented by a string, its value is returned.

Effect: The value of key $p$ from dictionary *pard* is returned. This effect is implemented in the Python specification of function *Get* [C.14].

```
# Function: get parameter
def Get(p):
    return pard[p]
```
[C.14]

**Function** *Control*

Function: To calculate the value of the optimal production rate.

Purpose: Based upon the present state and time, the value of the optimal continuous-time control law is returned.

Effect: When called for the first time, a MATLAB handle is opened by means of the *pymat* module. The parameter setup is placed into the MATLAB workspace as global variables. During the experiment, the MATLAB handle is kept open. Every time that *Control* is called, a MATLAB function-file specified by parameter key *'law'* is called with parameters $x$ and $t$. The calculated production rate $u$ is returned. This effect is implemented in the Python specification of function *Control* [C.15].

```
# Function: calculate value of control law
def Control(x, t):
    global G
    if G == None:
        G = pymat.open('matlab -nosplash')
        pymat.eval(G, 'clear all')
        for key in pard.keys():
            pymat.eval(G, 'global %s' % key)
            pymat.eval(G, '%s = %s;' % (key, pard[key]))
    pymat.eval(G, 'u = %s(%s, %s);' % (pard['law'], x, t))
    return pymat.get(G, 'u')[0]
```
[C.15]

As an example of a MATLAB function-file that is called by function *Control*, consider the following implementations of the optimal continuous-time control laws for the free and limited control problem ($u$), respectively Equations (A.39) and (B.48):

```
function y = fbfreecontrolu(x, t)
global q r tf d
y = d - d * sech(sqrt(q/r) * (t - tf))
    + sqrt(q/r) * tanh(sqrt(q/r) * (t - tf)) .* x;
```
[C.16]

and,

```
function y = fblimcontrolu(x, t)
global alpha
if fbfreecontrolu(x, t) > alpha
   y = alpha;
elseif fbfreecontrolu(x, t) < 0
   y = 0;
else
   y = fbfreecontrolu(x, t)
end
```
[C.17]

**Function** *StoreData*

Function: To temporarily store data.

Purpose: The given data is appended to a specified list.

Effect: Initially, a dictionary *stored* with empty lists is created. The received list *data* is appended to the list in *stored* with key *name*. This effect is implemented in the Python specification of function *StoreData* [C.18].

```
# Function: store data in list name
stored = {'path': [], 'td': []}
def StoreData(data, name):                                      [C.18]
    stored[name].append(data)
    return 1
```

**Function** *DumpStores*

Function: To permanently save stored data.

Purpose: The specified stored data is dumped into a file.

Effect: For every key in the received list *stores*, its value in dictionary *stored* is pickled to a file named by the key. This effect is implemented in the Python specification of function *DumpStores* [C.19].

```
# Function: dump store to file
def DumpStores(stores):
    for store in stores:
        output = open(store, 'w')                               [C.19]
        pickle.dump(stored[store], output)
        output.close()
    return 1
```

External functions [C.14, C.15, C.18, C.19], the import of other modules [C.11], and the loading of the parameter setup [C.13] are united into a single Python module *func.py*. The module *func.py* requires an input file *setup* and returns an output file *path* (and also *td* in the stochastic case, see the next section) for every performed experiment *exp_DE*. A change of *func.py* does not require re-compiling the $\chi$-model *exp_DE.chi* with the list of external specifications *func.ext* to an executable *exp_DE*. Multiple experiments with different parameters setups can then easily be performed by some kind of simulation module. It is left to the reader to specify such a simulation module, such that pre-processing and post-processing can be done in a favorite way. In this research, a Python module is generated to create the input file *setup* and call MATLAB to post-process the output files.

## C.3 Stochastic model

In the stochastic model of system $W$ with controller $C$, a stochastic demand rate is taken into account. Therefore, introduce variability to the mean inter request time $t_d$ of end process $E$. By means of a Gamma distribution, low, moderate, and high variability distributions can be modeled. The probability function of the Gamma distribution is defined by:

$$f(x) = \begin{cases} \frac{r(rx)^{q-1}\mathrm{e}^{-rx}}{\Gamma(q)}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases} \tag{C.1}$$

with the Gamma-function $\Gamma$ given by:

$$\Gamma(p) = \int_0^\infty t^{q-1}\mathrm{e}^{-t}\mathrm{d}t. \tag{C.2}$$

Here, the two parameters $q$ and $r$ define the shape and scale of the distribution. Changing the parameters results in low, moderate, or high variable distributions. The parameters $q$ and $r$ relate to the mean $\mu$ and squared coefficient of variation $c^2$ via $q = 1/c^2$ and $r = \mu c^2$ respectively, where the squared coefficient is defined by:

$$c^2 = \frac{\sigma^2}{\mu^2}, \tag{C.3}$$

and $\sigma^2$ denotes the variance. According to Hopp and Spearman [Hop00], a squared coefficient of variation substantially smaller than 1 indicates a lowly variable distribution, where highly variable distributions are indicated by a squared coefficient of variation substantially higher than 1. Distributions with a squared coefficient of variation near 1 are called moderately variable. The following description of the stochastic end process $E$ can be specified by $\chi$-specification [C.20].

**End process $E$**

Function: To model the stochastic demand for parts by the environment. Also, store inter request time sample data.

Purpose: Take parts from depot $D$ at a specified mean demand rate, see also Figure C.1.

Effect: First, a Gamma distribution $t$ for the inter request time $t_d$ is initialized. The distribution parameters $q$ and $r$ are calculated from the mean $m$ and squared coefficient of variation $c2$ returned by external function *GetReal*. The distribution is set to a particular seed determined by external function *GetNat*, such that subsequent experiments have equal sample results. Then, every $t_d$ period of time, a single part is taken from the depot. The sample $t_d$ is stored by returning $t_d$ to external function *StoreReal*. This effect is implemented in the $\chi$-specification of process $E$ [C.20].

$$
\begin{array}{l}
\mathsf{proc}\ E(a : ?\,\text{part}) = \\
[\![\ p : \text{part},\ \mathit{seed} : \mathit{nat},\ m,\ \mathit{c2},\ q,\ r,\ t_d : \text{real},\ t :\ \rightarrow \text{real} \\
\mid\ m := 1.0/\,\mathit{GetReal}(\mathbf{d})\,;\ \mathit{c2} := \mathit{GetReal}(\mathbf{c2}) \\
;\ \mathit{seed} := \mathit{GetNat}(\mathbf{seed}) \\
;\ q := 1.0/\mathit{c2}\ ;\ r := m * \mathit{c2} \\
;\ t := \text{gamma}(q,\ r)\,;\ \text{setseed}(t,\ \mathit{seed})\,;\ t_d := \sigma t \\
;\ *[\ \text{true}\,;\ \Delta\, t_d \\
\qquad \longrightarrow a\,?\,p \\
\qquad\quad ;\ [\ \mathit{StoreReal}(\mathit{td},\ \mathbf{td}) \longrightarrow \text{skip}\ ] \\
\qquad\quad ;\ t_d := \sigma t \\
\quad ] \\
]\!]
\end{array}
\tag{C.20}
$$

Little changes are needed to $\chi$-specification [C.8] to let the controller also function in the stochastic model. Besides dumping the control and state path data to a file, also the sample data of inter request time $t_d$ are dumped, see $\chi$-specification [C.21].

$$
\begin{array}{l}
[\!]\ \Delta\, t_f - \tau \\
\quad \longrightarrow \ldots \\
\qquad ;\ [\ \mathit{DumpStores}([\mathbf{path},\ \mathbf{td}]) \longrightarrow \text{skip}\ ] \\
\qquad ; \ldots
\end{array}
\tag{C.21}
$$

The specifications of external functions *GetNat* and *StoreReal*, see [C.22], are added to list *func.ext*.

```
// Python -> Chi
ext GetNat(p: string) -> nat = "Get"
// Chi -> Python
ext StoreReal(x: real, n: string) -> bool = "StoreData"
```
[C.22]

# Appendix D

# Simulation results

Several control designs have been derived in the first two appendices. The explicit behavior of the controlled system is determined by means of simulation. A simulation model has been presented in Appendix C in which the derived control laws are implemented.

## D.1 Parameter influence

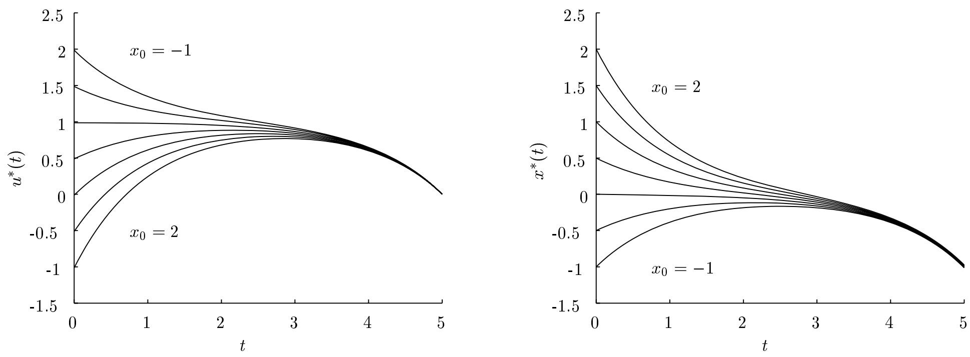
The simulation focuses on three subjects:

- parameter influence,
- validation of the flow model, and
- open loop and feedback control.

The parameter influence has been investigated in the continuous-time domain of the flow model. All other simulations have been performed in the discrete-event simulation model.

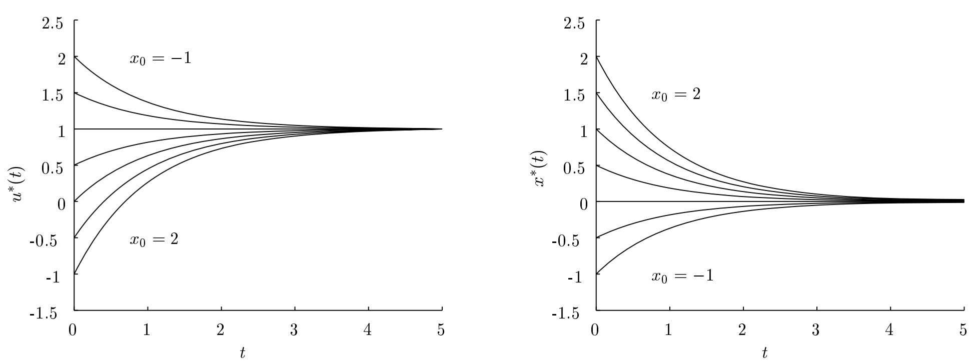
### Initial state $x_0$

The influence of the initial state is shown in Figures D.1 up to and including D.4. Here, the optimal control path $u^*(t)$ and optimal state path $x^*(t)$ are plotted for a range of initial states $x_0$. The final time $t_f$ is 5, cost parameters $q$ and $r$ are both $\frac{1}{2}$, the demand rate $d$ is 1, and the capacity $\alpha$ is set equal to $d$. For the range of control paths, left side, the state increases from top to bottom. This is opposite for the range of state paths, right side. From the figures follows that the controller reduces the initial offset in state to a desired optimal state path.



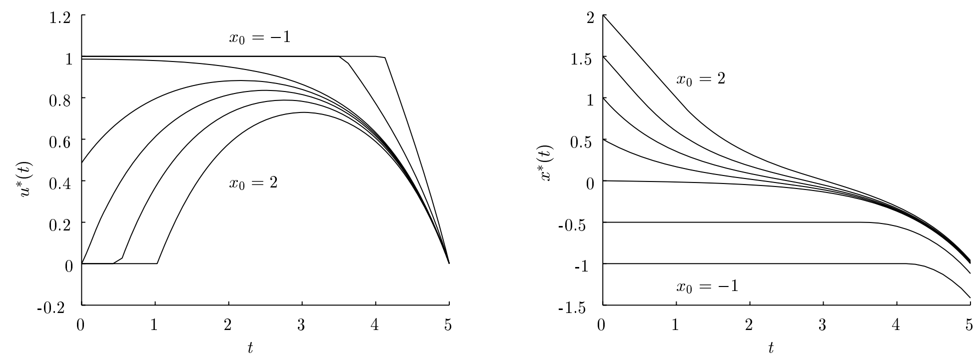

(a) (b)

Figure D.1: Influence of initial state $x_0$ with free control $(u)$: (a) optimal control path $u^*(t)$ and (b) optimal state path $x^*(t)$ for a range of initial states $x_0$ from -1 to 2 with stepsize $\frac{1}{2}$.



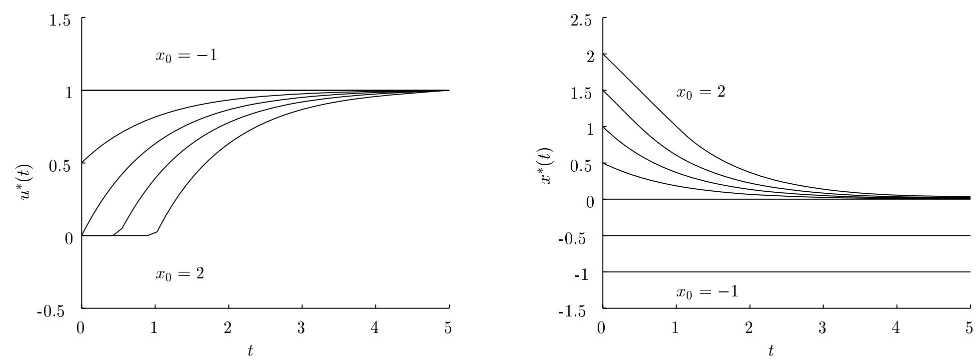

(a) (b)

Figure D.2: Influence of initial state $x_0$ with free control $(u - d)$: (a) optimal control path $u^*(t)$ and (b) optimal state path $x^*(t)$ for a range of initial states $x_0$ from -1 to 2 with stepsize $\frac{1}{2}$.




(a) (b)

Figure D.3: Influence of initial state $x_0$ with limited control ($u$): (a) optimal control path $u^*(t)$ and (b) optimal state path $x^*(t)$ for a range of initial states $x_0$ from -1 to 2 with stepsize $\frac{1}{2}$.

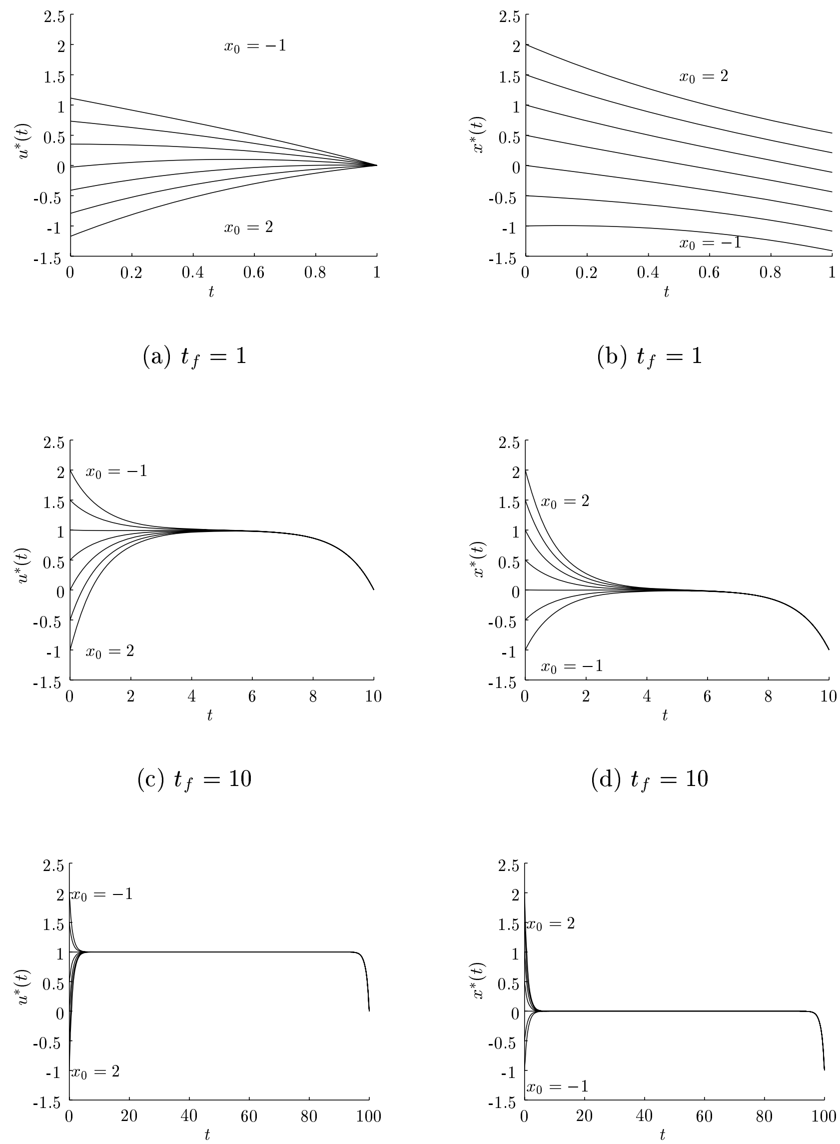



(a) (b)

Figure D.4: Influence of initial state $x_0$ with limited control ($u - d$): (a) optimal control path $u^*(t)$ and (b) optimal state path $x^*(t)$ for a range of initial states $x_0$ from -1 to 2 with stepsize $\frac{1}{2}$.

## Planning horizon $t_f$

A similar range of control and state paths is shown in Figure D.5 for free control problem ($u$) for three different planning horizons. As the planning horizon $t_f$ increases, the main effect of the controller is clearly to reduce the initial offset in state. The state is steered to a certain optimal value, similar to that in the hedging point concept as discussed in Chapter 4. The difference now is that the control variable changes gradually. The system behavior for the other control problems show the same effect.




(a) $t_f = 1$ (b) $t_f = 1$

(c) $t_f = 10$ (d) $t_f = 10$

(e) $t_f = 100$ (f) $t_f = 100$

Figure D.5: Influence of final time $t_f$ with free control $(u)$ on the optimal control path $u^*(t)$ and the optimal state path $x^*(t)$ for a range of initial states $x_0$ from -1 to 2 with stepsize $\frac{1}{2}$.

### Cost parameters $q$ and $r$

As mentioned, cost parameters $q$ and $r$ specify a preference for state and control costs respectively. Changing the ratio between these two parameters changes the system behavior substantially. Figures D.6 up to and including D.9 show the optimal system behavior for the free and limited control problems. The capacity $\alpha$ is set equal to the demand rate $d = 1$. Three different ratios for $\frac{q}{r}$ are considered. For a fair comparison, $q$ and $r$ are set such that for every ratio $\frac{q}{r}$ the same optimal performance is achieved, i.e., $J^* = 1$. For $\frac{q}{r} = \frac{1}{10}$, less costs for control actions are preferred than for control results. Consequently, an offset in state is accepted to let the control costs take smaller values. For $\frac{q}{r} = 10$, the opposite occurs. The offset in state is reduced much faster than for equal cost parameters. This results into relative extreme control actions at the beginning of the horizon.

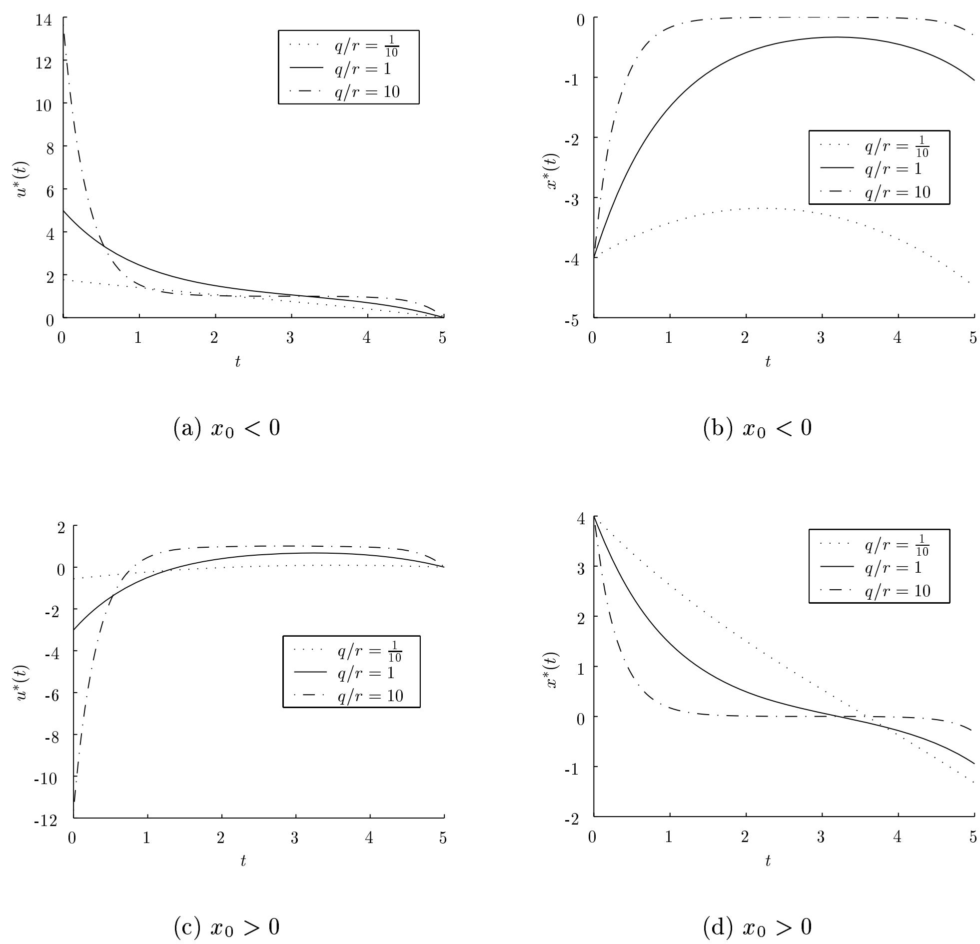
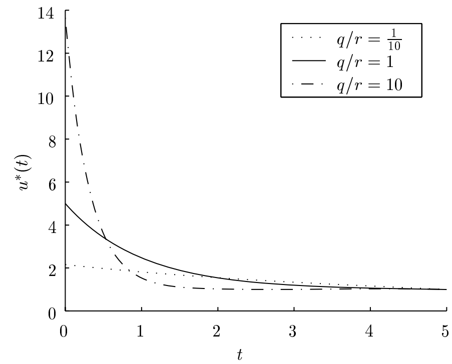
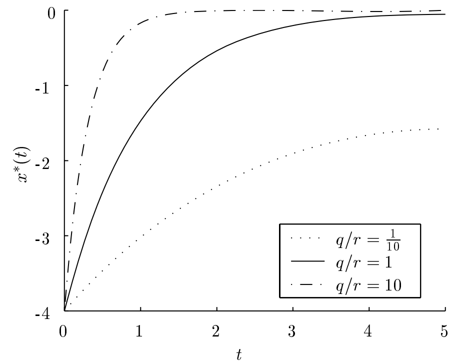


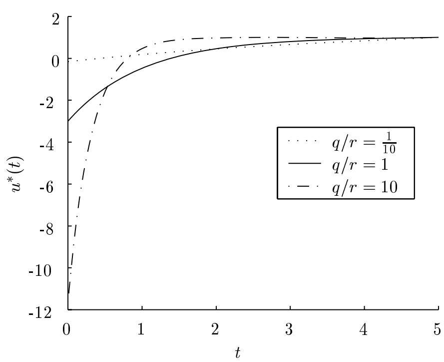

Figure D.6: Influence of cost parameters $q$ and $r$ with free control ($u$) on the optimal control path $u^*(t)$ and the optimal state path $x^*(t)$.



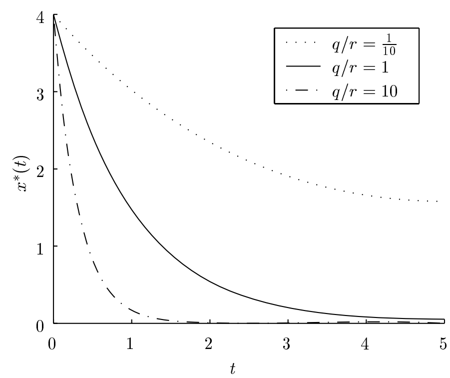

(a) $x_0 < 0$




(b) $x_0 < 0$

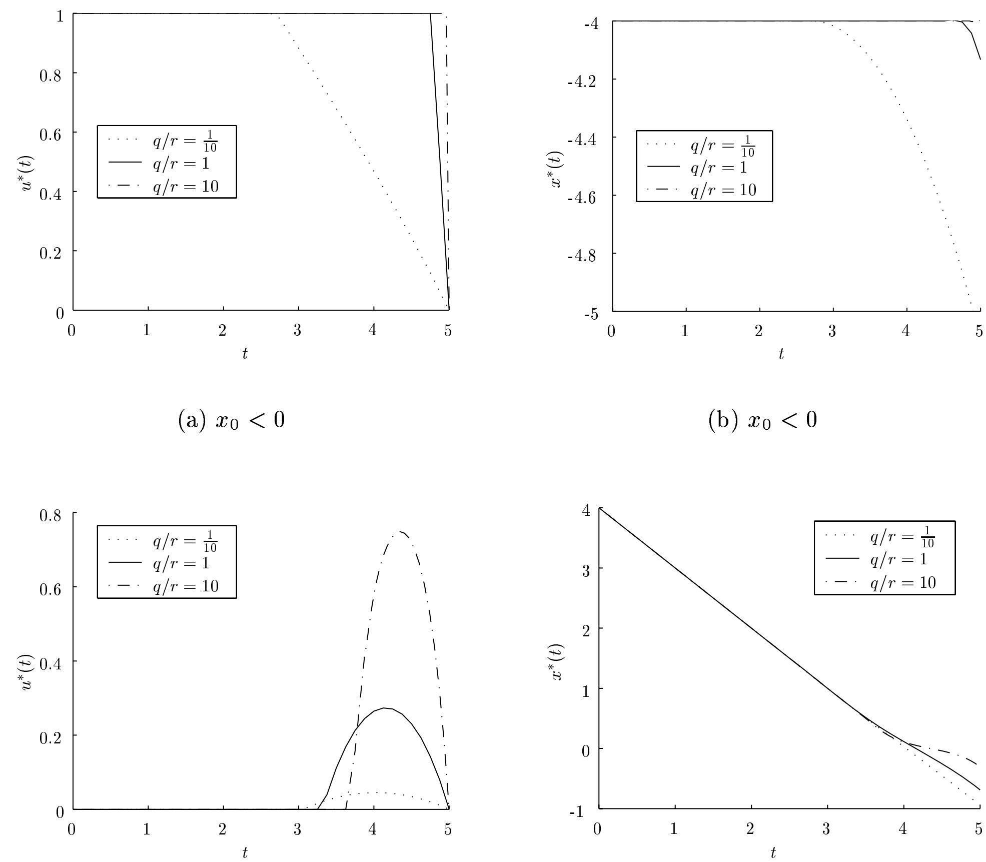



(c) $x_0 > 0$




(d) $x_0 > 0$

Figure D.7: Influence of cost parameters $q$ and $r$ with free control $(u - d)$ on the optimal control path $u^*(t)$ and the optimal state path $x^*(t)$.



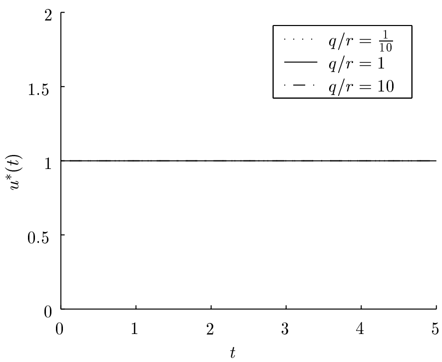

(a) $x_0 < 0$

(b) $x_0 < 0$

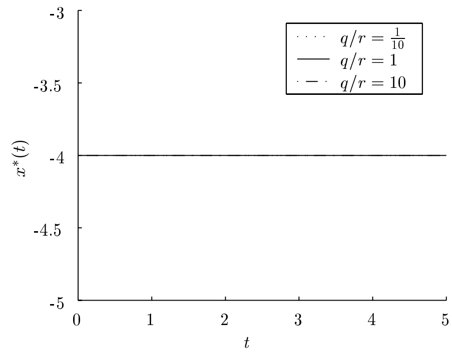
(c) $x_0 > 0$

(d) $x_0 > 0$

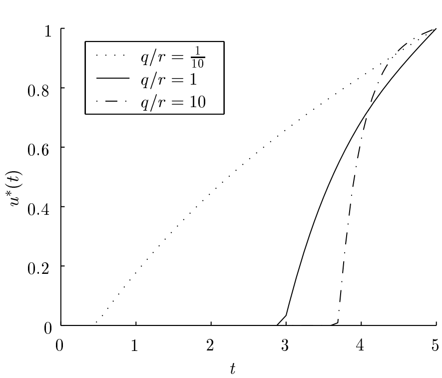
Figure D.8: Influence of cost parameters $q$ and $r$ with limited control $(u)$ on the optimal control path $u^*(t)$ and the optimal state path $x^*(t)$.



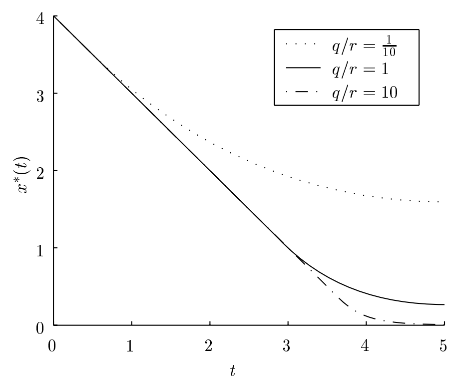

(a) $x_0 < 0$




(b) $x_0 < 0$




(c) $x_0 > 0$




(d) $x_0 > 0$

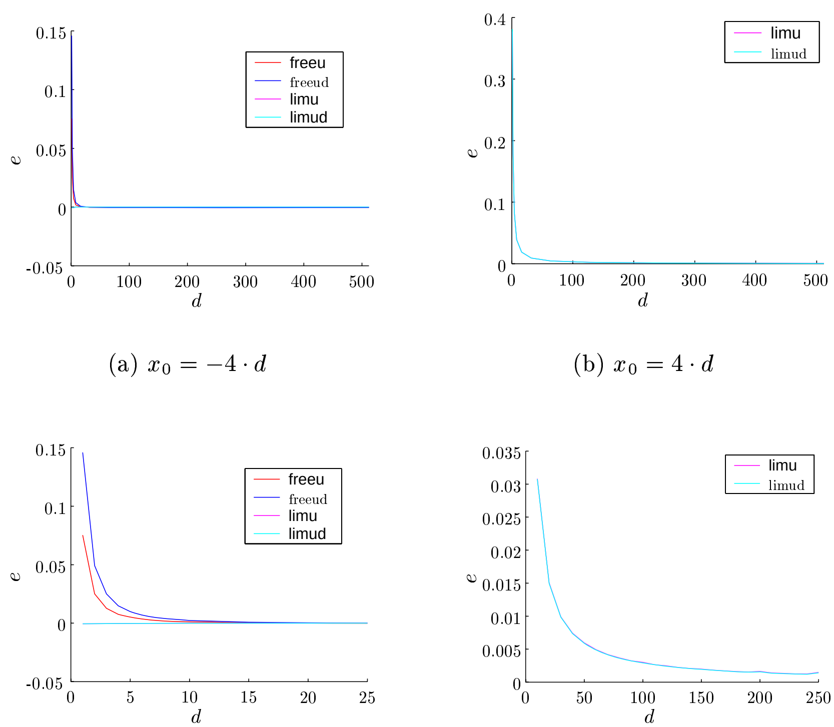
Figure D.9: Influence of cost parameters $q$ and $r$ with limited control $(u - d)$ on the optimal control path $u^*(t)$ and the optimal state path $x^*(t)$.

## D.2 Flow model validation

In Section 5.1 is assumed that the characteristic frequency of operations is much higher than that of planning. According to Assumption 2.3.2, the flow model is expected to be a valid approximation of the original, discrete-event model. The question that remains is *how much* higher the rate of operations must be. If the flow model is a valid approximation, then the performance of the approximate controlled system must be equal or close to that of the discrete-event controlled system. Introduce a relative performance error $e$ defined by:

$$e = \frac{J_{DE} - J_{CT}}{J_{CT}}, \tag{D.1}$$

where $J_{DE}$ denotes the performance of the discrete-event system and $J_{CT}$ denotes the performance of the continuous-time system. It is expected that for production rates not sufficiently high enough, the performance $J_{DE}$ is higher than performance $J_{CT}$. Recall that performance index $J$ is desired to be minimal.




(a) $x_0 = -4 \cdot d$

(b) $x_0 = 4 \cdot d$

(c) $x_0 = -4 \cdot d$

(d) $x_0 = 4 \cdot d$

Figure D.10: Relative error $e$ as a function of demand rate $d$ with final time $t_f = 10$.

Various simulations have been performed for a range of demand rates $d$. The final time $t_f$ is taken 10, cost parameters $q$ and $r$ are both $\frac{1}{2}$, initial state $x_0 = -4d$ or

$x_0 = 4d$, and capacity $\alpha$ is set equal to $d$. For every setup the relative error $e$ has been calculated. Results are plotted in Figure D.10. The negative values for $e$ are a result of the inaccuracy of the applied differential equation solver and integration method. From Figure D.10, Tables D.1 and D.2 are created. In these tables the demand rates are given for which the flow model is expected invalid, critical, and valid.

Table D.1: Invalid, critical, and valid demand rates $d$ for initial state $x_0 = -4d$.

| $\mathbf{x_0 = -4d}$ | **invalid** $\mathbf{e > 0.005}$ | **critical** $\mathbf{e \approx 0.005}$ | **valid** $\mathbf{e < 0.005}$ |
|---|---|---|---|
| free $(u)$ | 1 | 10 | 100 |
| free $(u - d)$ | 1 | 10 | 100 |
| limited $(u)$ | 1 | 200 | 500 |
| limited $(u - d)$ | 1 | 200 | 500 |

Table D.2: Invalid, critical, and valid demand rates $d$ for initial state $x_0 = 4d$.

| $\mathbf{x_0 = 4d}$ | **invalid** $\mathbf{e > 0.005}$ | **critical** $\mathbf{e \approx 0.005}$ | **valid** $\mathbf{e < 0.005}$ |
|---|---|---|---|
| limited $(u)$ | 1 | 200 | 500 |
| limited $(u - d)$ | 1 | 200 | 500 |

Figure D.11 shows the system behavior for free control problem $(u)$ for demand rates where the flow model is considered invalid, critical, and valid. This is also shown in Figure D.13 for limited control problem $(u)$. In both figures, the dotted line functions as a reference to the optimal, continuous-time behavior. From Tables 5.1 and 5.2 can be concluded for what time scales the flow model seems a good approximation. Consider the case of the free control problem with negative initial states. For a horizon of length 10, the time scale of operations must be at least 100 times smaller than that of the planning horizon. Similar effects and results for control problem $(u - d)$ are shown in Figures D.12 and D.14.

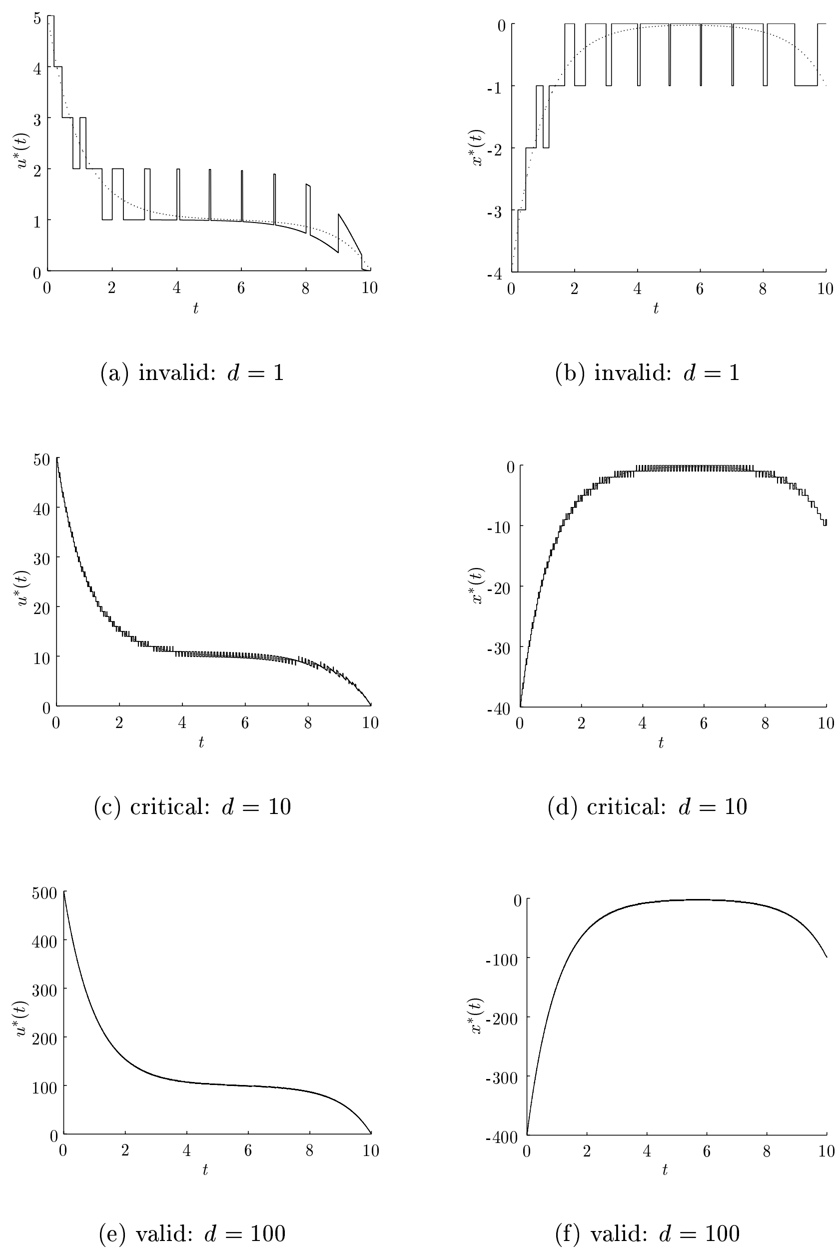
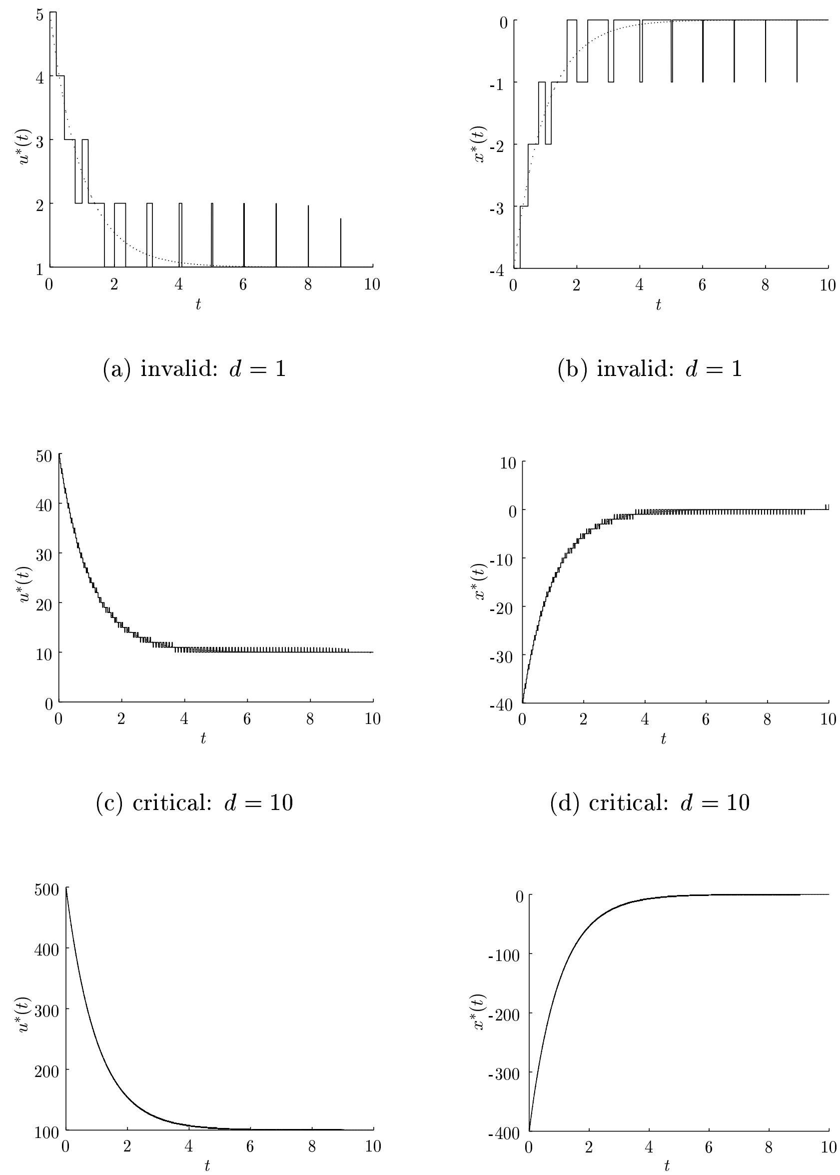



Figure D.11: Optimal control path $u^*(t)$ and optimal state path $x^*(t)$ with free control ($u$) for demand rates $d$ where the flow model is (a, b) invalid, (c, d) critical, and (e, f) valid and with negative initial states $x_0$.



(a) invalid: $d = 1$

(b) invalid: $d = 1$

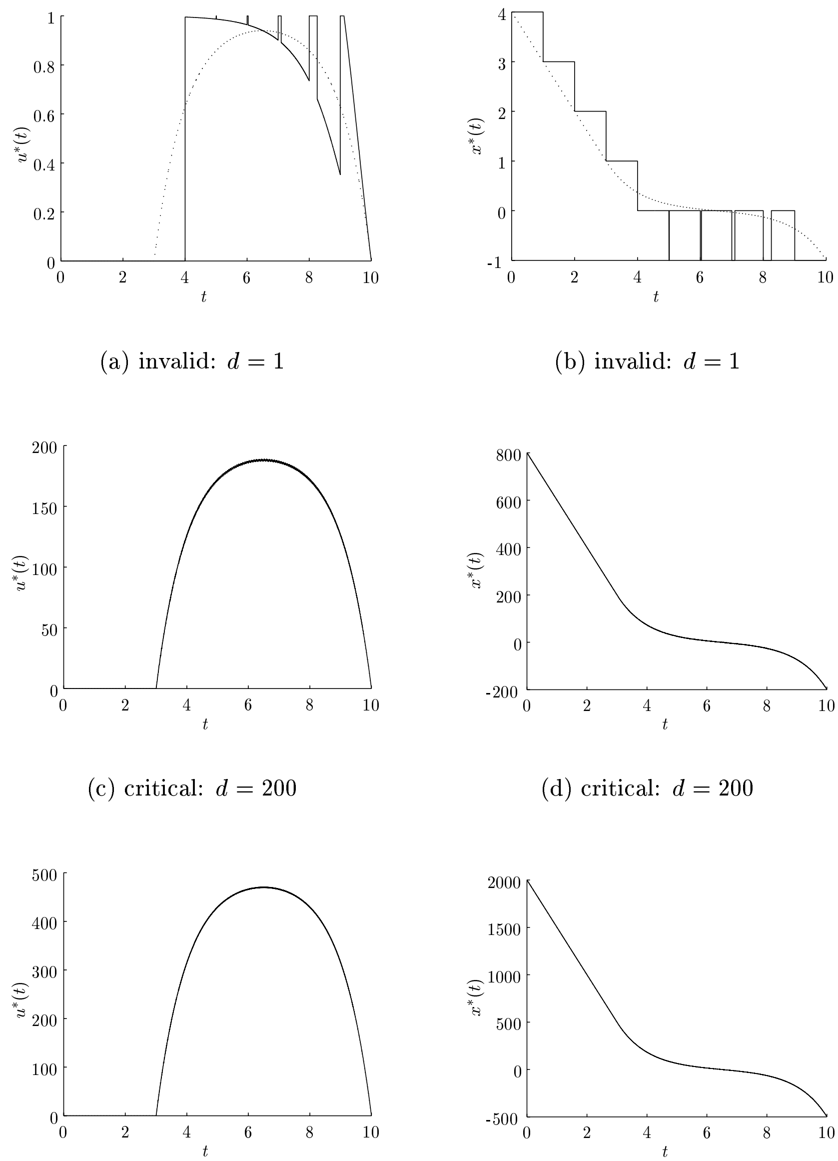
(c) critical: $d = 10$

(d) critical: $d = 10$

(e) valid: $d = 100$

(f) valid: $d = 100$

Figure D.12: Optimal control paths $u^*(t)$ and optimal state paths $x^*(t)$ with free control $(u - d)$ for demand rates $d$ where the flow model is (a, b) invalid, (c, d) critical, and (e, f) valid and with negative initial states $x_0$.




(a) invalid: $d = 1$

(b) invalid: $d = 1$

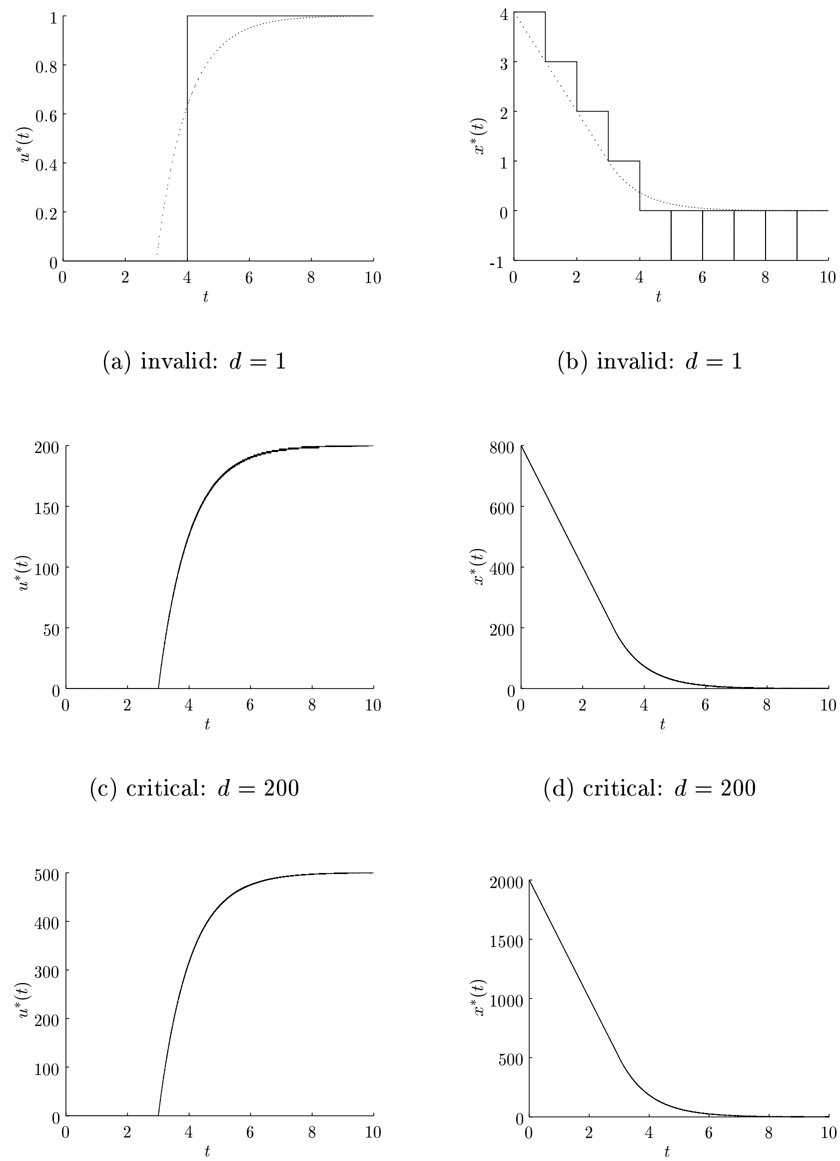
(c) critical: $d = 200$

(d) critical: $d = 200$

(e) valid: $d = 500$

(f) valid: $d = 500$

Figure D.13: Optimal control paths $u^*(t)$ and optimal state paths $x^*(t)$ with limited control $(u)$ for demand rates $d$ where the flow model is (a, b) invalid, (c, d) critical, and (e, f) valid and with positive initial states $x_0$.




(a) invalid: $d = 1$

(b) invalid: $d = 1$

(c) critical: $d = 200$

(d) critical: $d = 200$

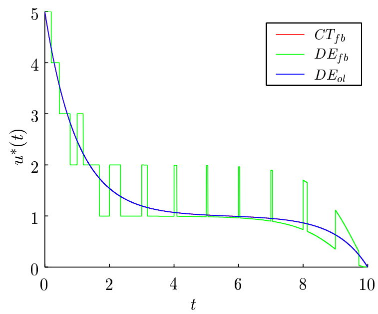
(e) valid: $d = 500$

(f) valid: $d = 500$

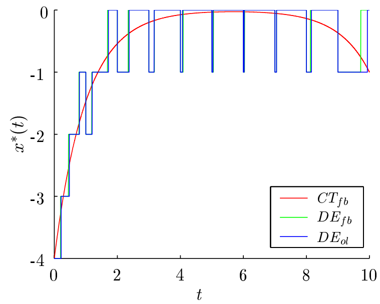
Figure D.14: Optimal control path $u^*(t)$ and optimal state path $x^*(t)$ with limited control $(u - d)$ for demand rates $d$ where the flow model is (a, b) invalid, (c, d) critical, and (e, f) valid and with positive initial states $x_0$.

## D.3 Open loop and feedback control

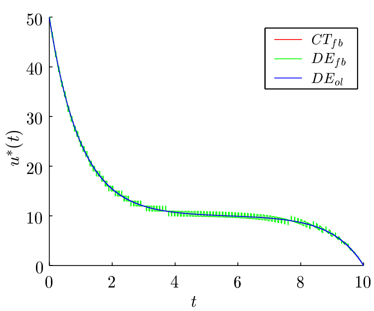
As mentioned in Chapter 2, feedback control takes unexpected events, small disturbances, or miscalculations due to uncertain parameters into consideration by means of the observed variables. The discrete-event flow of parts through the system leads to unexpected events that are not expected by the continuous-time controller. Therefore, only feedback control laws have been applied in the previous simulations. In this section, the effect of open loop and feedback control laws on the system behavior is investigated. In the deterministic simulation model, the control signal is sampled. In the remaining simulations, the sample time $t_s$ is set to 1/1000. Sampling is done at such a high rate, that the open loop control laws also achieve sufficiently well results, see Figure D.15. Here, the feedback paths are printed in green, and the open loop paths are printed in blue. For reference the continuous-time paths are printed in red. The simulations in this section are performed for a planning horizon $t_f$ of 10, with cost parameters $q$ and $r$ both $\frac{1}{2}$, the capacity $\alpha$ set equal to the demand rate $d$, and the initial state four times the demand. For the free control problems negative initial states are considered, and for the limited control problems positive initial states.



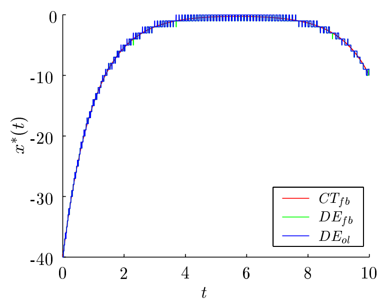

(a) invalid: $d = 1$



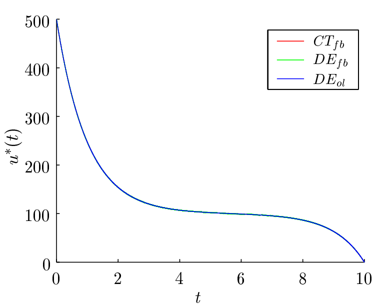

(b) invalid: $d = 1$



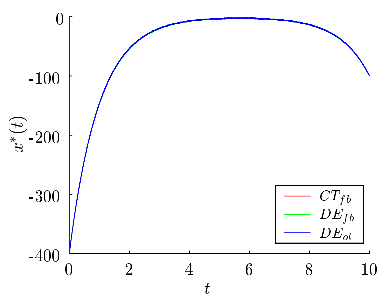

(c) critical: $d = 10$




(d) critical: $d = 10$




(e) valid: $d = 100$




(f) valid: $d = 100$

Figure D.15: Open loop and feedback control in the deterministic case with free control ($u$) for demand rates $d$ where the flow model is (a, b) invalid, (c, d) critical, and (e, f) valid and with negative initial states $x_0$.

To exploit the true benefit of feedback control, the controlled system is simulated with uncertain exogenous variables. Here, the constant demand rate is considered to be a stochastic variable. This is modeled by introducing variability to the inter request time, see the stochastic simulation model as presented in Appendix C.

Figures D.16 up to and including D.20 show the results for the free and limited control problems. The system behavior is simulated for demand rates for which the validity of the flow model is considered to be critical. Exception is Figure D.17, where the demand rate $d$ is set to valid value for the free control problem $(u)$. Three values of inter request time variability are considered:

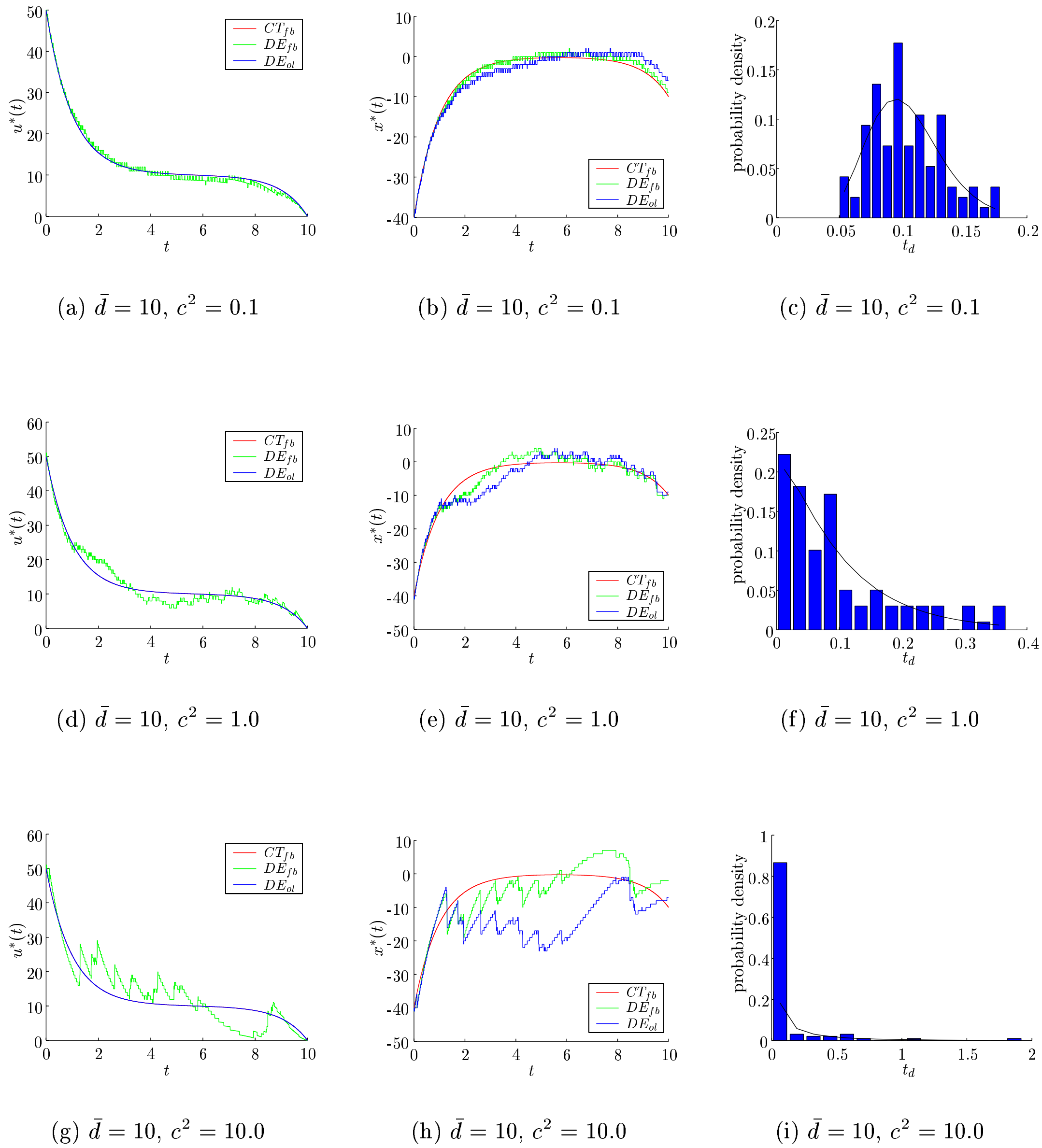
- lowly variable; $c^2 = 0.1$,
- moderately variable; $c^2 = 1.0$, and
- highly variable; $c^2 = 10$.

The resulting probability distributions of the inter request time are shown at the right side of the figure. Control and state paths are plotted for:

- the continuous-time model with feedback control; $CT_{fb}$,
- the discrete-time model with feedback control; $DE_{fb}$, and
- the discrete-time model with open loop control; $DE_{ol}$.

No variability is taken into account in the $CT_{fb}$-model. These control and state paths are only plotted as a reference.

All figures show the corrupting influence of variability. Because the open loop controller does not anticipate on the changes in the state path, these changes are not compensated. Due to its observations, the feedback controller adjusts its actions to compensate the external changes to the state path. Figure D.17 shows that for higher demand rates the effect of inter request time variability on the system behavior decreases relatively.

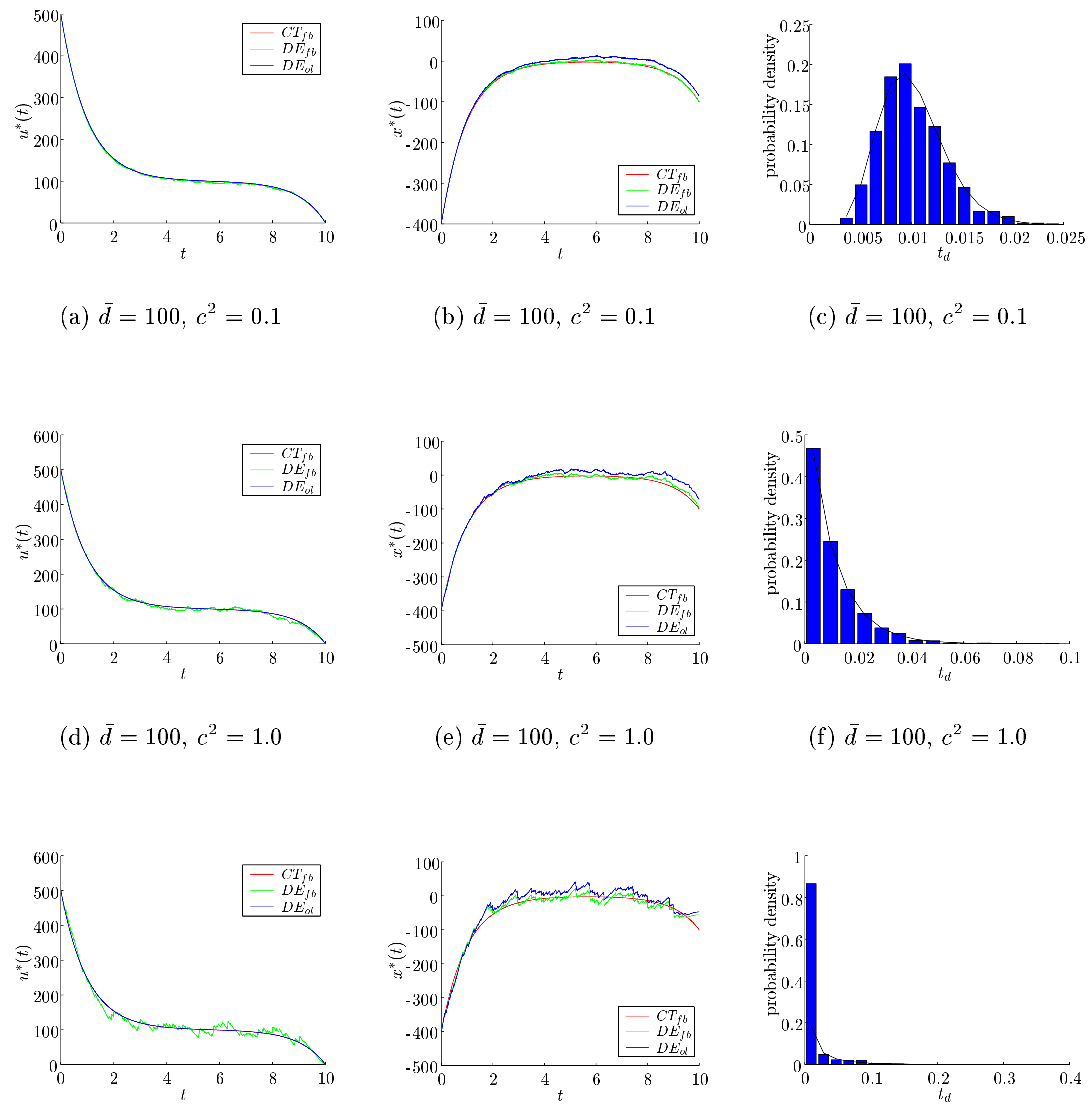



(a) $\bar{d} = 10$, $c^2 = 0.1$ (b) $\bar{d} = 10$, $c^2 = 0.1$ (c) $\bar{d} = 10$, $c^2 = 0.1$

(d) $\bar{d} = 10$, $c^2 = 1.0$ (e) $\bar{d} = 10$, $c^2 = 1.0$ (f) $\bar{d} = 10$, $c^2 = 1.0$

(g) $\bar{d} = 10$, $c^2 = 10.0$ (h) $\bar{d} = 10$, $c^2 = 10.0$ (i) $\bar{d} = 10$, $c^2 = 10.0$

Figure D.16: Open loop and feedback control in the stochastic case with free control ($u$) and negative initial state $x_0$. Control and state paths are shown for several values of inter request time variability: (a, b, c) lowly variable, (d, e, f) moderately variable, and (g, h, i) highly variable. Validity of the flow model is critical with an average demand rate $\bar{d} = 10$.

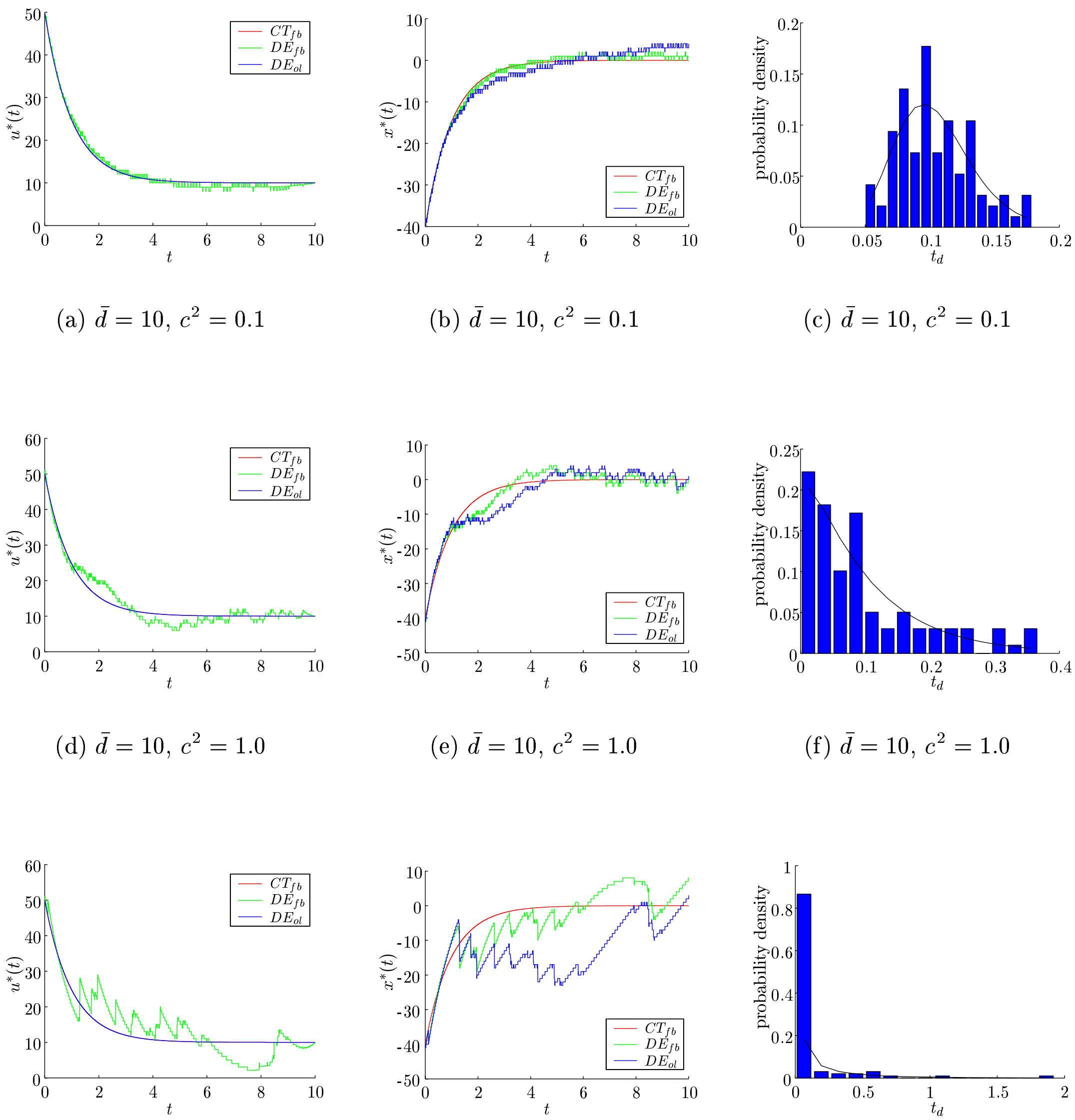



(a) $\bar{d} = 100,\ c^2 = 0.1$ (b) $\bar{d} = 100,\ c^2 = 0.1$ (c) $\bar{d} = 100,\ c^2 = 0.1$

(d) $\bar{d} = 100,\ c^2 = 1.0$ (e) $\bar{d} = 100,\ c^2 = 1.0$ (f) $\bar{d} = 100,\ c^2 = 1.0$

(g) $\bar{d} = 100,\ c^2 = 10.0$ (h) $\bar{d} = 100,\ c^2 = 10.0$ (i) $\bar{d} = 100,\ c^2 = 10.0$

Figure D.17: Open loop and feedback control in the stochastic case with free control ($u$) and negative initial state $x_0$. Control and state paths are shown for several values of inter request time variability: (a, b, c) lowly variable, (d, e, f) moderately variable, and (g, h, i) highly variable. The average demand rate $\bar{d} = 100$ gives a valid flow model.

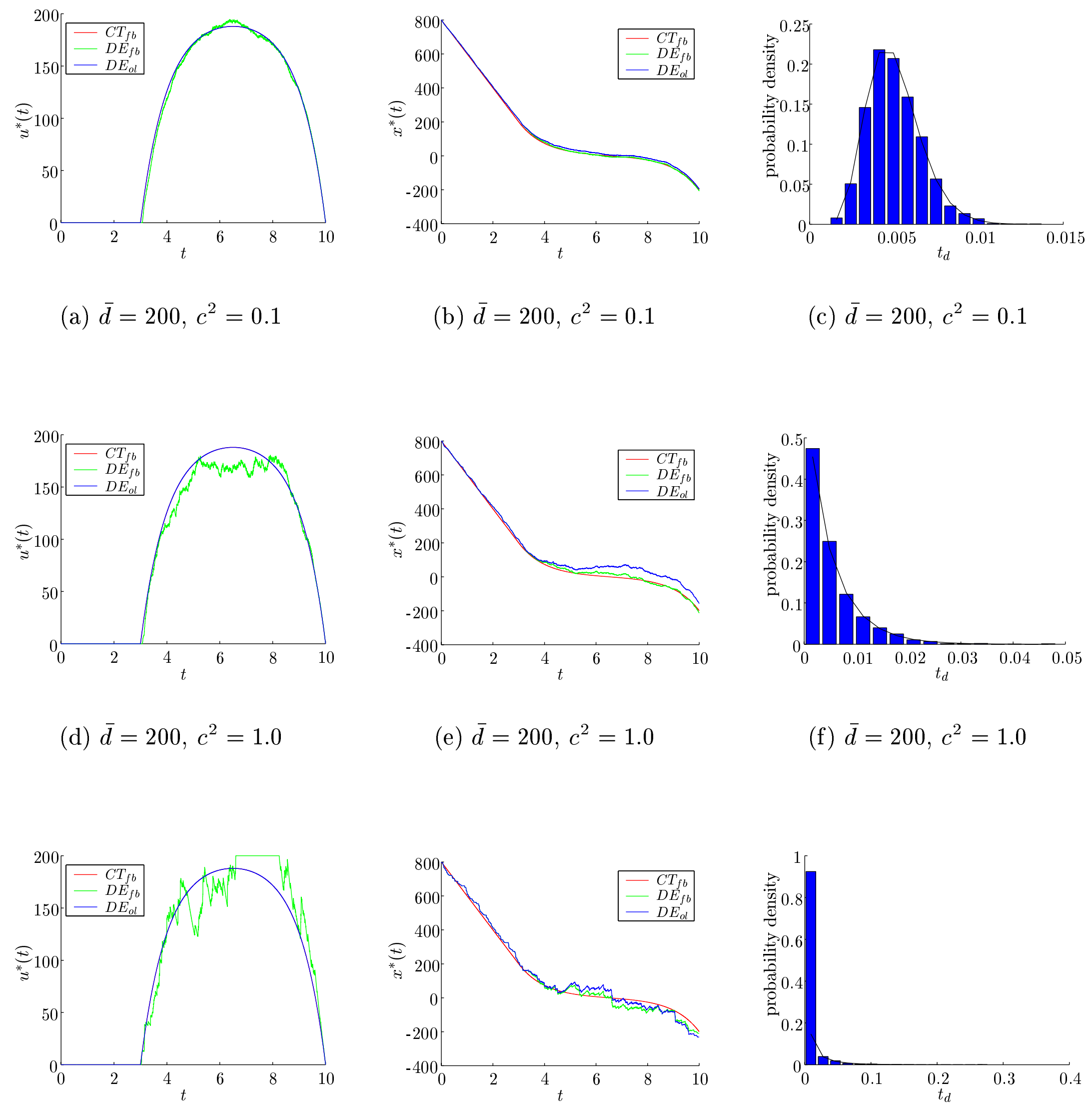



(a) $\bar{d} = 10$, $c^2 = 0.1$ (b) $\bar{d} = 10$, $c^2 = 0.1$ (c) $\bar{d} = 10$, $c^2 = 0.1$

(d) $\bar{d} = 10$, $c^2 = 1.0$ (e) $\bar{d} = 10$, $c^2 = 1.0$ (f) $\bar{d} = 10$, $c^2 = 1.0$

(g) $\bar{d} = 10$, $c^2 = 10.0$ (h) $\bar{d} = 10$, $c^2 = 10.0$ (i) $\bar{d} = 10$, $c^2 = 10.0$

Figure D.18: Open loop and feedback control in the stochastic case with free control $(u - d)$ and negative initial state $x_0$. Control and state paths are shown for several values of inter request time variability: (a, b, c) lowly variable, (d, e, f) moderately variable, and (g, h, i) highly variable. Validity of the flow model is critical with an average demand rate $\bar{d} = 10$.

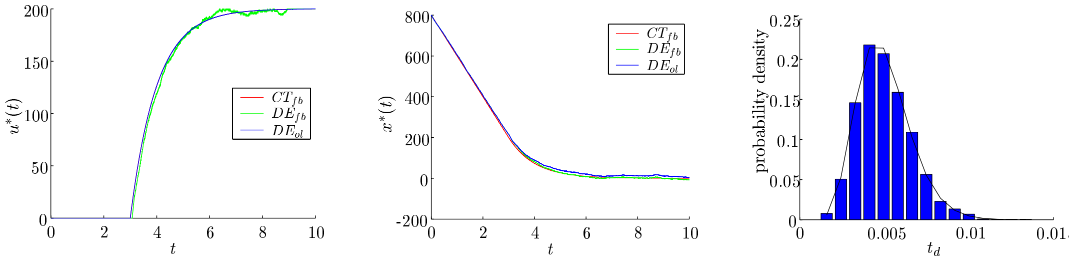



(a) $\bar{d} = 200,\ c^2 = 0.1$ (b) $\bar{d} = 200,\ c^2 = 0.1$ (c) $\bar{d} = 200,\ c^2 = 0.1$

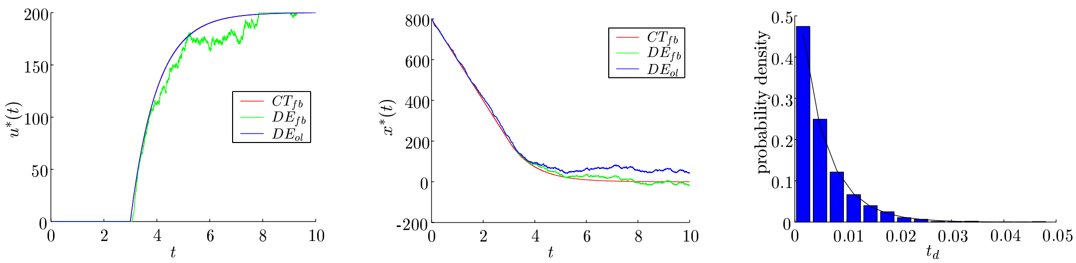
(d) $\bar{d} = 200,\ c^2 = 1.0$ (e) $\bar{d} = 200,\ c^2 = 1.0$ (f) $\bar{d} = 200,\ c^2 = 1.0$

(g) $\bar{d} = 200,\ c^2 = 10.0$ (h) $\bar{d} = 200,\ c^2 = 10.0$ (i) $\bar{d} = 200,\ c^2 = 10.0$

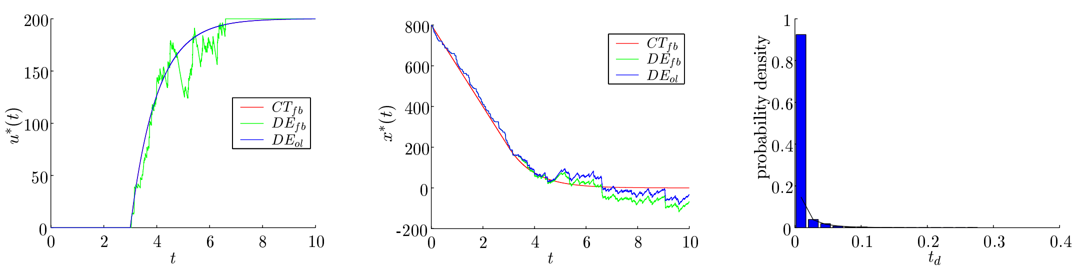
Figure D.19: Open loop and feedback control in the stochastic case with limited control ($u$) and positive initial state $x_0$. Control and state paths are shown for several values of inter request time variability: (a, b, c) lowly variable, (d, e, f) moderately variable, and (g, h, i) highly variable. Validity of the flow model is critical with an average demand rate $\bar{d} = 200$.




(a) $\bar{d} = 200,\ c^2 = 0.1$ (b) $\bar{d} = 200,\ c^2 = 0.1$ (c) $\bar{d} = 200,\ c^2 = 0.1$




(d) $\bar{d} = 200,\ c^2 = 1.0$ (e) $\bar{d} = 200,\ c^2 = 1.0$ (f) $\bar{d} = 200,\ c^2 = 1.0$




(g) $\bar{d} = 200,\ c^2 = 10.0$ (h) $\bar{d} = 200,\ c^2 = 10.0$ (i) $\bar{d} = 200,\ c^2 = 10.0$

Figure D.20: Open loop and feedback control in the stochastic case with limited control $(u - d)$ and positive initial state $x_0$. Control and state paths are shown for several values of inter request time variability: (a, b, c) lowly variable, (d, e, f) moderately variable, and (g, h, i) highly variable. Validity of the flow model is critical with an average demand rate $\bar{d} = 200$.