Observer based tracking controllers for a mobile car

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Abstract

In this paper the problem of global trajectory tracking control for the chained form system and the kinematic model of a mobile robot is considered. We develop an observer based controller using a cascaded system approach, resulting in $\mathcal{K}$-exponential convergence of tracking error. Simulations are provided to illustrate theoretical results.

keywords: mobile robot, chained form system, observer, trajectory tracking, cascaded system.

1 Introduction

Modeling and control of *nonholonomic systems* received significant attention during recent years. One of the problems connected to the subject of control is that of following a given reference trajectory (a trajectory tracking problem).

Several control schemes considering this issue have been proposed, based on different approaches. In [4], [8], [7] the linearization of the error model was used. In [1] the idea of dynamic feedback linearization was proposed. Global tracking results using integrator backstepping technique may be found in [3] and [2]. In [9] linear controllers based on cascaded systems approach have been designed.

The controllers mentioned above ensure local or global tracking of a reference trajectory, whereas most of them assume that all state variables of the considered system are directly measurable. However this is an ideal case, which can hardly be met in physical systems. In practical cases some state variables cannot be measured or the cost of measuring these variables is so high that one usually avoids using expensive sensors.

In this paper we design observers for a unicycle model and for an equivalent three-dimensional chained form system. For both systems we propose controllers which ensure $\mathcal{K}$-exponential convergence to zero of trajectory tracking errors in the closed-loop systems.

The full state observer for a three-dimensional chained form system is defined as a copy of the system being observed with additional terms which assure the convergence of observation error. Then using a lemma from adaptive control we show the exponential convergence of the observer error. In the design of the reduced order observer we use a result for linear systems, where the observer estimates a new variable defined as a linear combination of unknown and known variables. In our definition of the new variable we use a time-dependent term instead of the constant gains used customarily in linear systems. The proof of convergence is based on the cascaded systems approach.

The difficulty encountered in the case of the unicycle model is observing the orientation angle which appears in the position dynamics in the form of nonlinear terms. To avoid this difficulty, we define a four-dimensional system equivalent to the basic, three-dimensional model. Then we find an observer being a copy of the new system and use a Lyapunov function to prove the global asymptotic convergence of the observer error.

With regard to the problems mentioned in the case of full state observer we have resigned from searching for an observer of the orientation angle. Instead, we modify the controller given in [9] by replacing the angle with a certain its function and design the observer to estimate this function. Finally we demonstrate that the error in the closed-loop system with the new controller both using the original angle and the observer output tends to 0 $\mathcal{K}$-exponentially.

The paper is composed in the following way. Further in this section we present basic concepts and lemmas used in the paper and formulate the main problem. In section 2 the problem of observer design for a chain-form system is considered. In section 3 we design observers for the model of unicycle. Section 4 contains conclusions referring to the obtained results.

1.1 Preliminary results

Barbălat's Lemma. *If $f(t)$ is a differentiable function on $I\!R$, with $\lim_{t\to\infty} f(t) = m < \infty$ and if $\dot{f}(t)$ is uniformly continuous, then $\lim_{t\to\infty} \dot{f}(t) = 0$.*

Lemma 1. *As a consequence of Barbălat's lemma, if a function $f(t)$ is lower bounded on $I\!R$ and $\dot{f}(t)$ is negative semi-definite and uniformly continuous in t then $\lim_{t\to\infty} \dot{f} = 0$.*

Definition 1. *A continuous function $\phi : [0,a) \to [0,\infty)$ is said to be a class $\mathcal{K}$ function if it is strictly increasing and $\phi(0) = 0$.*

Definition 2. *A continuous function $\beta : [0,a) \times [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r,s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r,s)$ is decreasing with respect to s and $\beta(r,s) \to 0$ as $s \to \infty$.*

Consider the system

$$\dot{x} = f(t,x) \qquad \forall t \geqslant 0 \quad f(t,0) = 0, \tag{1}$$

where $f(t,x)$ is piecewise continuous in t and locally Lipschitz in x.

Definition 3. *A function $f : I\!R^+ \times I\!R^n \to I\!R$ is said to be* positive definite *if*

1. *if is continuous respect to t and x*
2. *$\forall t \in I\!R^+ \quad f(t,x) \geqslant f_0(x)$ where $f_0(x)$ is a time invariant positive definite function i.e. $\forall x \neq 0 \quad f_0(x) > 0$ and $f_0(0) = 0$.*

$f(t,x)$ is positive semidefinite *if $\forall x \quad f_0(x) \geqslant 0$. The function $f(t,x)$ is called* negative definite *or* negative semidefinite *when $-f(t,x)$ is, respectively, positive definite or positive semidefinite.*

Lemma 2 (See [3])**.** *If in the scalar system*

$$\dot{x} = -cx + p(t),$$

$c > 0$ and $p(t)$ is bounded and uniformly continuous, and if for any initial condition $x(t)$ is bounded and converges to 0 as $t \to \infty$ then

$$\lim_{t\to\infty} p(t) = 0.$$

Definition 4 (Persistence of excitation)**.** *A function $\boldsymbol{w}(t)$, $w : I\!R \to I\!R^n$ is said to be persistently exciting if there exist positive constants δ and k such that*

$$\int_t^{t+\delta} w(\tau)w^T(\tau)d\tau \geqslant kI_n, \qquad \forall t \geqslant t_0 \tag{2}$$

Lemma 3. *(cf. [5]). Consider the system*

$$\begin{bmatrix} \dot{e} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} A_m & b_m w^T(t) \\ -\gamma w(t) c_m^T & 0 \end{bmatrix} \begin{bmatrix} e \\ \phi \end{bmatrix} \tag{3}$$

where $e \in I\!R^n$, $\phi \in I\!R^m$, A_m is a $n \times n$ Hurwitz matrix, b_m and c_m are $n \times 1$ vectors $\gamma > 0$. Assume that $M(s) \triangleq c_m^T(sI - A_m)^{-1}b_m$ is a strictly positive real transfer function, i.e. $Re[M(i\omega)] > 0$ for all $\omega \in I\!R$. Then $\phi(t)$ is bounded and

$$\lim_{t\to\infty} e(t) = 0.$$

If in addition $\boldsymbol{w}(t)$ and $\dot{\boldsymbol{w}}(t)$ are bounded for all $t \geqslant t_0$, and $\boldsymbol{w}(t)$ is persistently exciting then the system (3) is globally exponentially stable.

Definition 5. *The system (1) is* uniformly stable *if for each $\varepsilon > 0$ there exist $\delta > 0$, independent of t_0, such that*

$$\forall t \geqslant t_0 \geqslant 0 \qquad \|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon$$

Definition 6. *The system (1) is* globally uniformly asymptotically stable (GUAS) *if it is uniformly stable and there exists a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ such that for all initial states $x(t_0)$*

$$\forall t \geqslant t_0 \geqslant 0 \qquad \|x(t)\| \leqslant \beta(\|x(t_0)\|, t - t_0)$$

Definition 7. *The system (1) is* globally exponentially stable (GES) *if there exist positive constants K and γ such that the solution $x(t)$ satisfies*

$$\forall t \geqslant t_0 \qquad \|x(t)\| \leqslant \|x(t_0)\| K e^{-\gamma(t-t_0)}$$

Definition 8. *The system (1) is said to be* globally $\mathcal{K}$-exponentially stable *if there exist $\gamma > 0$ and a class $\mathcal{K}$ function $k(\cdot)$ such that the solution $x(t)$ satisfies*

$$\forall t \geqslant t_0 \qquad \|x(t)\| \leqslant k(\|x(t_0)\|) e^{-\gamma(t-t_0)}$$

Theorem 4 (Converse Lyapunov theorem cf. [5])**.** *Consider the system (1). Assume that the equilibrium $x(t) = 0$ is globally exponentially stable. Then there exists a class C^1 function $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ that satisfies*

$$\begin{aligned} c_1\|x\|^2 \leqslant V(t,x) &\leqslant c_2\|x\|^2 \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) &\leqslant -c_3\|x\|^2 \\ \left\| \frac{\partial V}{\partial x} \right\| &\leqslant c_4\|x\|^2 \end{aligned}$$

for some positive constants c_1, c_2, c_3 and c_4.

Theorem 5 (Cascaded systems, see [10])**.** *Consider the system*

$$\begin{aligned} \Sigma_1 &: \ \dot{x} = f_1(t,x) + g(t,x,y)y \\ \Sigma_2 &: \ \dot{y} = f_2(t,y) \end{aligned} \tag{4}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $f_1(t,x)$ is continuously differentiable in (t,x) and $f_2(t,y)$, $g(t,x,y)$ are continuous in their arguments, and locally Lipschitz in y and (x,y), respectively.

We can view the system (4) as the system $\dot{x} = f_1(t,x)$ that is perturbed by the state of the system $\dot{y} = f_2(t,y)$. The cascaded system (4) is GUAS if the following three assumptions hold:

1. For Σ_1: *the system $\dot{x} = f_1(t,x)$ is GUAS and there exists a continuously differentiable function $V(t,x) : \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n \to \mathbb{R}$ that satisfies*

$$\begin{aligned} W(x) &\leqslant V(t,x), \\ \tfrac{\partial V}{\partial t} + \tfrac{\partial V}{\partial x} \cdot f_1(t,x) &\leqslant 0, \quad \forall \|x\| \geqslant \eta, \\ \left\| \tfrac{\partial V}{\partial x} \right\| \|x\| &\leqslant cV(t,x), \quad \forall \|x\| \geqslant \eta, \end{aligned}$$

   where $W(x)$ is a positive definite proper function, $c > 0$ and $\eta > 0$ are constants, $\|\cdot\|$ means Euclidean norm of a vector,

2. For the interconnection: *the function $g(t,x,y)$ satisfies for all $t \geqslant t_0$:*

$$\|g(t,x,y)\| \leqslant \theta_1(\|y\|) + \theta_2(\|y\|)\|x\|,$$

   where $\theta_1, \theta_2 : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ are continuous functions, *3.* For Σ_2: *the system $\dot{y} = f_2(t, y)$ is GUAS and for all $t_0 \geqslant 0$:*

$$\int_{t_0}^{\infty} \|y(t, t_0, y(t_0))\| dt \leqslant \kappa(\|y(t_0)\|),$$

where the function $\kappa(\cdot)$ is a class $\mathcal{K}$ function,

Lemma 6. *If the assumptions of Theorem 5 hold and in addition $\dot{x} = f_1(t, x)$ and $\dot{y} = f_2(t, y)$ are globally $\mathcal{K}$-exponentially stable, then the cascaded system (4) is globally $\mathcal{K}$-exponentially stable.*

Remark 1. When the system $\dot{y} = f_2(t, y)$ is $\mathcal{K}$-exponentially stable

$$\int_{t_0}^{\infty} \|y(t)\| dt \leqslant \int_{t_0}^{\infty} k(\|y_0\|) e^{-\gamma t} dt = \frac{1}{\gamma} k(\|y_0\|) e^{-\gamma t_0}$$

then the assumption on Σ_2 of Theorem 5 is satisfied.

Remark 2. If the system $\dot{x} = f_1(t, x)$ is GES the existence of a proper function $V(t, x)$ in Theorem 5 is guaranteed by Theorem 4.

1.2 Problem formulation

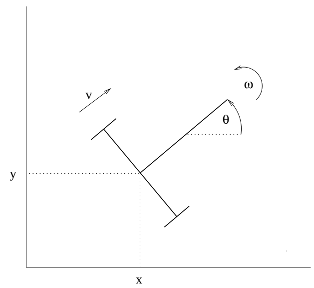


Figure 1: The mobile car

A kinematic model of a mobile car with two degrees of freedom is given by:

$$\begin{aligned} \dot{x} &= v \, \cos\theta \\ \dot{y} &= v \, \sin\theta\,, \\ \dot{\theta} &= \omega \end{aligned} \tag{5}$$

where (x, y) are the midpoint of the rear axis coordinates, θ is the heading direction, v is the forward velocity of the car and ω is the angular velocity (see Figure 1).

With the following change of coordinates and the feedback transformation

$$\begin{cases} x_1 = \theta \\ x_2 = x\cos\theta + y\sin\theta \\ x_3 = x\sin\theta - y\cos\theta \end{cases} \tag{6a}$$

$$\begin{cases} u_1 = \omega \\ u_2 = v - \omega x_3 \end{cases} \tag{6b}$$

the system (5) is transformed into the chained form:

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= u_1 x_2 \end{aligned} \tag{7}$$

The problem studied in the sequel is to find controllers for systems (5) and (7) which guarantee tracking a given reference trajectory when one of the state variables cannot be measured.

With this purpose we first design an observer estimating the unknown coordinates or the whole state vector. Next, we modify one of the known controllers replacing unmeasurable states with their estimates and determine the conditions under which the combination of the observer and the controller tracks the reference trajectory asymptotically.

2 Tracking a chained form system

2.1 Controller

A reference trajectory for the chained form system (7) is given by the following equations

$$\begin{aligned} \dot{x}_{1r} &= u_{1r} \\ \dot{x}_{2r} &= u_{2r} \\ \dot{x}_{3r} &= u_{1r}\, x_{2r} \end{aligned} \tag{8}$$

We define the tracking error as $\boldsymbol{x_e} = \boldsymbol{x} - \boldsymbol{x_r}$ with the dynamics

$$\begin{aligned} \dot{x}_{1e} &= u_1 - u_{1r} \\ \dot{x}_{2e} &= u_2 - u_{2r} \\ \dot{x}_{3e} &= u_1\, x_2 - u_{1r}\, x_{2r} \end{aligned} \tag{9}$$

Proposition 1. *Consider the system (9) with the state feedback*

$$\begin{aligned} u_1 &= u_{1r} - c_1\, x_{1e} \\ u_2 &= u_{2r} - c_2\, x_{2e} - c_3\, u_1\, x_{3e} \end{aligned} \tag{10}$$

If c_1, c_2, c_3 are positive constants, x_{2r} is bounded, and if u_{1r} is persistently exciting then the closed-loop system (9, 10) is globally $\mathcal{K}$-exponentially stable

Proof. From (9) and (10) we obtain the following closed loop dynamics

$$\dot{x}_{1e} = -c_1\, x_{1e} \tag{11a}$$
$$\dot{x}_{2e} = -c_2\, x_{2e} - c_3\, u_1\, x_{3e} \tag{11b}$$
$$\begin{aligned} \dot{x}_{3e} &= u_1\, x_2 - x_{2r}\, u_1 + x_{2r}\, u_1 - u_{1r}\, x_{2r} = \\ &= u_1\, x_{2e} + (u_1 - u_{1r})x_{2r} = u_1\, x_{2e} - x_{2r}\, c_1\, x_{1e} \end{aligned} \tag{11c}$$

Here the subsystem (11a) is exponentially stable with the solution $x_{1e}(t) = x_{1e}(0)e^{-c_1 t}$. Since x_{1e} tends to 0 and thus is not persistently exciting and since we have assumed that u_{1r} is persistently exciting, it follows that u_1 defined as in (10) satisfies the persistence of excitation condition.

Denote $\xi = [x_{2e}\ x_{3e}]^T$, then the subsystems (11b) and (11c) can be put in the form

$$\dot{\xi} = \begin{bmatrix} \dot{x}_{2e} \\ \dot{x}_{3e} \end{bmatrix} = \begin{bmatrix} -c_2 & -c_3 u_1 \\ u_1 & 0 \end{bmatrix} \begin{bmatrix} x_{2e} \\ x_{3e} \end{bmatrix} - \begin{bmatrix} 0 \\ c_1 x_{2r} \end{bmatrix} x_{1e} = \mathbf{F}(t)\xi + \mathbf{G}(t)x_{1e} \tag{12}$$

We can treat (12) as a system $\dot{\xi} = \mathbf{F}(t)\xi$ perturbed by the state of (11a). Since x_{1e} converges to 0 exponentially, it satisfies the integrability condition of Σ_2 of Lemma 5.

Then using Lemma 3 we find that system $\dot{\xi} = \mathbf{F}(t)\xi$ is globally exponentially stable. Hence from converse Lyapunov theory it follows that there exists a function V satisfying assumptions on Σ_1 of Theorem 5. The boundedness of x_{2r} guarantees that the interconnection assumption is also fulfilled.

Hence, using Theorem 6 we conclude that the system (9) is globally $\mathcal{K}$-exponentially stable. □

2.2 A full state observer for a chained form system

In this section a system in chained form (7) with two outputs is considered:

$$\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= u_1 x_2 \\
y_1 &= x_1 \\
y_2 &= x_3.
\end{aligned} \tag{13}$$

The goal is to find an observer for the system (13). The idea is to find an observer defined as a copy of the system (13) with an additional correction term f. We try to determine f such that the state of the observer converges to the state of the system

$$\begin{aligned}
\dot{\hat{x}}_1 &= u_1 + f_1(\hat{x}_1, \hat{x}_2, \hat{x}_3, u_1, u_2, y_1, y_2) \\
\dot{\hat{x}}_2 &= u_2 + f_2(\hat{x}_1, \hat{x}_2, \hat{x}_3, u_1, u_2, y_1, y_2) \\
\dot{\hat{x}}_3 &= u_1 \hat{x}_2 + f_3(\hat{x}_1, \hat{x}_2, \hat{x}_3, u_1, u_2, y_1, y_2),
\end{aligned} \tag{14}$$

where f_1, f_2, f_3 are to be determined.

If we define the observer error as $\tilde{\boldsymbol{x}} = \boldsymbol{x} - \hat{\boldsymbol{x}}$ then the error dynamics become

$$\begin{aligned}
\dot{\tilde{x}}_1 &= -f_1(\hat{x}_1, \hat{x}_2, \hat{x}_3, u_1, u_2, y_1, y_2) \\
\dot{\tilde{x}}_2 &= -f_2(\hat{x}_1, \hat{x}_2, \hat{x}_3, u_1, u_2, y_1, y_2) \\
\dot{\tilde{x}}_3 &= u_1 \tilde{x}_2 - f_3(\hat{x}_1, \hat{x}_2, \hat{x}_3, u_1, u_2, y_1, y_2).
\end{aligned} \tag{15}$$

When the system (15) is globally asymptotically stable at 0, the estimated state $\hat{\boldsymbol{x}}$ tends to the real state $\boldsymbol{x}$.

Proposition 2. *Consider the system (15) with functions f_1, f_2 and f_3 defined as follows*

$$\begin{aligned}
f_1 &= a_1 \tilde{x}_1 \\
f_2 &= a_2 u_1 \tilde{x}_3 \\
f_3 &= a_3 \tilde{x}_3
\end{aligned} \tag{16}$$

where $a_1, a_2, a_3 > 0$ are tuning gains. Then, assuming that u_1 and $\dot{u}_1$ are bounded and u_1 is persistently exciting, the system (15) is globally exponentially stable at 0 and thus (14, 16) form an observer for the system (13).

Proof. We notice that system (15) with f defined as in (16) may be treated as two separate systems $\tilde{x}_1$ and $[\tilde{x}_2\ \tilde{x}_3]^T$, in which the subsystem $\dot{\tilde{x}}_1 = -a_1\ \tilde{x}_1$ is globally exponentially stable.

When we represent the second subsystem in the form

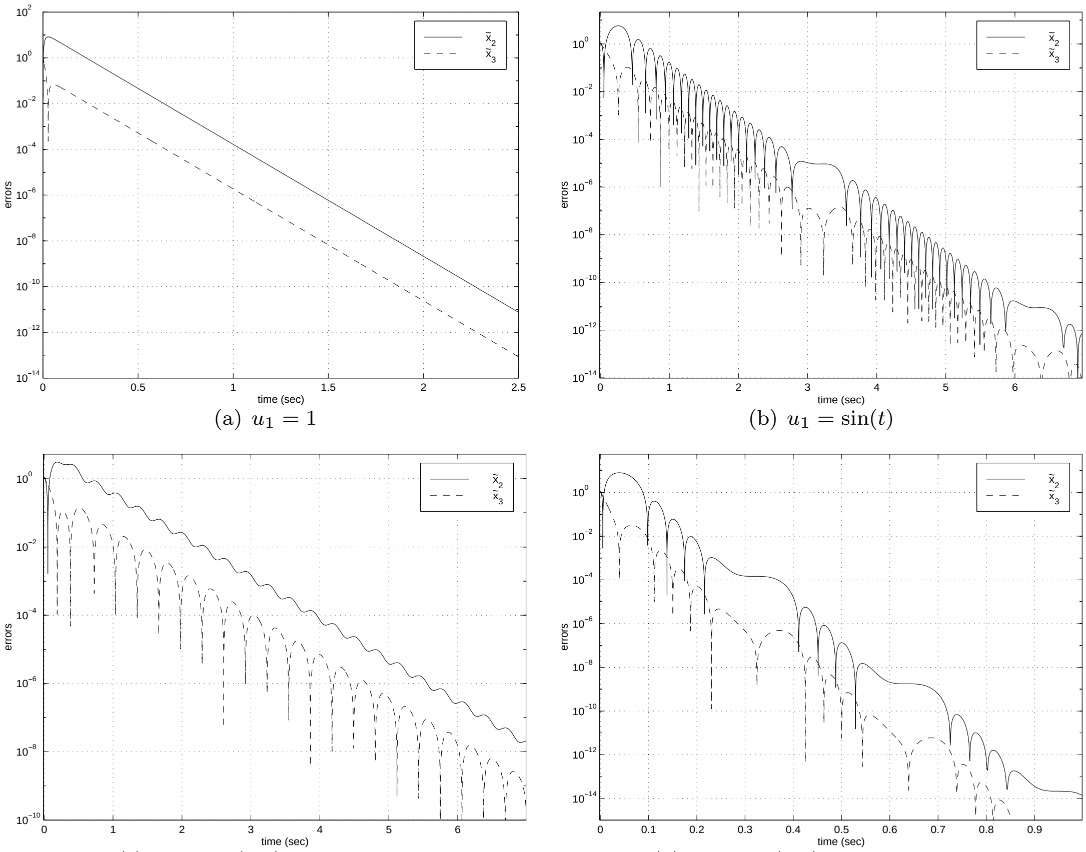
$$\begin{bmatrix} \dot{\tilde{x}}_3 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} -a_3 & u_1 \\ -a_2 u_1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_3 \\ \tilde{x}_2 \end{bmatrix} \tag{17}$$

hence, using Lemma 3, we find $M(i\omega) = (i\omega + a_3)^{-1}$ which has real part greater than 0 for all $\omega \in I\!R$. From the assumption that u_1 and $\dot{u}_1$ are bounded and u_1 is persistently exciting from Lemma 3 we conclude that the system (17) is globally exponentially stable. $\square$

2.3 Observer performance

The behaviour of the observer was tested using MATLAB. The figures contain time characteristics of $\tilde{x}_2$ and $\tilde{x}_3$ only, because $\tilde{x}_1$ is an independent variable with the solution $\tilde{x}_1(t) = \tilde{x}_1(0)\, e^{-a_1 t}$.

To show exponential convergence all graphs were plotted in semilogarithmic scale. Initial values of the errors were set at 1.



(a) $u_1 = 1$

(b) $u_1 = \sin(t)$

(c) $u_1 = \sin(10t)$, small gains

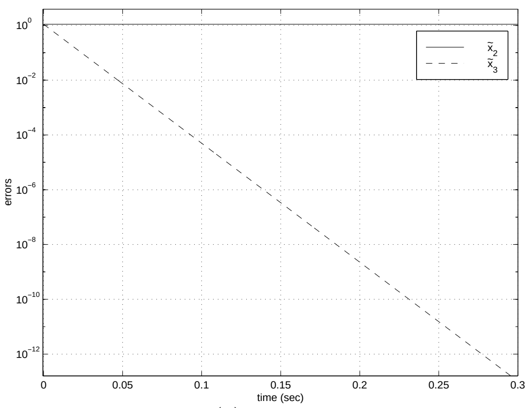
(d) $u_1 = \sin(10t)$, increased gains

Figure 2: Performance of the full state observer for a chained form system

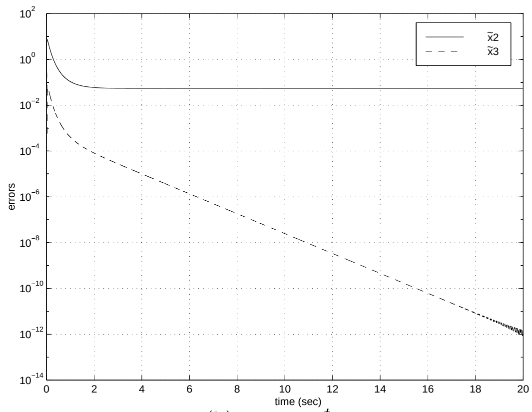
Figure 2 shows the errors of the observer for different inputs fulfilling the assumptions. Figure 2(a) presents time characteristics for constant control input $u_1 = 1$ with gains $a_2 = 1000$ and $a_3 = 100$. In Figure 2(b) errors for input $u_1 = \sin t$ and gains $a_2 = 1000$ and $a_3 = 10$ is shown.

Next figures present the influence of increasing gains. In Figure 2(c) $a_2 = 100$, $a_3 = 10$, comparing to Figure 2(d) where $a_2 = 10000$, $a_3 = 100$, one can notice smaller oscillations of the variable x_2, but also longer convergence time.

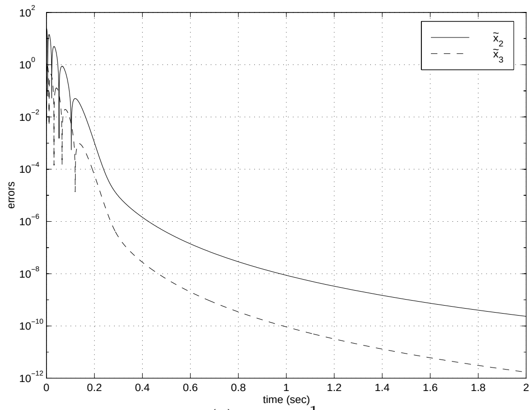
As one can see choosing proper values for gains improve performance of the observer remarkably.



(a) $u_1 = 0$



(b) $u_1 = e^{-t}$



(c) $u_1 = \frac{1}{\sqrt{t}}$

Figure 3: The observer (14, 16) in case when input u_1 is not persistantly exciting

To check if the assumptions on persistant excitation of u_1 are necessary simulations for inputs $u_1 = 0$, $u_1 = e^{-t}$ and $u_1 = 1/\sqrt{t}$ were made (see Figure 3).

In the cases of $u_1 = 0$ (Figure 3(a)) and $u_1 = e^{-t}$ (Figure 3(b)) when the $\int_0^\infty u_1^2(t)dt = c < \infty$ the error $\tilde{x}_2$ does not converge to 0. In the case of $u_1 = 1/\sqrt{t}$ (Figure 3(c)) the integral is infinite, but u_1 is not persistently exciting therefore $\tilde{x}_2$ tends to 0 although not exponentially.

2.4 A reduced order observer

The observer (14, 16) reconstructs all three state variables although two of them are known. However, it is possible to use a reduced order observer instead of the full state observer and estimate only the unknown variable x_2.

With the purpose of combining an observer with the controller (10), an observer estimating the closed loop system error (11) is proposed.

The error dynamics (11) can be put into the following form

$$\begin{aligned}
\dot{\boldsymbol{x}}_{\boldsymbol{e}} &= \begin{bmatrix}\dot{x}_{1e}\\ \dot{x}_{2e}\\ \dot{x}_{3e}\end{bmatrix} = \begin{bmatrix}-c_1 & 0 & 0\\ 0 & -c_2 & -c_3u_1\\ -c_1x_{2r} & u_1 & 0\end{bmatrix}\begin{bmatrix}x_{1e}\\ x_{2e}\\ x_{3e}\end{bmatrix} \qquad (18)\\
\boldsymbol{y} &= \begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}x_{1e}\\ x_{3e}\end{bmatrix}. \qquad (19)
\end{aligned}$$

A reduced order observer for this system comes from a result for linear systems (see [6]).

First we define a new variable z

$$z = x_{2e} - \begin{bmatrix}b_1(t) & b_2(t)\end{bmatrix}\begin{bmatrix}x_{1e}\\ x_{3e}\end{bmatrix} = x_{2e} - \mathbf{B}(t)\boldsymbol{y}, \qquad (20)$$

where $\mathbf{B} = [b_1(t)\ b_2(t)]$ and b_1, b_2 are function we choose to guarantee asymptotic stability of the observer. The derivative of z is given by

$$\begin{aligned}
\dot{z} &= -c_2x_{2e} - c_3u_1x_{3e} - \dot{\mathbf{B}}\boldsymbol{y} - \mathbf{B}\dot{\boldsymbol{y}} =\\
&= -(b_2u_1+c_2)(x_{2e}-\mathbf{B}\boldsymbol{y}) - (b_2u_1+c_2)\mathbf{B}\boldsymbol{y} - c_3u_1x_{3e} - \dot{\mathbf{B}}\boldsymbol{y} + b_1c_1x_{1e} + b_2c_1x_{2r}x_{1e} = \qquad (21)\\
&= -(b_2u_1+c_2)z - (b_2u_1+c_2)\mathbf{B}\boldsymbol{y} - c_3u_1x_{3e} - \dot{\mathbf{B}}\boldsymbol{y} + b_1c_1x_{1e} + b_2c_1x_{2r}x_{1e}.
\end{aligned}$$

An estimate of z can be calculated as

$$\dot{\hat{z}} = -(b_2u_1+c_2)\hat{z} - (b_2u_1+c_2)\mathbf{B}\boldsymbol{y} - c_3u_1x_{3e} - \dot{\mathbf{B}}\boldsymbol{y} + b_1c_1x_{1e} + b_2c_1x_{2r}x_{1e} \qquad (22)$$

With estimation error $\tilde{z} = z - \hat{z}$, the error dynamics become

$$\dot{\tilde{z}} = -(b_2u_1+c_2)\tilde{z} \qquad (23)$$

where c_2 is a positive constant. Then choosing $\mathbf{B}(t) = [0\ a_1u_1(t)]$ with $a_1 > 0$ and assuming that u_1 is bounded we find that the system (23) is globally exponentially stable. The value of $b_1(t)$ does not influence the convergence, however choosing $b_1(t) = 0$ simplifies the update law for $\hat{z}$.

Using this result we define an estimate of x_{2e}

$$\hat{x}_{2e} = \hat{z} + \mathbf{B}(t)\boldsymbol{y} \qquad (24)$$

From (24) and (22) with $\tilde{x}_{2e} = x_{2e} - \hat{x}_{2e}$ we obtain

$$\begin{aligned}
\dot{\hat{x}}_{2e} &= \dot{\hat{z}} + a_1u_1\dot{x}_{3e} + a_1\dot{u}_1x_{3e} =\\
&= a_1u_1^2\tilde{x}_{2e} - c_2\hat{x}_{2e} - c_3u_1x_{3e}\\
\dot{\tilde{x}}_{2e} &= -(a_1u_1^2+c_2)\tilde{x}_{2e}
\end{aligned}$$

2.5 The controller using the reduced order observer

In the controller (10) the input u_2 is a function of the unknown state x_{2e}. However, we are able to obtain an estimate $\hat{x}_{2e}$ for x_{2e} using the observer (22, 24). We prove that after replacing x_{2e} with $\hat{x}_{2e}$ in the control law, the trajectory tracking error still converges to 0.

Proposition 3. *Consider the system (9) with the following control law*

$$\begin{aligned}
u_1 &= u_{1r} - c_1\,x_{1e}\\
u_2 &= u_{2r} - c_2\,\hat{x}_{2e} - c_3\,u_1\,x_{3e}
\end{aligned} \qquad (25)$$

and $\hat{x}_{2e}$ defined by

$$\begin{aligned}
\hat{x}_{2e} &= \hat{z} + a_1u_1x_{3e} \qquad (26a)\\
\dot{\hat{z}} &= -(a_1u_1^2+c_2)\hat{z} - (a_1u_1^2+c_2)a_1u_1x_{3e} - c_3u_1x_{3e} - a_1\dot{u}_1x_{3e} - a_1c_1u_1x_{2r}x_{1e}. \qquad (26b)
\end{aligned}$$

If c_1, c_2, c_3 and a_1 are positive constants, x_{2r}, u_{1r} and $\dot{u}_{1r}$ are bounded and if u_{1r} is persistently exciting, then the combined system (9, 25 and 26) is globally $\mathcal{K}$-exponentially stable, hence the closed loop system trajectory converges to the reference trajectory.

Proof. From the dynamics (9) and the control law (25) we obtain the following equations

$$\dot{x}_{1e} = -c_1\, x_{1e} \tag{27a}$$
$$\dot{x}_{2e} = -c_2\, \hat{x}_{2e} - c_3\, u_1\, x_{3e} \tag{27b}$$
$$\dot{x}_{3e} = u_1\, x_{2e} - x_{2r}\, c_1\, x_{1e}, \tag{27c}$$

where (27b) can be rewritten as

$$\begin{aligned}\dot{x}_{2e} &= -c_2\, x_{2e} + c_2\, x_{2e} - c_2\, \hat{x}_{2e} - c_3\, u_1\, x_{3e} = \\ &= -c_2\, x_{2e} - c_3\, u_1\, x_{3e} + c_2\, \tilde{x}_{2e}.\end{aligned} \tag{28}$$

Since the x_{2e} update law depends on $\tilde{x}_{2e}$, we replace the observer equations (26) with $\tilde{x}_{2e}$ dynamics. Hence combining (27), (28) and (25) we obtain the following set of equations suitable to apply the theorem on cascaded systems.

$$\begin{aligned}\begin{bmatrix}\dot{x}_{2e}\\ \dot{x}_{3e}\end{bmatrix} &= \begin{bmatrix}-c_2 & -c_3 u_{1r}\\ u_{1r} & 0\end{bmatrix}\begin{bmatrix}x_{2e}\\ x_{3e}\end{bmatrix} + \begin{bmatrix}c_1 c_3 x_{3e} & c_2\\ -c_1 x_{2r} - c_1 x_{2e} & 0\end{bmatrix}\begin{bmatrix}x_{1e}\\ \tilde{x}_{2e}\end{bmatrix} = \\ &= \mathbf{F}(t)\begin{bmatrix}x_{2e}\\ x_{3e}\end{bmatrix} + \mathbf{G}(t, x_{2_e}, x_{3e})\begin{bmatrix}x_{1e}\\ \tilde{x}_{2e}\end{bmatrix}\end{aligned} \tag{29a}$$

$$\begin{bmatrix}\dot{x}_{1e}\\ \dot{\tilde{x}}_{2e}\end{bmatrix} = \begin{bmatrix}-c_1 x_{1e}\\ -(a_1 u_1^2 + c_2) x_{2e}\end{bmatrix} \tag{29b}$$

The solution of x_{1e} converges exponentially to 0 and is bounded. The solution of $\tilde{x}_{2e}$ dynamics is given by

$$\tilde{x}_{2e} = \tilde{x}_{2e}(0)\, e^{-\int_0^t \left(c_2 + a_1 u_1^2(\tau)\right) d\tau} \leqslant \tilde{x}_{2e}(0)\, e^{-c_2 t}.$$

Since constants c_2 and a_1 are assumed to be positive, we conclude that $\tilde{x}_{2e}$ converges exponentially to 0. Hence the subsystem (29b) is GES and it meets the assumptions on subsystem Σ_2 of Theorem 5.

According to Lemma 3, if u_1 and $\dot{u}_1$ are bounded and u_1 is persistently exciting, the subsystem (29a) is globally exponentially stable.

The assumption on linearity of the interconnection term is satisfied if x_{2r} is bounded. Hence from Lemma 6 we conclude that the system (29) is globally $\mathcal{K}$-exponentially stable. □

3 An observer for a mobile car

In this section we consider the problem of the system (5) following a reference trajectory (x_r, y_r, θ_r) (see [4]). Let us define the error coordinates in a local coordinate system attached to the mobile car

$$\begin{bmatrix}x_e\\ y_e\\ \theta_e\end{bmatrix} = \begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_r - x\\ y_r - y\\ \theta_r - \theta\end{bmatrix}.$$

The trajectory tracking error is described by the equations

$$\begin{aligned}\dot{x}_e &= \omega y_e + v_r \cos\theta_e - v\\ \dot{y}_e &= -\omega x_e + v_r \sin\theta_e\\ \dot{\theta}_e &= \omega_r - \omega.\end{aligned} \tag{30}$$

As a controller we use the result from [9].

$$\begin{aligned}v &= v_r + c_2 x_e\\ \omega &= \omega_r + c_1 \theta_e.\end{aligned} \tag{31}$$

Combining (30) and (31) we obtain the following closed loop error dynamics

$$\begin{aligned} \dot{x}_e &= \ (\omega_r + c_1\theta_e)y_e + v_r(\cos\theta_e - 1) - c_2 x_e \\ \dot{y}_e &= -(\omega_r + c_1\theta_e)x_e + v_r \sin\theta_e \\ \dot{\theta}_e &= -c_1\theta_e. \end{aligned} \tag{32}$$

Remark 3. Clearly, since the $\dot{\theta}_e$ equation depends only on θ_e one cannot reconstruct x_e and y_e when only the angle is measurable.

3.1 A full state observer

3.1.1 A four dimensional observer

The full state observer is usually defined as a copy of a system with additional terms which assure the error convergence. However, in the case of the system (5) finding the proper terms is not trivial because of the trigonometric functions of θ present in x and y dynamics. Therefore, we increase the dimension of the system (5) by defining new variables s and c which replace the angle θ.

$$\begin{aligned} c &= \cos\theta \\ s &= \sin\theta \end{aligned} \tag{33}$$

Then we transform the system equations (5) to the following form

$$\begin{aligned} \dot{x} &= \quad v\, c \\ \dot{y} &= \quad v\, s \\ \dot{s} &= \quad \omega\, c \\ \dot{c} &= - \quad \omega\, s, \end{aligned} \tag{34}$$

where x and y are assumed to be measured.

Remark 4. Since we have transformed the three dimensional system to the one with higher dimension, some constraint equations should appear. In our case the constrains have the form $s^2 + c^2 = 1$, so the state space of (34) is an infinite cylinder. For the new system we can find an observer although it does not necessarily satisfy the constraint equation. It appears that the dimension of the observer is greater than the dimension of the system.

Similarly to Proposition 2 we define an observer being a copy of the system (34) with an additional vector function $f(x, y, \hat{x}, \hat{y}, \hat{c}, \hat{s}, v, \omega)$.

$$\begin{aligned} \dot{\hat{x}} &= \quad v\, \hat{c} + f_1 \\ \dot{\hat{y}} &= \quad v\, \hat{s} + f_2 \\ \dot{\hat{s}} &= \quad \omega\, \hat{c} + f_3 \\ \dot{\hat{c}} &= - \quad \omega\, \hat{s} + f_4. \end{aligned} \tag{35}$$

When we define the observation error as the difference between (34) and (35), we obtain the following error dynamics

$$\begin{aligned} \dot{\tilde{x}} &= \dot{x} - \dot{\hat{x}} = v\tilde{c} - f_1 \\ \dot{\tilde{y}} &= \dot{y} - \dot{\hat{y}} = v\tilde{s} - f_2 \\ \dot{\tilde{s}} &= \dot{s} - \dot{\hat{s}} = \omega\tilde{c} - f_3 \\ \dot{\tilde{c}} &= \dot{c} - \dot{\hat{c}} = \omega\tilde{s} - f_4. \end{aligned} \tag{36}$$

If the error converges to 0, we can conclude that the estimates $\hat{x}$, $\hat{y}$, $\hat{s}$ and $\hat{c}$ tend to the state of the system (34).

Proposition 4. *Consider an observer in the form*

$$\begin{aligned} \dot{\hat{x}} &= & v\,\hat{c} + & \quad a_1\,\tilde{x} \\ \dot{\hat{y}} &= & v\,\hat{s} + & \quad a_2\,\tilde{y} \\ \dot{\hat{s}} &= & \omega\,\hat{c} + & \quad a_3 v\,\tilde{y} \\ \dot{\hat{c}} &= - & \omega\,\hat{s} + & \quad a_4 v\,\tilde{x} \end{aligned} \tag{37}$$

with error equations

$$\begin{aligned} \dot{\tilde{x}} &= & v\,\tilde{c} - & \quad a_1\,\tilde{x} \\ \dot{\tilde{y}} &= & v\,\tilde{s} - & \quad a_2\,\tilde{y} \\ \dot{\tilde{s}} &= & \omega\,\tilde{c} - & \quad a_3 v\,\tilde{y} \\ \dot{\tilde{c}} &= - & \omega\,\tilde{s} - & \quad a_4 v\,\tilde{x}, \end{aligned} \tag{38}$$

where $a_1, a_2, a_3, a_4 > 0$. If ω, v and $\dot{v}$ are bounded, and if there exists an $\varepsilon > 0$ such that

$$\liminf_{t\to\infty} v(t) \;\geqslant\; \varepsilon \tag{39}$$

then the system (38) is globally asymptotically stable at 0.

Proof. Consider the positive definite function V

$$V = \frac{1}{2}\left(a_4\,\tilde{x}^2 + a_3\,\tilde{y}^2 + \tilde{s}^2 + \tilde{c}^2\right). \tag{40}$$

Differentiating V along the dynamics (38) yields

$$\begin{aligned} \dot{V} &= a_4\,\tilde{x}\dot{\tilde{x}} + a_3\,\tilde{y}\dot{\tilde{y}} + \tilde{s}\dot{\tilde{s}} + \tilde{c}\dot{\tilde{c}} = \\ &= a_4 v\,\tilde{x}\tilde{c} - a_4 a_1\,\tilde{x}^2 + a_3\,v\,\tilde{y}\,\tilde{s} - a_3\,a_2\,\tilde{y}^2 + \omega\,\tilde{s}\tilde{c} - a_4 v\,\tilde{x}\tilde{c} - \omega\,\tilde{s}\tilde{c} - a_3 v\,\tilde{y}\tilde{s} = \\ &= -a_4 a_1\,\tilde{x}^2 - a_3 a_2\,\tilde{y}^2. \end{aligned} \tag{41}$$

Since $V > 0$ and $\dot{V} \leqslant 0$ we conclude that $\tilde{x}$, $\tilde{y}$, $\tilde{s}$ and $\tilde{c}$ are bounded. Differentiating $\dot{V}$ we find

$$\ddot{V} = -a_1 a_4\,\tilde{x}(v\tilde{c} - a_1\tilde{x}) - a_2 a_3\,\tilde{y}(v\tilde{s} - a_1\tilde{y})$$

which, when $|v| \leqslant v_{max}$, is also bounded. Therefore we conclude that $\dot{V}$ is uniformly continuous. Hence, from Lemma 1 we know that $\dot{V} \to 0$, what results with $\tilde{x}$ and $\tilde{y}$ converging to 0.

From (38) and the assumption on boundedness of ω we conclude that $\tilde{s}$ and $\tilde{c}$ are uniformly continuous. Hence we apply Lemma 2 to the $\tilde{x}_1$ and $\tilde{x}_2$ dynamics in (38) and obtain

$$\lim_{t\to\infty} v\,\tilde{s} = 0, \qquad\qquad \lim_{t\to\infty} v\,\tilde{c} = 0.$$

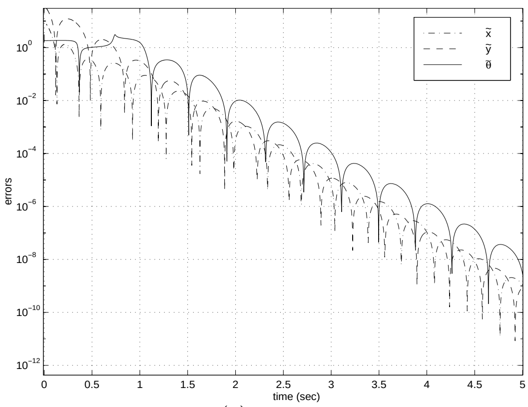
With the assumption (39) on v we conclude that $\tilde{s}$ and $\tilde{c}$ tend asymptoticaly to 0 as $t \to \infty$. $\square$

The main disadvantage of this observer is its redundancy, because two variables are used to estimate θ. Moreover the observer estimates only functions of θ and the orientation angle has to be calculated separately (taking for example $\hat{\theta} = \arctan(\frac{\hat{s}}{\hat{c}})$).

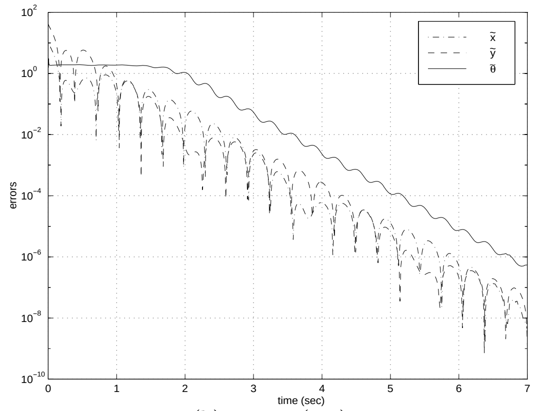
3.1.2 Simulations

The following simulations show the performance of the observer proposed in the previous subsection. Since in most cases the angle $\hat{\theta}$, and not $\hat{s}$ or $\hat{c}$, is needed, plots were plotted for $\bar{\theta}$ defined as follows

$$\bar{\theta} = \begin{cases} \arccos \frac{\tilde{c}}{\tilde{s}^2 + \tilde{c}^2} & \text{for} \quad \tilde{s} \geqslant 0 \\ \\ -\arccos \frac{\tilde{c}}{\tilde{s}^2 + \tilde{c}^2} & \text{for} \quad \tilde{s} < 0 \end{cases}$$

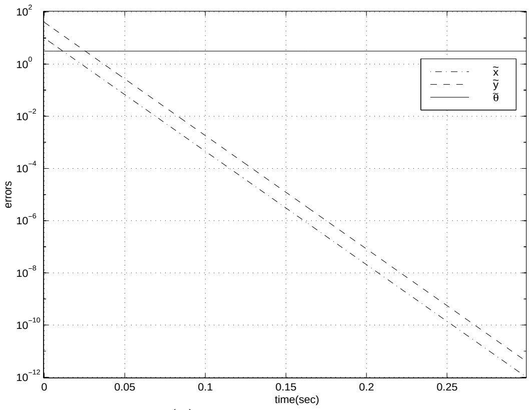


(a) $v = 1$

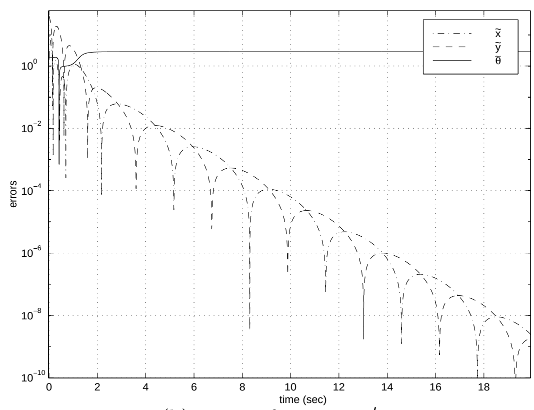


(b) $v = \sin(10t)$

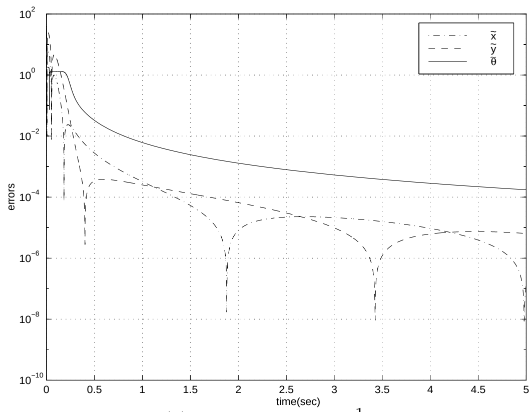
Figure 4: A four dimensional observer for a mobile car



(a) Errors for $v = 0$



(b) Errors for $v = e^{-t}$



(c) Errors for $v = \frac{1}{\sqrt{t}}$

Figure 5: Observation's error with not persistently exciting input

All plots were made in semilogarithmic scale. Initial values of errors were equal to $\bar{\theta} = \frac{\pi}{2}$ ($\tilde{s} = 1, \tilde{c} = 0$), $\tilde{x} = 10$ and $\tilde{y} = 40$.

In Figure 4 observer errors for inputs $v = 1$ and $v = \sin(t)$ are shown. The gains used are $a_1 = a_2 = 10$ and $a_3 = a_4 = 100$. One can see that the errors converge to 0 exponentially.

Figure 5 presents error characteristics when the input v is not persistently exciting. If the input v satisfies condition

$$\int_0^\infty v^2(t)dt = \infty \tag{42}$$

the observer still tends to 0, but not exponentially (Figure 5(c)). Otherwise, the error of the angle converges to a non-zero constant (Figures 5(a) and 5(a)).

3.2 A reduced-order observer

3.2.1 Estimating position coordinates

In order to present a possiblity of estimating each of the state variables given in (32), we shall design the observers for different sets of available outputs.

To observe y_e we assume the following outputs

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_e \\ \theta_e \end{bmatrix}.$$

In a similar way as in section 2.4 we introduce a new variable z being a linear combination of known variable x_e and unknown y_e

$$z = y_e + b(t)\ x_e,$$

where $b(t)$ is a function we choose to obtain the convergence of the observer.

Differentiating z along the dynamics (32), with $\omega = \omega_r + c_1\theta_e$, yields

$$\begin{aligned}
\dot{z} &= -\omega x_e + v_r \sin\theta_e + \dot{b}x_e + b\omega y_e + bv_r(\cos\theta_e - 1) - c_2 x_e = \\
&= b\omega(y_e + bx_e) - (b^2\omega + \omega + c_2 - \dot{b})x_e + v_r \sin\theta_e + bv_r(\cos\theta_e - 1) = \\
&= b\omega z - (b^2\omega + \omega + c_2 - \dot{b})x_e + v_r \sin\theta_e + bv_r(\cos\theta_e - 1).
\end{aligned}$$

To simplify the equations we denote

$$f(t, x_e, \theta_e) = -(b^2\omega + \omega + c_2 - \dot{b})x_e + v_r \sin\theta_e + bv_r(\cos\theta_e - 1). \tag{43}$$

Then the z dynamics become

$$\dot{z} = b\omega z + f(t, x_e, \theta_e). \tag{44}$$

We use a copy of (44) as the observer

$$\dot{\hat{z}} = b\omega\hat{z} + f(t, x_e, \theta_e). \tag{45}$$

Denote the estimation error $\tilde{y}_e = y_e - \hat{y}_e$. Then we have to find $b(t)$ to stabilize the error dynamics

Proposition 5. *Consider the system (32) with outputs x_e and θ_e and the following observer estimating y_e*

$$\hat{y}_e = \hat{z} - b(t)x_e,$$

where $\hat{z}$ is the solution of (45, 43), $b(t) = -c_y\omega(t) = -c_y(\omega_r(t) + c_1\theta_e)$, $c_y > 0$ and

$$\lim_{t\to\infty} \int_0^t \omega_r^2(\tau)d\tau = \infty. \tag{46}$$

Then $\tilde{y}_e = y_e - \hat{y}_e$ converges to 0 as t tends to ∞. If in addition $\omega_r(t)$ is persistently exciting then $\tilde{y}_e$ converges to 0 exponentially.

Proof.

$$\begin{aligned}\dot{\tilde{y}}_e = \dot{y}_e - \dot{\hat{y}}_e &= \dot{z} - \tfrac{d}{dt}(b(t)x_e) - \dot{\hat{z}} + \tfrac{d}{dt}(b(t)x_e) = \\ &= b(t)\omega z + f(t, x_e, \theta_e) - b(t)\omega\hat{z} - f(t, x_e, \theta_e) = \\ &= b(t)\omega(z - \hat{z} - b(t)x_e + b(t)x_e) = \\ &= b(t)\omega\tilde{y}_e\end{aligned} \tag{47}$$

When $b(t) = -c_y\omega$ and $c_y > 0$ the solution of (47) is

$$\tilde{y}_e(t) = y_e(0)e^{-c_y\int_0^t \omega^2(\tau)d\tau} \tag{48}$$

which tends to 0 if the integral goes to ∞.

Furthermore, if $\omega(t)$ is persistently exciting $\tilde{y}_e$ converges to 0 exponentially (cf. [11]).

In the closed loop system obtained from (30), (31) the solution $\theta_e(t) = \theta_e(0)e^{-c_1 t}$. Hence θ_e and $\dot{\theta}_e$ converge exponentially to 0. Since $\omega = \omega_r + c_1\theta_e$, we conclude that the assumptions on ω given in Proposition 5 are equivalent to the corresponding conditions on ω_r.

□

The control law given in (31) is a function of available outputs and therefore the observer of y_e does not influence the performance of the system.

Analogously to the above result, we can define an observer for x_e when the available outputs are

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_e \\ \theta_e \end{bmatrix}$$

$$z = x_e + b(t)y_e$$

$$\begin{aligned}\dot{z} &= \omega(t,\theta_e)y_e - c_2x_e + v_r(\cos\theta_e - 1) + \dot{b}y_e - b(t)\omega(t,\theta_e)x_e + b(t)v_r\sin\theta_e = \\ &= -(c_2 + b(t)\omega)(x_e + b(t)y_e) + f(t, y_e, \theta_e) = \\ &= -(c_2 + b(t)\omega)z + f(t, y_e, \theta_e)\end{aligned}$$

$$\begin{aligned}\hat{x}_e &= \hat{z} - b(t)y_e \\ \dot{\hat{z}} &= -(c_2 + b(t)\omega)\hat{z} + f(t, y_e, \theta_e)\end{aligned} \tag{49}$$

Hence we obtain the following observer error dynamics

$$\begin{aligned}\dot{\tilde{x}}_e = \dot{x}_e - \dot{\hat{x}}_e &= \dot{z} - \tfrac{d}{dt}(b(t)y_e) - \dot{\hat{z}} + \tfrac{d}{dt}(b(t)y_e) = \\ &= -(c_2 + b(t)\omega)(z - \hat{z}) = \\ &= -(c_2 + b(t)\omega)\tilde{x}_e.\end{aligned}$$

If $b(t) = c_e\omega$ and if $c_2, c_e > 0$, then the observer error converges exponentially to 0.

After we have found an observer estimating the unknown variable x_e, we replace x_e by $\hat{x}_e$ in the controller (31).

Proposition 6. *Consider the system*

$$\begin{aligned}\dot{x}_e &= (\omega_r + c_1\theta_e)y_e + v_r(\cos\theta_e - 1) - c_2\hat{x}_e \\ \dot{y}_e &= -(\omega_r + c_1\theta_e)x_e + v_r\sin\theta_e \\ \dot{\theta}_e &= -c_1\theta_e \\ \dot{\tilde{x}}_e &= -(c_2 + c_e\omega^2)\tilde{x}_e,\end{aligned} \tag{50}$$

where c_1, c_2, c_e are positive constants and $\tilde{x}_e = x_e - \hat{x}_e$.

Under the assumption that ω_r and $\dot{\omega}_r$ are bounded and ω_r is persistently exciting the system (50) is $\mathcal{K}$-exponentially stable.

Proof. We can prove stability of (50) in a similar way as in the proof of Proposition 3.

Putting the x_e equation in form

$$\dot{x}_e = (\omega_r + c_1\theta_e)y_e + v_r(\cos\theta_e - 1) - c_2x_e + c_2\tilde{x}_e$$

we can treat first three equations included in the system (50) as the system (32) perturbed by the observer error $\tilde{x}_e$ where (32) is $\mathcal{K}$-exponentially stable under the assumptions on ω_r, $\dot{\omega}_r$ and c_1 (cf. [9]), and $\tilde{x}_e$ is exponentially stable if the conditions on c_2 and c_e are satisfied. Hence applying Lemma 6 we conclude that the system (50) is globally $\mathcal{K}$-exponentially stable. □

3.2.2 An observer estimating the orientation angle

With the purpose of finding a reduced order observer for the system (32) when x_e and y_e are measured, we first try to find an observer for the simplified system

$$\begin{aligned}\dot{x}_e &= v_r\cos\theta_e\\ \dot{y}_e &= v_r\sin\theta_e\\ \dot{\theta}_e &= 0.\end{aligned}\tag{51}$$

Then we expect the observer for the system (51) may be extended afterwards to an observer for the system (32).

In a similar way as in section 3.1.1 we transform (51) by setting $\sin\theta_e = s_e$ and $\cos\theta_e = c_e$

$$\begin{aligned}\dot{x}_e &= v_rc_e\\ \dot{y}_e &= v_rs_e\\ \dot{c}_e &= 0\\ \dot{s}_e &= 0.\end{aligned}$$

A reduced order observer for this system is defined via

$$\begin{bmatrix}\hat{c}_e\\ \hat{s}_e\end{bmatrix} = \begin{bmatrix}\hat{z}_1\\ \hat{z}_2\end{bmatrix} + \begin{bmatrix}b_1(t) & 0\\ 0 & b_2(t)\end{bmatrix}\begin{bmatrix}x_e\\ y_e\end{bmatrix} = \begin{bmatrix}\hat{z}_1\\ \hat{z}_2\end{bmatrix} + \mathbf{B}\begin{bmatrix}x_e\\ y_e\end{bmatrix}\tag{52}$$

$$\begin{bmatrix}\dot{\hat{z}}_1\\ \dot{\hat{z}}_2\end{bmatrix} = -v_r\mathbf{B}\begin{bmatrix}\hat{z}_1\\ \hat{z}_2\end{bmatrix} - v_r\mathbf{B}^2(t)\begin{bmatrix}x_e\\ y_e\end{bmatrix} - \dot{\mathbf{B}}(t)\begin{bmatrix}x_e\\ y_e\end{bmatrix}.\tag{53}$$

From (52) and (53) defining $\tilde{s}_e = s_e - \hat{s}_e$ and $\tilde{c}_e = c_e - \hat{c}_e$ and choosing $b_1(t) = a_1v_r$ and $b_2(t) = a_2v_r$ where $a_1, a_2 > 0$ yields

$$\dot{\tilde{c}}_e = -b_1(t)v_r\tilde{c}_e = -a_1v_r^2\tilde{c}_e \tag{54a}$$

$$\dot{\tilde{s}}_e = -b_2(t)v_r\tilde{s}_e = -a_2v_r^2\tilde{s}_e, \tag{54b}$$

which are asymptotically stable if v_r satisfies the condition (42) and exponentially stable if v_r is persistently exciting (compare the proof of Proposition 5).

It is worth of noting that the equations (54) are separated. Hence, we are able to use only one of them as an observer for θ_e.

When we compare θ_e equations in (30) and (32) we remark that for $|\theta_e| < \pi$ it is possible to use $\omega = \omega_r + c_1\sin\theta_e$ instead of (31) to stabilize θ_e. Hence we obtain

$$\dot{\theta}_e = -c_1\sin\theta_e.\tag{55}$$

Define the Lyapunov function

$$V(\theta_e) = 1 - \cos\theta_e,\tag{56}$$

and differentiate it along the dynamics (55)

$$\dot{V} = -c_1\sin^2\theta_e \leqslant 0.\tag{57}$$

If c_1 is a positive constant and $|\theta_e(0)|< \pi$, the system (55) is asymptoticaly stable.

We can also find $\alpha(\theta_e(0)) > 0$ such that

$$\sin^2\theta_e \geqslant \alpha(\theta_e(0))(1-\cos\theta_e).$$

Then (57) satisfies

$$\dot{V} \leqslant -c_1\alpha(\theta_e(0))(1-\cos\theta_e) = -c_1\alpha(\theta_e(0))V$$

and the system (55) is $\mathcal{K}$-exponentially stable.

It turns out that we can use $\sin\theta_e$ instead of θ_e in the control law. Then we should extend the observer estimating $\sin\theta_e$ for the simplified system (51) to the case of the system (30).

Define the new variable z

$$z = \sin\theta_e - av_r y_e.$$

Its derivative along the dynamics (30) is given by

$$\dot{z} = (\omega-\omega_r)\cos\theta_e - a\dot{v}_r y_e + av_r\omega x_e - av_r^2\sin\theta_e.$$

Denote $\psi = \sin\theta_e$ and $\hat{\psi} = \widehat{\sin}\theta_e$. Hence, we define the observer

$$\hat{\psi} = \hat{z} + av_r y_e \tag{58a}$$
$$\dot{\hat{z}} = -a\dot{v}_r y_e + av_r\omega x_e - av_r^2\hat{z} - a^2v_r^3 y_e. \tag{58b}$$

With the observer error $\tilde{\psi} = \psi - \hat{\psi}$, the observer error dynamics are given by

$$\dot{\tilde{\psi}} = (\omega-\omega_r)\cos\theta_e - av_r^2\tilde{\psi}. \tag{59}$$

3.2.3 A controller based on the reduced order observer

The equation (59) describes an observer for the system (30). Since the dynamics of $\tilde{\psi}$ contain a term depending on θ_e, we need to study the stability of the combined observer (59) and the update law for θ_e coming from (30). We consider the case of the input ω is given by

$$\omega = \omega_r + c_1(t)\hat{\psi}. \tag{60}$$

Combining the equations for θ_e and $\tilde{\psi}$ with the control law (60) yields

$$\dot{\theta}_e = -c_1(t)\hat{\psi}$$
$$\dot{\tilde{\psi}} = -c_1(t)\hat{\psi}\cos\theta_e - av_r^2\tilde{\psi},$$

where $c_1(t)$ is a non-negative function of time. Then since $\hat{\psi} = \sin\theta_e - \tilde{\psi}$ we obtain

$$\begin{aligned}\dot{\theta}_e &= -c_1(t)\big(\sin\theta_e - \tilde{\psi}\big)\\ \dot{\tilde{\psi}} &= -c_1(t)\frac{1}{2}\sin 2\theta_e + c_1(t)\tilde{\psi}\cos\theta_e - av_r^2\tilde{\psi}.\end{aligned} \tag{61}$$

For the system (61) define a Lyapunov function V

$$V = (1-\cos\theta_e) + \frac{1}{2}\tilde{\psi}^2. \tag{62}$$

The derivative of V along the trajectories (61) yields

$$\begin{aligned}\dot{V} &= -c_1(t)\sin^2\theta_e + c_1(t)\tilde{\psi}\sin\theta_e - \big(-c_1(t)\cos\theta_e + av_r^2\big)\tilde{\psi}^2 - c_1(t)\frac{1}{2}\sin 2\theta_e\tilde{\psi} \leqslant\\ &\leqslant -c_1(t)\sin^2\theta_e - \big(-c_1(t)\cos\theta_e + av_r^2\big)\tilde{\psi}^2 + c_1(t)(|\sin\theta_e|+|\frac{1}{2}\sin 2\theta_e|)|\tilde{\psi}|.\end{aligned}$$

Since $\frac{1}{2}|\sin 2\theta_e|\leqslant|\sin\theta_e|$ and $c_1(t)\cos\theta_e\leqslant c_1(t)$

$$\dot{V}\leqslant -c_1(t)\sin^2\theta_e-\big(-c_1(t)+av_r^2\big)\tilde{\psi}^2+2c_1(t)|\sin\theta_e||\tilde{\psi}|$$

Assume $c_1(t)=\frac{1}{2}\gamma av_r^2$, where $0<\gamma<1$. Then

$$\dot{V}\leqslant -av_r^2\left(\frac{\gamma}{2}\big(|\sin\theta_e|-|\tilde{\psi}|\big)^2+(1-\gamma)\tilde{\psi}^2\right)\leqslant 0. \tag{63}$$

When $\liminf_{t\to\infty}|v_r(t)|\geqslant\varepsilon>0$ the system (61) is asymptotically stable for all initial conditions.

We notice that for small δ and $\theta_e\in(-\arccos(\delta-1),\arccos(\delta-1))$, $\sin^2\theta_e>1-\cos\theta_e$. Hence we transform (63) into the following form

$$\begin{aligned}\dot{V}\leqslant&-av_r^2\left(\alpha^2\sin^2\theta_e+\beta^2\tilde{\psi}^2-2\alpha\beta|\sin\theta_e||\tilde{\psi}|+\frac{\eta}{\delta}\sin^2\theta_e+(\frac{\eta}{2}+\kappa)\tilde{\psi}^2\right)\leqslant\\ \leqslant&-av_r^2\left(\alpha|\sin\theta_e|-\beta|\tilde{\psi}|\right)^2-\eta av_r^2\left(1-\cos\theta_e+\frac{1}{2}\tilde{\psi}^2\right)\leqslant\\ \leqslant&-\eta av_r^2V\end{aligned}$$

where α, β, η and κ satisfy the set of equations

$$\begin{aligned}\alpha\beta&=\frac{\gamma}{2}\\ \alpha^2+\frac{\eta}{\delta}&=\frac{\gamma}{2}\\ \beta^2+\eta+\kappa&=1-\frac{\gamma}{2}.\end{aligned}$$

From the (3.2.3) we conclude that the system (61) is gllobally $\mathcal{K}$- exponentially stable.

Proposition 7. *Consider the system (30) with the control law*

$$\begin{aligned}v&=v_r+c_2x_e\\ \omega&=\omega_r+\frac{1}{2}\gamma av_r^2\hat{\psi}\end{aligned}\tag{64}$$

and the observer given by

$$\begin{aligned}\hat{\psi}&=\hat{z}+av_ry_e\\ \dot{\hat{z}}&=av_r^2\hat{z}-a^2v_r^3y_e-a\dot{v}_ry_e,\end{aligned}\tag{65}$$

where c_2 and a are positive constants, $0<\gamma<1$. If v_r and ω_r are bounded and persistently exciting and $\dot{\omega}_r$, $\dot{v}_r$ are bounded, the closed loop system obtained from (30), (64) and (65)

$$\begin{aligned}\dot{x}_e&=\ (\omega_r+\frac{\gamma}{2}av_r^2\hat{\psi})y_e+v_r(\cos\theta_e-1)-c_2x_e\\ \dot{y}_e&=-(\omega_r+\frac{\gamma}{2}av_r^2\hat{\psi})x_e+v_r\sin\theta_e\\ \dot{\theta}_e&=-\frac{\gamma}{2}av_r^2\hat{\psi}\\ \dot{\tilde{\psi}}&=-\frac{\gamma}{2}av_r^2\big(\frac{1}{2}\sin 2\theta_e-\tilde{\psi}\cos\theta_e\big)-av_r^2\tilde{\psi}\end{aligned}\tag{66}$$

is $\mathcal{K}$-exponentially stable.

Proof. We notice that

$$f(x)-f(0)=\int_0^1 df(sx)=\sum_i\int_0^1\frac{\partial f}{\partial x_i}(sx)x_i ds.$$

We can put the closed loop dynamics in cascaded form

$$\begin{bmatrix}\dot{x}_e\\ \dot{y}_e\end{bmatrix} = \begin{bmatrix}-c_2 & \omega_r\\ -\omega_r & 0\end{bmatrix}\begin{bmatrix}x_e\\ y_e\end{bmatrix} + \mathbf{G}(x_e, y_e, \theta_e, \tilde{\psi})\begin{bmatrix}\theta_e\\ \tilde{\psi}\end{bmatrix} \tag{67a}$$

$$\begin{bmatrix}\dot{\theta}_e\\ \dot{\tilde{\psi}}\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}\gamma a v_r^2\big(\sin\theta_e - \tilde{\psi}\big)\\ -\frac{\gamma}{2}a v_r^2\big(\frac{1}{2}\sin 2\theta_e - \tilde{\psi}\cos\theta_e\big) - a v_r^2\tilde{\psi}\end{bmatrix}, \tag{67b}$$

where $\mathbf{G}(x_e, y_e, \theta_e, \tilde{\psi})$ is given by

$$\mathbf{G} = \left[\begin{array}{c|c} \frac{\gamma}{2}a v_r^2 y_e \int_0^1 \cos s\theta_e ds + v_r \int_0^1 \sin s\theta_e ds & -\frac{\gamma}{2}a v_r^2 y_e\\ -\frac{\gamma}{2}a v_r^2 x_e \int_0^1 \cos s\theta_e ds + v_r \int_0^1 \cos s\theta_e ds & -\frac{\gamma}{2}a v_r^2 x_e \end{array}\right].$$

If v_r is persistently exciting and the constants used are strictly positive, the subsystem (67b) is globally $\mathcal{K}$-exponentially stable

Then we treat the states θ_e and $\tilde{\psi}$ as terms perturbing a system in the form (3), which according to Lemma 3 is globally exponentially stable when the assumptions on ω_r and $\dot{\omega}_r$ are satisfied. To apply Lemma 6 we need the interconnection term $\mathbf{G}$ to be at most linear in respect to $\|[x_e\ y_e]^T\|$.

$$\|\mathbf{G}\|_\infty = \max_i \sum_j |g_{ij}| \leqslant \gamma a v_r^2 \max\{|x_e|, |y_e|\} + 2|v_r| \leqslant \gamma a v_r^2 \left\|\begin{bmatrix}x_e\\ y_e\end{bmatrix}\right\| + 2|v_r|.$$

The condition is fulfilled if v_r is bounded.

Hence we conclude that the system (66) is globally $\mathcal{K}$-exponentially stable. □

Remark 5. We notice that both forward and angular velocities need to be persistently exciting. The assumption on v_r is needed to ensure convergence of the observer, while the condition on ω_r results from the controller used.

3.2.4 Simulation results

In proposition 7 we proved $\mathcal{K}$-exponential stability of the output based controller. Here we present time characteristics obtained from simulations.

Figures 6 7 and 8 show position and angle tracking error and the observer error. Initial values of errors were set on $(x_e, y_e, \theta_e) = (10, 20, \pi/2)$ and $\tilde{\psi} = 1$. As gains we used $c_2 = 2$, $a = 10$ and $\gamma = 0.9$.

Applying constant reference controls $v_r = \omega_r = 1$ (Figure 6) one can notice exponential convergence of the system (67). Next figures present necessity of the assumption of persistent excitation of v_r and ω_r.

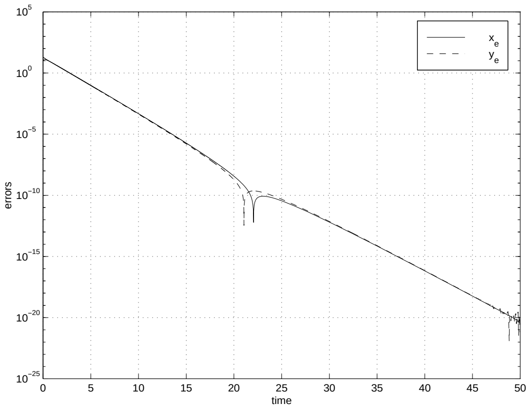
If the reference angular velocity does not satisfies the assumption (Figure 7), the position error does not tend exponentially to 0, although similarly to the previous observers if $\int_0^\infty \omega_r^2 dt = \infty$ they are still asymptotically stable. Also if the forward velocity v_r is not persistently exciting (Figure 8), tracking error and observer error does not converge to 0.

4 Conclusions

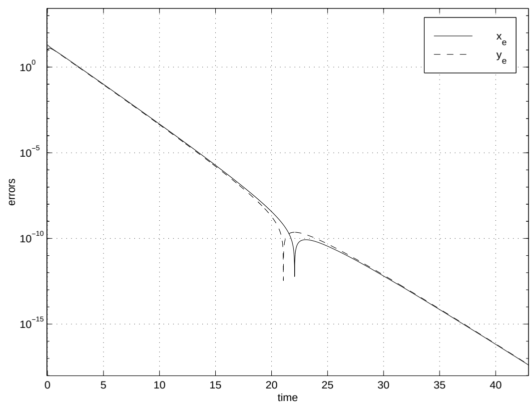
In Section 2 we have proposed observers for a three-dimensional chained form system. A further subject of interest may be extending obtained results to a general chained form system.

The observers developed in Section 3 consider a kinematic model of a mobile car. A more general model used usually to describe a wheeled mobile robot is *simplified dynamic model*(cf. [9], [3]). Since the full state observer used in Section 3 may be applied also to the generalized model, it may be useful to study the possibility of extending the designed observers and output based controller to the case of the simplified dynamic model.

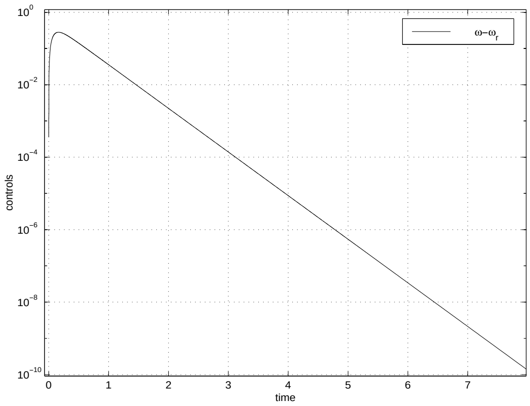
In all observers estimating the orientation angle we have assumed either persistently exciting or non-zero value of the reference forward velocity. When this condition is not met, the observer



(a) Position error

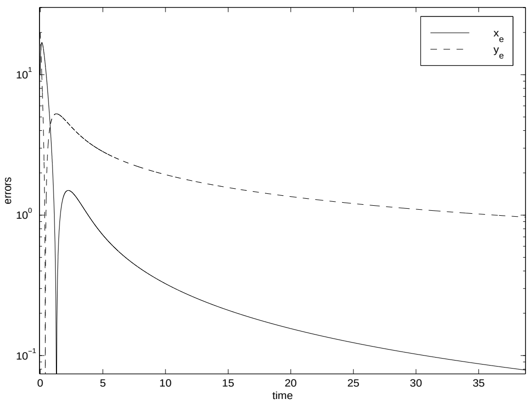


(b) Angle and observer error

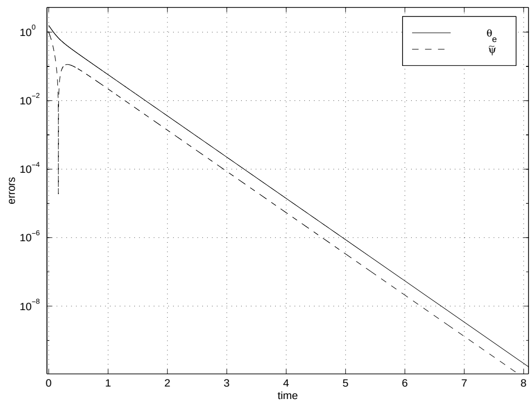


(c) Angle and observer error

Figure 6: Time characteristic of the controller (67)

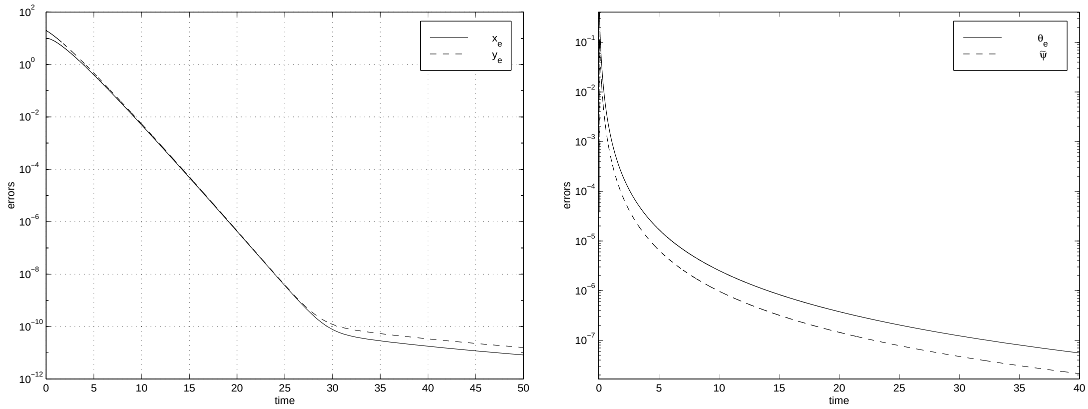


(a) Position error



(b) Angle and observer error

Figure 7: Performance of (67) when $\omega_r = \frac{1}{\sqrt{t}}$, $v_r = 1$



(a) Position error

(b) Angle and observer error

Figure 8: Performance of (67) when $v_r = \frac{1}{\sqrt{t}}$, $\omega_r = 1$

error and the angle error do not converge to 0. It means that in the stabilization task we are able to reach the desired position, but the angle remains unknown.

In observers for a chained form system (Section 2) and in combined observer and controller for a mobile car (Section 3.2) we find another restriction on the reference trajectory — the assumption on persistent excitation of the angular velocity. If this condition is not met, the position error of the systems does not tend to 0. This results in impossibility of tracking straight lines with the use of an observer, what is a common task for mobile cars.

Another problem not discussed here, but very important for the practical realization of controllers is choosing tuning gains. When the task for a mobile car is specified, the gains should be optimized.

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