1 Nonlinear Models for Control of Manufacturing Systems

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Current literature on modeling and control of manufacturing systems can roughly be divided into three groups: fluid models, queuing theory, and discrete event models. Most fluid models describe linear time-invariant controllable systems without any dynamics. These models mainly focus on throughput and are not concerned with cycle time. Queuing theory deals with relationships between throughput and cycle time, but is mainly concerned with steady state analysis. In addition, queuing models are not suitable for control theory. Discrete event models suffer from state-explosion. Simple models of manufacturing systems can be studied and analyzed, but for larger problems the dimension of the state grows exponentially. In addition, most control problems studied are supervisory control problems: the avoidance of undesired states. An important class of interesting manufacturing control problems asks for proper balancing of both throughput and cycle time for a large nonlinear dynamical system that never is in steady state. None of the mentioned models is able to deal with these kind of control problems. In this paper, models are presented which are suitable for addressing this important class of interesting manufacturing control problems.

1.1 Introduction

In this paper we are interested in the problem of how to ramp up a manufacturing line, or to be more precise: we are interested in models that are suitable for obtaining a proper solution to this problem. For that reason we propose a *computationally feasible*, *dynamic* model that incorporates both *throughput* and *cycle time*.

The model we propose is a flow model, based on the theory of modeling traffic flow. The idea is to consider the flow of products as a compressible fluid flow. The flow model we propose is not to be confused with the flow model as initiated by Kimemia and Gershwin [21] for modeling failure-prone manufacturing systems, nor with the fluid models or fluid queues as proposed by queuing theorists [16, 33], nor with the stochastic fluid model as introduced by Cassandras [10].

All of the three mentioned fluid models from literature are throughput oriented. These models do not explicitly contain information about cycle time. Also the processing times of machines are assumed to be deterministic. As a result, a property of these models is that any feasible throughput can be achieved by means of zero inventory.

An class of models available in the literature are models based on relations from queuing theory [22, 23], see e.g. [9, 32]. Although these results give valuable insight in steady state behavior of manufacturing lines, a disadvantage is that only steady state is concerned. No dynamic relations are available. Therefore, these models can not be used for studying the problem of how to ramp up a manufacturing line.

A third class of models are discrete event models like for instance the class of discrete event systems as studied by Ramadge and Wonham [28]. These models do include dynamics, and both throughput and cycle time are incorporated. Unfortunately, as all states in which a manufacturing system can be have to be considered, these models are almost unsuitable for practical use. However, a promising modeling approach consists of the so-called max-plus-linear discrete event systems with variability expansion, as studied in [8].

To summarize: roughly three classes of models for manufacturing lines have been studied in the literature so far: discrete event models that suffer from state explosion, queuing theory that contains only steady state results, and fluid models that do not incorporate cycle times. In this paper we propose a new class of flow models, by considering the flow of products to be a compressible fluid flow. However, first we recall the fluid models as currently available in the literature and illustrate some of its shortcomings. We also prepare ways to (partially) overcome these shortcomings.

1.2 Extensions to the standard fluid model

As mentioned in the introduction, one of the advantages of fluid models is that these models incorporate the dynamical behavior of manufacturing systems. Unfortunately, these models do not take into account cycle times. In this section we present extensions to the fluid model that (partially) overcome this disadvantage. However, before we can present this extension we first have to present the fluid model as currently used in literature.

1.2.1 A common fluid model

The current standard way of deriving fluid models is most easily explained by means of an example. Therefore, consider a simple manufacturing system consisting of two machines in series, as displayed in Figure 1.1. Let $u_0(t)$ denote the rate at which jobs arrive to the system

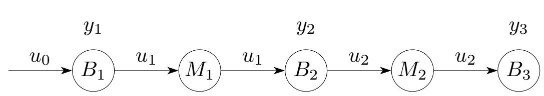

Figure 1.1: A simple manufacturing system

at time t, let $u_i(t)$ denote the rate at which machine M_i produces lots at time t, let $y_i(t)$ denote the number of lots in buffer B_i at time t ($i \in \{1, 2\}$), and let $y_3(t)$ denote the number of lots produced by the manufacturing system at time t. Assume that machines M_1 and M_2 have a maximum capacity of respectively μ_1 and μ_2 lots per time unit. This provides us with all information for deriving a fluid model.

Clearly the rate of change of the buffer contents is given by the difference between the rates at which lots enter and leave the buffer. Under the assumption that the number of lots can be considered continuous, this observation leads to the following fluid model:

$$\begin{aligned}
\dot{y}_1(t) &= u_0(t) - u_1(t)\\
\dot{y}_2(t) &= u_1(t) - u_2(t)\\
\dot{y}_3(t) &= u_2(t)
\end{aligned} \tag{1.1}$$

which can also be expressed as follows:

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} u(t) \tag{1.2a}$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} u(t), \tag{1.2b}$$

where $u = [u_0, u_1, u_2]^T$ and $y = [y_1, y_2, y_3]^T$. We also have capacity constraints on the input, as well as the constraint that the buffer contents should remain positive. These constraints can be expressed means of the following equations:

$$0 \leq u_1(t) \leq \mu_1, 0 \leq u_2(t) \leq \mu_2 \quad \text{and} \quad y_1(t) \geq 0, y_2(t) \geq 0, y_3(t) \geq 0. \tag{1.3}$$

System (1.2) is a controllable linear system of the form $\dot{x} = Ax + Bu, y = Cx + Du$ as extensively studied in control theory. Note that the description (1.2) is not the only possible input/output/state model which yields the input/output behavior (1.1). To illustrate this, consider the change of coordinates

$$x(t) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \bar{x}(t), \tag{1.4}$$

which results in the following input/output/state model:

$$\dot{\bar{x}}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \bar{x}(t) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u(t) \tag{1.5a}$$

$$y(t) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \bar{x}(t) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} u(t). \tag{1.5b}$$

We would like to study the response of the output of the system (1.2), or equivalently (1.5). Assume that initially we start with an empty production line (i.e. $x(0) = 0$), that both machines have a capacity of 1 lot per unit time (i.e. $\mu_1 = \mu_2 = 1$) and that we feed the line at a rate of 1 lot per time-unit (i.e. $u_0 = 1$). Furthermore, assume that machines produce at full capacity, but only in case something is in the buffer in front of it, i.e.

$$u_i(t) = \begin{cases} \mu_i & \text{if } y_i(t) > 0 \\ 0 & \text{otherwise} \end{cases} \qquad i \in \{1, 2\}. \tag{1.6}$$

Under these assumptions, the resulting contents of buffer B_3 are as displayed in Figure 1.2. Notice that immediately lots start coming out of the system. Clearly this is not what happens

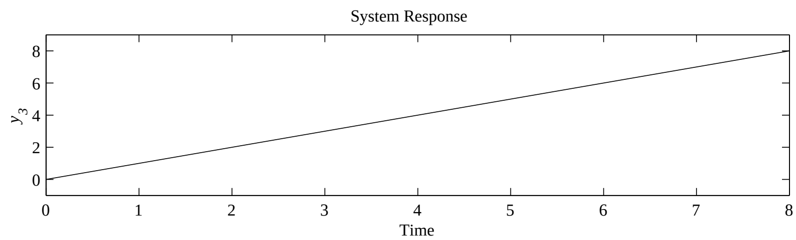

Figure 1.2: Output of the manufacturing system using model (1.1)

in practice. Since both machines M_1 and M_2 need to process the first lot, it should take the system at least $\frac{1}{\mu_1} + \frac{1}{\mu_2}$ time units before lots can come out. This illustrates our statement that cycle times are not incorporated in fluid models as currently available in literature. Now we are ready for formulating an extension to the standard fluid model as presented.

1.2.2 An extension

In the previous subsection we noticed that in the standard fluid model lots immediately come out of the system, once we start producing. A way to overcome this problem is to explicitly take into account the required delay. Whenever we decide to change the production rate of

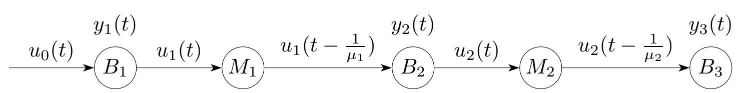

Figure 1.3: A simple manufacturing system revisited

machine M_1, buffer B_2 notices this $\frac{1}{\mu_1}$ time units later. As a result the rate at which lots arrive to buffer B_2 at time t is equal to the rate at which machine M_1 was processing at time $t - \frac{1}{\mu_1}$. This observation results in the following model (see also Figure 1.3):

$$\begin{aligned}
\dot{y}_1(t) &= u_0(t) - u_1(t) \\
\dot{y}_2(t) &= u_1(t - \frac{1}{\mu_1}) - u_2(t) \\
\dot{y}_3(t) &= u_2(t - \frac{1}{\mu_2})
\end{aligned} \tag{1.7}$$

Clearly the constraints (1.3) also apply to the model (1.7).

We expect that this model shows a response which is closer to reality. Assume that for the system (1.7) we also have $\mu_1 = \mu_2 = 1$ lot per time unit, and that we perform the same experiments as in the previous subsection, i.e. start from $x(0) = 0$, apply $u_0 = 1$ and (1.6). The resulting response of buffer B_3 is displayed in Figure 1.4. If we compare the results from

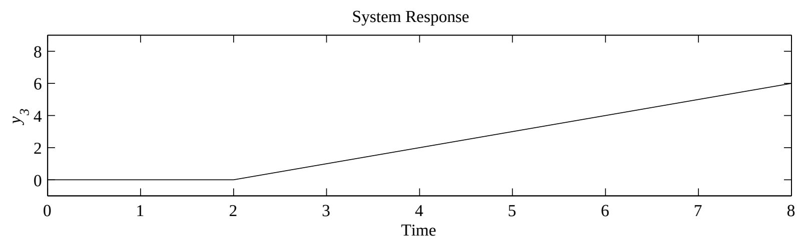

Figure 1.4: Output of the manufacturing system using model (1.7)

Figure 1.4 to that of Figure 1.2 we see that no products enter buffer B_3 during the first 2.0 time units in case we use the extended fluid model. Clearly the extended fluid model produces more realistic results than the standard fluid model.

1.2.3 An approximation to the extended fluid model

In the previous subsection we proposed an extended version of the standard fluid model. Although the model (1.7) still is a linear model, standard linear control theory is not able to deal with this model, due to the time delay. For controlling the model (1.7) we have to rely on control theory of infinite dimensional linear systems. For a good introduction to infinite dimensional linear systems, see e.g. [11].

Instead of using infinite dimensional linear systems theory, another possibility would be to approximate the time delays by means of an Padé approximation. When we use second order Padé approximations, the model (1.7) can be approximated as:

$$\dot{x} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & 0 & 0 & \mathbf{0} & 0 & 0 \\ \mathbf{0} & \mathbf{0} & 0 & 0 & \mathbf{0} & 0 & 0 \\ 0 & 0 & 4 & 6 & -3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & 0 & 0 & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 6 & -3 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 \end{bmatrix} x + \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} u \tag{1.8a}$$

$$y = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & 0 & 0 & \mathbf{0} & 0 & 0 \\ \mathbf{0} & \mathbf{1} & -3 & 0 & \mathbf{-1} & 0 & 0 \\ \mathbf{0} & \mathbf{0} & 0 & 0 & \mathbf{1} & -3 & 0 \end{bmatrix} x + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} u, \tag{1.8b}$$

Notice the structure in (1.8). In bold face we can easily recognize the dynamics (1.5). The additional dynamics is needed for approximating the time delays.

If we initiate the system (1.8) from $x(0) = 0$ and feed it at a rate $u_0 = 1$ while using (1.6), we obtain the system response as depicted in Figure 1.5. It is clear that we do not get the same response as in Figure 1.4, but the result is rather acceptable from a practical point of view. At least it is closer to reality than the response as displayed in Figure 1.2.

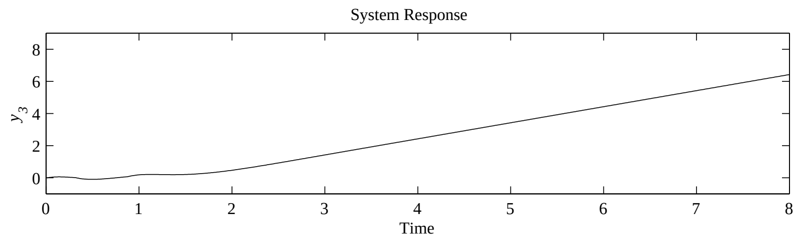

Figure 1.5: Output of the manufacturing system using model (1.8)

1.2.4 A hybrid model

In the previous subsections, we provided some extensions to the standard fluid model by taking into account the time delay lots encounter due to the processing of machines. We also mentioned the constraints (1.3) that have to be obeyed. These are constraints that we have to take into account when designing a controller for our manufacturing system. The way we dealt with these constraints in the previous subsections, was by requiring the machines to produce only in case the buffer contents in front of that machine were positive, cf. (1.6).

An way to extend the standard fluid model (1.2) is to think of these constraints in a different way. As illustrated in subsection 1.2.1, when we turn on both machines, immediately lots start coming out of the system. This is an undesirable feature that we would like to avoid. In practice, the second machine can only start producing when the first machine has finished a lot. Keeping this in mind, why do we allow machine M_2 to start producing as soon as the buffer contents of the buffer in front of it are positive? Actually, machine M_2 should only start producing as soon as a whole product has been finished by the machine M_1. In words: machine M_2 should only start producing as soon as the buffer contents of the buffer in front of it becomes 1. Therefore, we should not allow for a positive u_2 as soon as $y_2 > 0$, but only in case $y_2 \geq 1$.

When we consider the initially empty system (1.2), i.e. $x(0) = 0$, and assume

$$u_i(t) = \begin{cases} \mu_i & \text{if } y_i(t) \geq 1 \\ 0 & \text{otherwise} \end{cases} \qquad i \in \{1, 2\}, \tag{1.9}$$

the resulting system response to an input of $u_0 = 1$ is shown in Figure 1.6. Notice that we obtain exactly the same response as in Figure 1.4.

Unfortunately, this is not all. The change in the constraints as proposed is not sufficient. It is in case we ramp up our manufacturing systems, but in case we ramp down it is not. Suppose that after a while we do not feed the manufacturing line anymore, i.e. after a while we have $u_0 = 0$. In that case machine M_1 builds off the contents of the buffer B_1, until exactly one product remains. As soon as $y_1 = 1$, the machine is not allowed to produce anymore due to the constraint we imposed. This is not what we would like to have. Therefore, in case $u_1 = 0$, machine M_1 should be allowed to produce until $y_1 = 0$.

Under these conditions, we could also think of our model operating in different modes.

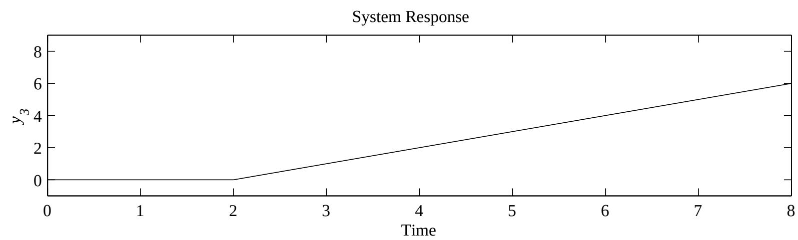

Figure 1.6: Output of the manufacturing system

For the manufacturing system under consideration we can distinguish the following modes:

$$\begin{array}{llllll}
\text{mode 1:} & 0 \le y_1 \le 1, & 0 \le y_2 \le 1, & u_0 = 0, & u_1 \ge 0, & u_2 = 0. \\
\text{mode 2:} & 0 \le y_1 \le 1, & 0 \le y_2 \le 1, & u_0 \ge 0, & u_1 = 0, & u_2 \ge 0. \\
\text{mode 3:} & 1 \le y_1, & 0 \le y_2 \le 1, & & u_1 = 0, & u_2 \ge 0. \\
\text{mode 4:} & 1 \le y_1, & 0 \le y_2 \le 1, & & u_1 \ge 0, & u_2 = 0. \\
\text{mode 5:} & 0 \le y_1 \le 1, & 1 \le y_2, & u_0 = 0, & u_1 \ge 0. & \\
\text{mode 6:} & 0 \le y_1 \le 1, & 1 \le y_2, & u_0 \ge 0, & u_1 = 0. & \\
\text{mode 7:} & 1 \le y_1, & 1 \le y_2. & & &
\end{array}$$

In all of these modes, the systems dynamics are described by (1.2).

In fact, what we just presented is a hybrid systems model of the manufacturing system under consideration. The description as just presented is also known as that of Piecewise Affine (PWA) systems [31]. Other well known descriptions are Linear Complementarity (LC) systems [17, 30] and Mixed Logical Dynamical (MLD) systems [7]. In [5, 18] it was shown that (under certain assumptions like well-posedness) these three descriptions are equivalent. This knowledge is useful, as each modeling class has its own advantages (cf. [3]). Stability criteria for PWA systems were proposed in [15, 20], and control and state-estimation techniques for MLD hybrid models have been presented in [4, 6, 7]. These results can now be applied for controlling the hybrid systems model of our manufacturing system.

1.3 A new flow model

In the previous section we proposed to replace the standard fluid model (1.1) with the model (1.7) which contains a time-delay. In that way we could overcome the shortcoming of the standard fluid model that once we start producing, immediately lots come out of the system. We also presented a Padé-approximation of this time-delayed model, as well as a hybrid systems model that produced the desired delays.

Although the proposed models do not suffer from the problem that lots come out of the system as soon as we start producing, cycle times are not truly present in these models. It is not possible to determine the time it takes lots to leave once they have entered the system. As mentioned in the introduction, we are interested in dynamic models that incorporate both throughput *and* cycle time.

Therefore, the models presented in the previous section are (still) not satisfactory. Furthermore, according to these models any feasible throughput can be achieved by means of zero inventory. In this section we present a dynamic model that does incorporate both throughput and cycle time. This dynamic model is inspired by the continuum theory of highway traffic. Therefore, before presenting this dynamic model we first present some results from traffic theory.

1.3.1 Introduction to traffic flow theory: the LWR model

In the mid 1950's Lighthill and Whitham [25] and Richards [29] proposed a first order fluid approximation of traffic flow dynamics. This model nowadays is known in traffic flow theory as the LWR model.

Traffic behavior for a single one-way road can be described using three variables that vary in time t and space x: flow $u(x,t)$, density $\rho(x,t)$, and speed $v(x,t)$. The first observation is that flow is the product of speed and density:

$$u(x,t) = \rho(x,t) \cdot v(x,t) \qquad \forall x, t. \tag{1.10}$$

Second, for a highway without entrances or exits, the number of cars between any two locations x_1 and x_2 ($x_1 < x_2$) needs to be conserved at any time t, i.e. the change in the number of cars between x_1 and x_2 is equal to the flow entering via x_1 minus the flow leaving via x_2:

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2} \rho(x,t)\mathrm{d}x = u(x_1,t) - u(x_2,t), \tag{1.11a}$$

or in differential form:

$$\frac{\partial \rho}{\partial t}(x,t) + \frac{\partial u}{\partial x}(x,t) = 0. \tag{1.11b}$$

The two relations (1.10) and (1.11) are basic relations that any model must satisfy. As we have three variables of interest, a third relation will be needed. For this third relation, several choices can be made. The LWR model assumes in addition to the relations (1.10) and (1.11) that the relation between flow and density observed under steady state conditions also holds when flow and density vary with x and/or t; i.e. for a homogeneous highway:

$$u(x,t) = S(\rho(x,t)). \tag{1.12}$$

The model (1.10, 1.11, 1.12) can predict some things encountered in traffic rather well. In order to overcome some of the deficiencies of the LWR model, in the early 1970's higher order theories have been proposed where the relation (1.12) has been replaced by an other partial differential equation, containing diffusion or viscosity terms. Unfortunately, these extended models experience some undesirable properties, as made clear in [13]. The most annoying of these properties is the fact that in these second order models cars can travel back wards. Second order models that do not suffer from this deficiency have been presented in [19, 34]. However, for our modeling purposes the first order LWR model (1.10, 1.11, 1.12) is sufficient.

1.3.2 A traffic flow model for manufacturing flow

In the previous subsection we introduced the LWR model from traffic flow theory. This model describes the dynamic behavior of cars along the highway at a macroscopic level and contains information both about the number of cars passing a certain point and about the time it takes cars to go from one point to the. The observation we make in this paper is that we can not only use this model for describing the flow of cars along the highway, but also for describing the flow of products through a manufacturing line.

Consider, instead of a homogeneous highway, a homogeneous manufacturing line, i.e. a manufacturing line that consists of a lot of identical machines. Let t denote the time and let x the position in the manufacturing line. The behavior of lots flowing through the manufacturing line can also be described by three variables that vary with time and position: flow $u(x,t)$ measured in unit lots per unit time, density $\rho(x,t)$ measured in unit lots per unit machine, and speed $v(x,t)$ measured in unit machines per unit time. Now we can relate these three variables by means of the equations (1.10), (1.11), and (1.12), where in (1.12) the function S describes the relation between flow and density observed under steady state conditions.

To make this last statement more explicit, consider a manufacturing line where all machines have exponentially distributed processing times and an average capacity of μ lots per unit time. Furthermore, consider a Poisson arrival process where lots arrive to the first machine with a rate of λ lots per unit time ($\lambda < \mu$), and assume that buffers have infinite capacity. Then we know from queuing theory [22] that the average number of lots in each workstation (consisting of a buffer and a machine) in steady state is given by

$$N = \frac{\frac{\lambda}{\mu}}{1 - \frac{\lambda}{\mu}} = \frac{\lambda}{\mu - \lambda}. \tag{1.13}$$

In words: in steady state we have

$$\rho(x,t) = \frac{u(x,t)}{\mu - u(x,t)} \tag{1.14}$$

from which we can conclude that in steady state:

$$u(x,t) = \frac{\mu\rho(x,t)}{1 + \rho(x,t)}. \tag{1.15}$$

For this example, this is the mentioned function $S(\rho)$.

With this information we can conclude that the dynamics of this manufacturing line might be described by means of the partial differential equation

$$\frac{\partial \rho}{\partial t} + \mu \frac{\partial}{\partial x}\left(\frac{\rho}{1+\rho}\right) = 0 \tag{1.16a}$$

Together with the relations

$$u = \frac{\mu\rho}{1+\rho} \text{ and } v = \frac{u}{\rho} \text{ or } v = \frac{\mu}{1+\rho} \tag{1.16b}$$

this completes our model.

Notice that contrary to the fluid models presented in the previous sections, the dynamic model (1.16) is able to incorporate the stochasticity as experienced in manufacturing lines. If the manufacturing line would be in steady state, the throughput and cycle time as predicted by the model (1.16) will be exactly the same as those predicted by queuing theory. However, contrary to queuing theory, the model (1.16) is not a steady state model, but also incorporates dynamics. Therefore, the model (1.16) is a dynamic model that incorporates both throughput and cycle time. Furthermore, given the experience in the field of fluid dynamics, the model is computationally feasible as well.

1.4 The manufacturing flow model revisited

In section 1.3 we noticed that for the standard fluid model (1.1) it is possible to achieve any feasible throughput by means of zero inventory. Even when we are not interested in cycle times, this is still a major shortcoming of the standard fluid models. Using insight from the flow model as derived in the previous section, this shortcoming of standard fluid models can be overcome.

Consider the fluid model (1.16). Discretization of this model (with respect to x only, see also [12]) yields

$$\begin{aligned} \dot{x}_1 &= u_0 - \frac{\mu x_1}{1+x_1} \\ \dot{x}_2 &= \frac{\mu x_1}{1+x_1} - \frac{\mu x_2}{1+x_2} \\ \dot{x}_3 &= \frac{\mu x_2}{1+x_2}. \end{aligned} \tag{1.17}$$

Notice that the discretized model (1.17) can also be seen as a system of the form (1.1) where instead of (1.6) we use

$$u_i(t) = \frac{\mu_i y_i}{1+y_i} \qquad i \in \{1, 2\}. \tag{1.18}$$

What we can learn from this observation is that in case we move from deterministic processing times to stochastic processing times, apparently we should replace the inputs (1.6) with (1.18). In that case, to each throughput rate corresponds a non-zero steady state WIP level which is equal to the one predicted by queuing theory. Furthermore, notice that whenever we start from a feasible initial condition, i.e. the buffer contents initially are nonnegative, the conditions (1.3) are always met.

More can be said about the model (1.17). In section 1.2 we mainly were considered with the output of the manufacturing line, i.e. we were mainly concerned with the signal $y_3(t) = x_3(t)$. Even though the model (1.17) clearly is a nonlinear model, it has a nice structure: the model is feedback linearizable [26, 27]. To make this statement more explicit, consider the following change of coordinates:

$$
\begin{aligned}
z_1 &= \frac{\mu^2(x_1 - x_2)}{(1+x_1)(1+x_2)^3} \\
z_2 &= \frac{\mu x_2}{1+x_2} \\
z_3 &= x_3
\end{aligned}
\tag{1.19a}
$$

together with the input

$$
u_0 = \frac{(1+x_1)^2(1+x_2)^2}{\mu^2} v - \frac{2\mu(x_1-x_2)}{1+x_2} + \frac{3\mu(x_1-x_2)(1+x_1)}{(1+x_2)^3} + \frac{\mu x_1}{1+x_1} \tag{1.19b}
$$

where v can be an arbitrary signal. If we apply (1.19) to the system (1.17) we obtain the system

$$
\dot{z}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} z(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} v(t) \tag{1.20a}
$$

$$
y_3(t) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} z(t) + \begin{bmatrix} 0 \end{bmatrix} v(t) \tag{1.20b}
$$

which is a *linear* system. After applying the nonlinear change of coordinates and feed forward (1.19) we can control the output of the manufacturing line by means of standard linear control theory, as made clear by the system (1.20).

Another standard nonlinear control technique that can be used for controlling the system (1.17) is backstepping, cf. [24, 26].

1.5 Concluding remarks

In the literature roughly three classes of models for manufacturing lines have been studied so far: fluid models that do not incorporate cycle times, queuing theory that contains only steady state results, and discrete event models that suffer from state explosion.

In this paper we presented a flow model for modeling manufacturing lines, based on the theory of modeling traffic flow. The presented model is the first computationally feasible dynamic model that incorporates both throughput and cycle time. This model is a suitable model for addressing dynamic control questions like how to ramp up a given manufacturing line.

We also illustrated that the presented flow model can give valuable insights on how to modify the standard fluid models from literature in case we would like to deal with non-deterministic processing times of machines.

The idea to use traffic flow models for modeling the dynamics of manufacturing systems emerged only recently. Related work can be found in [1, 2]. Also the book [14] provides a good introduction to the subject.

Issues like the relation between variability of manufacturing systems and turbulence, the influence of scheduling policies on the relation (1.15), extensions to higher order models (like [19, 34], while keeping in mind the observations in [13]), correct discretization schemes (cf. [12]), control of these flow models, and last but not least the validity of these models will be subject of future study.

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