Faculty of Applied Mathematics

University of Twente
University for Technical and Social Sciences

P.O. Box 217
7500 AE Enschede
The Netherlands
Phone +31-53-893400
Fax +31-53-356695
Telex 44200

(Adaptive) Control of Chaotic and Robot Systems via Bounded Feedback Control

A.A.J. Lefeber

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Prof. dr. ir. H. Kwakernaak
Dr. H. Nijmeijer
Dr. A.J. van der Schaft
Dr. ir. H. Berghuis

Abstract

So far, the research on controlling chaos has mainly been directed to controlling the system towards fixed points or periodic orbits of the system. Nijmeijer and Berghuis for the first time considered the problem of controlling the chaotic forced Duffing system towards *any* desired trajectory, using feedback control. Since in practice one has to deal with input limitations, bounded control laws have to be developed.

This thesis introduces the concept of a composite controller, in order to achieve global convergence of the systems state to any desired trajectory. In case no suitable velocity measurements are available, suitable observers will be introduced. Also adaptive control laws will be proposed, in case some parameters are unknown.

The systems considered are the forced Duffing equation, the forced van der Pol equation, and rigid robot systems. All proposed controllers satisfy input limitations, independent of the initial conditions, such that global trajectory tracking will be achieved.

Preface

September, 1995: There are some remaining courses I still have to complete, but there is one problem: How? Since I have been asked to participate in the *Onderwijs Visitatie Commissie*, I have very little spare time to study. Three days a week in some city, visiting its University, or to be more precise the Department of Mathematics of that University, and that does not even include the time needed for preparation! What should I do? Waste two months of study? I prefer not to.
Fortunately, a solution has been found: I will start with my MSc-project: Adaptive and robust control of chaotic systems. Since I first have to read about the subject of controlling chaotic systems, I can use the few days left to get familiar with the subject. The remaining days, during the visitations, can be used for reflection. My only order is: *Just think about it.*
Well, that worked. While taking a shower at my parents home on a Saturday morning, the idea of a composite controller has been borne. Exploiting that idea takes a lot of time, and it has already become December. I was supposed to pick up my study in November but my advisor, Henk Nijmeijer, will soon leave for three months to Australia (as I beforehand knew). What to do now?
We decided that I should go on with my project, and use the three months of absence of my advisor to complete my study. Easy said, easy done.

April 1996: My advisor is back and I (almost) completed my study. The results of my Msc-project have been gathered in this report. I want to thank Henk Nijmeijer for all the advices he gave, and the attention he paid to me. I hope reading this report, will enjoy you as much as I enjoyed my conversations (face to face, or by email) with him.

Erjen Lefeber

Contents

Chapter 1

Introduction

Recently, much theoretical and experimental work has been done on the use of control theory to improve the performance of a system that behaves chaotically in the absence of control. This chapter will briefly introduce this subject. The chaotic systems considered in this thesis will be described.

1.1 Chaotic systems

Chaotic systems are characterised, amongst others, by their extreme sensitivity to small perturbations in their initial conditions. This hypersensitivity is called the butterfly effect, an allusion to the suggestion that a butterfly fluttering its wings somewhere in, say, Canberra can influence the weather in the Netherlands a few days later. In mathematical terms, two trajectories that appear to start from the same initial conditions, will diverge, due to some tiny, impossible-to-measure difference. Thus, even if we know the equations that describe the behaviour of a chaotic system, we are unable to predict a trajectory for any significant length of time.
Two important paradigms for the study of chaos are the forced Duffing and van der Pol systems.

1.1.1 The Duffing system

In the mass-spring mechanical system, shown in figure 1.1, we consider a mass m sliding on a horizontal surface and attached to a vertical surface by a spring.

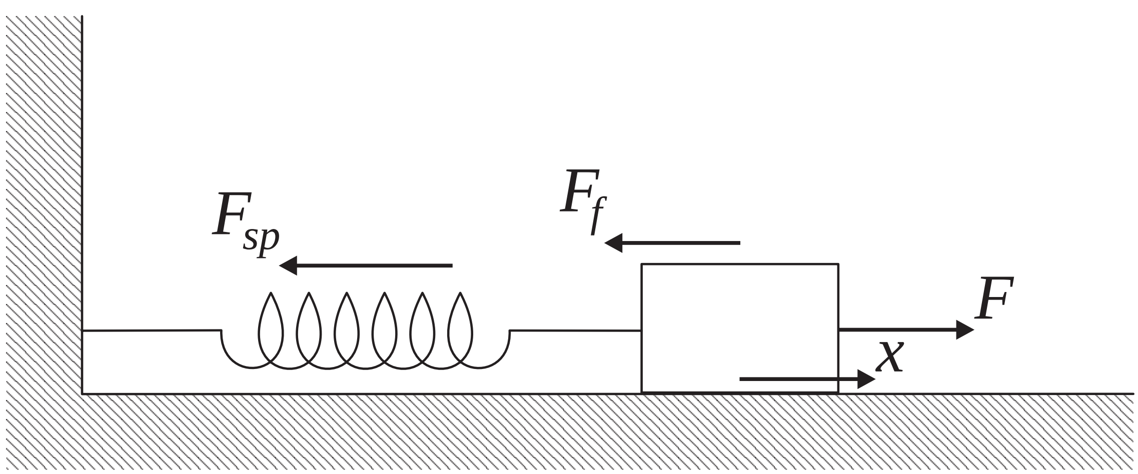

Figure 1.1: Mass-spring mechanical system

An external force F is applied to the mass. Denoting the resistive force due to friction as F_f and the restoring force of the spring as F_{sp} we can write this system as

$$m\ddot{x} + F_f + F_{sp} = F$$

For a relatively small displacement, the restoring force of the spring can be modeled as a linear function $F_{sp} = kx$, where k is the spring constant. For a large displacement however, the restoring force may depend nonlinearly on x. The function

$$F_{sp} = k(1 + a^2x^2)x$$

for example models the so-called hardening spring, where, beyond a certain displacement, a small displacement increment produces a large force increment.
As the mass moves in the air, a viscous medium, there will be a frictional force due to the viscosity. For small velocity, we can assume that $F_f = c\dot{x}$.
The combination of a hardening spring, linear viscous damping, and a periodic external force $F = A\cos\omega t$ results in the forced Duffing equation

$$m\ddot{x} + c\dot{x} + kx + ka^2x^3 = A\cos\omega t \tag{1.1}$$

This equation was first studied in 1918 by Duffing and can also be used to describe a pendulum moving in a viscous medium.
In order to describe the dynamics of a buckled beam which is set in a nonuniformed field of two fixed permanent magnets, when only one mode of vibration is considered, we can use the model

$$\ddot{x} + p\dot{x} - x + x^3 = q\cos\omega t$$

with $p > 0$ and q some constants, which is of a similar structure as (1.1).
For certain parameters both systems exhibit chaotic behaviour. Since in this thesis we consider the problem of controlling chaotic systems, we consider a controlled version of the forced Duffing equation:

$$\ddot{x} + p\dot{x} + p_1x + x^3 = u + q\cos\omega t \tag{1.2}$$

where $p > 0$, p_1, q, and ω are (un)known constants and $u(\cdot)$ is the, physically realisable, control input. Notice that (1.1) can be written this way by rescaling time as well as state.
In case of no control ($u \equiv 0$) some typical periodic and chaotic solutions of (1.2) are [4] $p = 0.4$, $p_1 = -1.1$, $\omega = 1.8$ and

1. $q = 0.620$ (period 1)
2. $q = 1.498$ (period 2)
3. $q = 1.800$ (chaotic)
4. $q = 2.100$ (chaotic)

1.1.2 The van der Pol system

In the electrical circuit, shown in figure 1.2, we assume the inductor and capacitor to be linear, time-invariant and passive, that is, $L > 0$ and $C > 0$. The resistive element is an active circuit characterized by the voltage-controlled i-v characteristic $i = h(v)$.
Using Kirchhoff's current law, we obtain $i_C + i_L + i = 0$, resulting in

$$C\frac{dv}{dt} + \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau + h(v) = 0$$

Differentiating with respect to t and rescaling the time-variable, results in :

$$\ddot{v} + \sqrt{L/C}h'(v)\dot{v} + v = 0$$

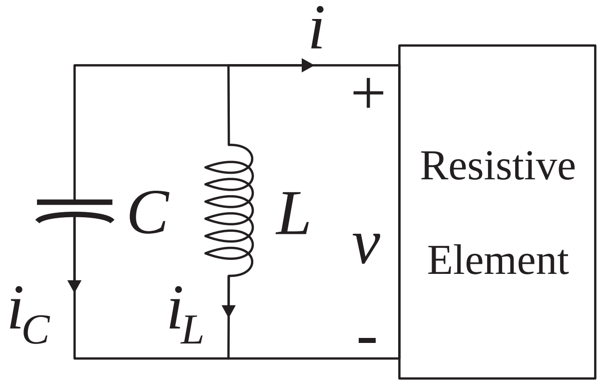

Figure 1.2: Electrical circuit

In case $h(v) = -v + \frac{1}{3}v^3$ and a source of alternating voltage is added to the circuit, the circuit equation takes the form

$$\ddot{v} - \mu(1 - v^2)\dot{v} + v = q\cos\omega t$$

with $\mu > 0$, which is known as the forced van der Pol oscillator. Van der Pol in 1922 used this equation to model an electrical circuit with a triode valve.
This equation also exhibits chaotic behaviour. According to [13] using the parameters $\mu = 5$, $q = 5$, and $\omega = 2.463$ results in chaotic behaviour.
Since in this thesis we consider the problem of controlling chaotic systems, we consider a controlled version of the forced van der Pol oscillator:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = u + q\cos\omega t \tag{1.3}$$

where $\mu > 0$, q, and ω are (un)known constants and $u(\cdot)$ is the, physically realisable, control input.

1.2 Control of chaos

The research on controlling chaos started with an article of Ott, Grebogi and Yorke [12] in which they described a general method to control a nonlinear system by stabilizing one of the unstable periodic orbits embedded in its chaotic attractor, via small time-dependent perturbations of a variable system parameter. Their idea, in short, is to wait until the chaotic system approaches the desired orbit. When the system is close enough, we change one of the accessable parameters for one period of the chosen orbit and the system settles onto the desired orbit. When the system inevitably starts to stray from this orbit, we have to repeat the process.
All literature on controlling chaos can roughly be devided into three categories:

OGY-method The above mentioned idea of Ott, Grebogi and York has been extended and applied to laser systems, electrical circuits, chemical reactions and many other systems. Another method based on a similar idea and worth to be mentioned is the Occasional Proportional Feedback (OPF) method.

External perturbation By injecting external signals to the system, independent of the state or even structure of the system, it is possible to force a chaotic system to perform in a desired way. Two main ideas are entrainment (a special chosen feedforward controller) and weak periodic perturbation.

Feedback Also external signals are injected to the system but dependent of the state and structure of the system.

A more extensive overview is presented in [8]. It contains a review on the literature in controlling chaos. Two other reviews on this subject are [3, 11]. It is worth to mention that there exists an extensive bibliography on this subject which can be found on internet [2].
This thesis belongs to the third category: feedback. A lot of interesting work on this subject has been done by Chen and Dong (e.g. [3, 4]), who were able to prove (global) asymptotic stability of their proposed feedbacks using Lyapunov theory. In continuation of their results, Nijmeijer and Berghuis [10] used robotic control ideas to control Duffing's equation not only towards a fixed point or periodic orbit of the system but towards *any* desired trajectory $x_d(t)$ in $\mathcal{C}^2$. Their results also include a robust system-parameter-independent controller-observer, that guarantees practical stability towards the desired trajectory x_d.

1.3 About this thesis

This thesis is an attempt to extend the results of Nijmeijer and Berghuis in connection with input constraints:

$$|u(t)| \leq u_{max} \qquad t \geq 0 \tag{1.4}$$

i.e. we consider the problem of developing a feedback control law $u(t)$, satisfying (1.4), for the systems (1.2) and (1.3) such that the resulting trajectory $x(t)$ will follow any desired trajectory $x_d(t)$ in $\mathcal{C}^2$, i.e.

$$\lim_{t \to \infty} |x(t) - x_d(t)| = 0$$

Furthermore, in [10], Nijmeijer and Berghuis proposed a robust system-parameter-independent controller-observer which guarantees that the tracking errors tend towards a closed region around zero, under a high-gain assumption. Drawbacks of high-gain feedback are noice amplification and large control efforts. Therefore, in this thesis also adaptive controllers are included.
In order to achieve global convergence of the state to any desired trajectory, the concept of a composite control is introduced. In case locally stable results are available, the problem of finding a globally stable control law can in most cases be reduced to finding a globally ultimately uniformly bounded first phase controller.
Finally, the notion of composite control will also be used in the tracking control problem for rigid robots.

Chapter 2

Preliminaries

This chapter introduces the basic definitions and results, to be used in this theses. The greater part of its contents has been taken from a similar chapter of Sastry & Bodson [14].

2.1 Stability of systems

This section is concerned with differential equations of the form

$$\dot{x} = f(t, x) \qquad x(t_0) = x_0 \tag{2.1}$$

where $x \in I\!R^n$, $t \geq 0$.

Definition 2.1.1 The system defined by (2.1) is said to be **autonomous** or **time-invariant** if f does not depend on t, and **non-autonomous** or **time-varying**, otherwise.

We define by B_h the closed ball of radius h centered at 0 in $I\!R^n$. Properties will be said to be true

locally if true for all x_0 in some ball B_h.

globally if true for all $x_0 \in I\!R^n$.

in any closed ball if true for all $x_0 \in B_h$ with h arbitrary.

uniformly if true for all $t_0 \geq 0$.

Definition 2.1.2 x is called an **equilibrium point** of (2.1) if $f(t, x) = 0$ for all $t \geq 0$.

If there exists an equilibrium point x_0, we can translate it to the origin. This is of great notational help, and we will assume henceforth that 0 is an equilibrium point or (2.1).

Definition 2.1.3 The equilibrium point $x = 0$ is called a **stable** equilibrium point of (2.1), if, for all $t_s \geq 0$ and $\epsilon > 0$ there exists $\delta(t_0, \epsilon)$ such that

$$|x_0| < \delta(t_0, \epsilon) \Rightarrow |x(t)| < \epsilon \qquad \forall t \geq t_0.$$

where $x(t)$ is the solution of (2.1) starting from x_0 at t_0.

Definition 2.1.4 The equilibrium point $x = 0$ is called a **uniformly stable** equilibrium point of (2.1) if, in the preceding definition, δ can be chosen independent of t_0.

Definition 2.1.5 The equilibrium point $x = 0$ is called an **asymptotically stable** equilibrium point of (2.1) if

1. $x = 0$ is a stable equilibrium point of (2.1)

2. $x = 0$ is attractive, that is, for all $t_0 \geq 0$ there exists $\delta(t_0)$ such that

$$|x_0| < \delta \Rightarrow \lim_{t\to\infty} |x(t)| = 0$$

Definition 2.1.6 The equilibrium point $x = 0$ is called an **uniformly asymptotically stable** equilibrium point of (2.1) if

1. $x = 0$ is a uniformly stable equilibrium point of (2.1)
2. There exists $\delta > 0$ and a function $\gamma(\tau, x_0) : I\!R_+ \times I\!R^n \mapsto I\!R_+$, such that $\lim_{t\to\infty} \gamma(\tau, x_0) = 0$ for all x_0 and
$$|x_0| < \delta \Rightarrow |x(t)| \leq \gamma(t - t_0, x_0) \qquad \forall t \geq 0$$

Definition 2.1.7 The equilibrium point $x = 0$ is called a **globally asymptotically stable** equilibrium point of (2.1), if it is asymptotically stable and $\lim_{t\to\infty} |x(t)| = 0$ for all $x_0 \in I\!R^n$. **Globally uniformly asymptotically stability** is defined likewise.

Definition 2.1.8 The equilibrium point $x = 0$ is called an **exponentionally stable** equilibrium point of (2.1) if there exist m, $\alpha > 0$ such that the solution $x(t)$ satisfies

$$|x(t)| \leq me^{-\alpha(t-t_0)}|x_0| \tag{2.2}$$

for all $x_0 \in B_h$, $t \geq t_0 \geq 0$.

Definition 2.1.9 **Global exponential stability** means that (2.2) is satisfied. for any $x_0 \in I\!R^n$

Definition 2.1.10 We say the system (2.1) is **ultimately bounded** if there is a $b > 0$ such that corresponding to each solution $x(t)$ of (2.1) there is a $T > 0$ with the property that $|x(t)| < b$ for all $t > T$.

2.2 Lyapunov stability theory

We now review some of the key concepts and results of Lyapunov stability theory for ordinary differential equations of the form (2.1). The method is basically a generalization of the idea that if some "measure of the energy" associated with a system is decreasing, then the system will tend to its equilibrium.

Definition 2.2.1 A function $\alpha(\cdot) : I\!R_+ \to I\!R_+$ **belongs to class** K (denote $\alpha(\cdot) \in K$), if it is continuous, strictly increasing, and $\alpha(0) = 0$.

Definition 2.2.2 A continuous function $v(\cdot,\cdot) : I\!R_+ \times I\!R^n \to I\!R_+$ is called a **locally positive definite function** if, for some $h > 0$, and some $\alpha(\cdot) \in K$

$$v(t, 0) = 0 \text{ and } v(t, x) \geq \alpha(|x|) \qquad \forall x \in B_h, \;\; t \geq 0$$

Definition 2.2.3 A continuous function $v(\cdot,\cdot) : I\!R_+ \times I\!R^n \to I\!R_+$ is called a **positive definite function** if, for some $\alpha(\cdot) \in K$

$$v(t, 0) = 0 \text{ and } v(t, x) \geq \alpha(|x|) \qquad \forall x \in I\!R^n, \;\; t \geq 0$$

and the function $\alpha(p) \to \infty$ as $p \to \infty$.

Definition 2.2.4 The function $v(\cdot,\cdot)$ is called **decrescent**, if there exists a function $\beta(\cdot) \in K$, such that

$$v(t, x) \leq \beta(||x||) \qquad \forall x \in B_h, \;\; t \geq 0$$

Theorem 2.2.5 *When $v(\cdot,\cdot)$ is a locally positive definite continuously differential function, and locally $\frac{dv}{dt}(t,x) \leq 0$, then $x = 0$ is a stable equilibrium point, where the derivative of v is taken along trajectories of (2.1), i.e.*

$$\frac{dv}{dt}(t,x) = \frac{\partial v(t,x)}{\partial t} + \frac{\partial v(t,x)}{\partial x} f(t,x)$$

Theorem 2.2.6 *When $v(\cdot,\cdot)$ is a* decrescent locally positive definite *continuously differential function, and locally $\frac{dv}{dt}(t,x) \leq 0$, then $x = 0$ is a* uniformly stable *equilibrium point, where the derivative of v is taken along trajectories of (2.1).*

Theorem 2.2.7 *When $v(\cdot,\cdot)$ is a* locally positive definite *continuously differential function, and $\frac{dv}{dt}(t,x)$ is a locally positive definite function, then $x = 0$ is an* asymptotically stable *equilibrium point, where the derivative of v is taken along trajectories of (2.1).*

Theorem 2.2.8 *When $v(\cdot,\cdot)$ is a* decrescent locally positive definite *continuously differential function, and $\frac{dv}{dt}(t,x)$ is a locally positive definite function, then $x = 0$ is a* uniformly asymptotically stable *equilibrium point, where the derivative of v is taken along trajectories of (2.1).*

Theorem 2.2.9 *When $v(\cdot,\cdot)$ is a* decrescent positive definite *continuously differential function, and $\frac{dv}{dt}(t,x)$ is a positive definite function, then $x = 0$ is a* globally uniformly asymptotically stable *equilibrium point, where the derivative of v is taken along trajectories of (2.1).*

In case $v(\cdot,\cdot)$ is a decrescent positive definite continuously differential function, and $\frac{dv}{dt}(t,x)$ is a positive **semi**definite function there are two theorems that can be of great help.
In case of an autonomous system

$$\dot{x} = f(x) \qquad f(0) = 0 \tag{2.3}$$

La Salle's theorem is often used.

Theorem 2.2.10 La Salle's Theorem *Consider the autonomous system (2.3). If there exists a radially unbounded positive definite differentiable function $V(x)$ such that*

$$\frac{dV}{dt}(x) = \frac{\partial V(x)}{\partial t} + \frac{\partial V(x)}{\partial x} f(x) \leq 0$$

Then every trajectory of (2.3) converges to the largest invariant subset of (2.3) in

$$\Omega = \{x \in I\!R^n | \dot{V}(x) = 0\}.$$

For non-autonomous systems, a useful result is Barbalat's Lemma:

Theorem 2.2.11 Barbalat's Lemma *If $f(t)$ is a uniformly continuous function, such that $\lim_{t\to\infty} \int_0^t f(\zeta)d\zeta$ exists and is finite, then $f(t) \to 0$ as $t \to \infty$.*

2.3 Rigid robot systems

In this thesis we will study the dynamics of a serial n-link rigid robot manipulator, that can be descibed by [1]

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau \tag{2.4}$$

where q is the $n \times 1$ vector of joint displacements, τ is the $n \times 1$ vector of applied joint torques, $M(q)$ is the $n \times n$ symmetric positive definite manipulator inertia matrix, $C(q,\dot{q})\dot{q}$ is the $n \times 1$ vector of centripetal and Coriolis torques, and $g(q)$ is the $n \times 1$ vector of gravitational torques obtained as the gradient of the robot potential energy $\mathcal{U}(q)$. We assume that the links are connected with revolute joints.
This system has many properties, which can be found in [1] and [6]. Amongst others, we mention

Property 2.3.1 The time derivative of the inertia matrix, and the centripetal and Coriolis matrix satisfy:

$$x^T \left[\frac{1}{2}\dot{M}(q) - C(q,\dot{q})\right] x = 0 \qquad \forall x \in I\!R^n$$

Property 2.3.2 The matrices $M(q)$, $C(q,\dot{q})$ and $g(q)$ are bounded with respect to q:

$$\begin{aligned} 0 < M_m \leq ||M(q)|| \leq M_M & \quad \text{for all} \quad q \in I\!R^n \\ ||C(q,x)|| \leq C_M ||x|| & \quad \text{for all} \quad q,x \in I\!R^n \\ ||g(q)|| \leq g_M & \quad \text{for all} \quad q \in I\!R^n \end{aligned}$$

Property 2.3.3 The Coriolis matrix satisfies $C(x,y)z = C(x,z)y$.

Property 2.3.4 There exists a constant k_G such that the gravitational torque vector $g(q)$ satisfies:

$$||g(x) - g(y)|| \leq k_G ||x - y||$$

Property 2.3.5 There exists a reparametrization of all unknown parameters into a parameter vector $\theta \in I\!R^p$ that enters linearly in the system dynamics (2.4). Therefore the following holds

$$M(u,\theta)x + C(u,v,\theta)w + g(q,\theta) \equiv M_0(u)x + C_0(u,v)w + g_0(q) + Y(u,v,w,x)\theta$$

where $M_0(\cdot)$, $C_0(\cdot)$, $g_0(\cdot)$ represent the known part of the system dynamics, and $Y(u,v,w,x)$ is a $n \times p$ regressor matrix that contains nonlinear but known functions.

2.4 Other results

This section contains two results, that are also important for this thesis.

2.4.1 The class $\mathcal{F}$

Let $\mathcal{F}$ denote the class of non-decreasing continuous differentiable functions $f : I\!R \to I\!R$ satisfying $f(0) = 0$, $f'(0) > 0$ and $\max_{x \in I\!R} |f(x)| \leq 1$.
Some examples of functions $f(x) \in \mathcal{F}$ are $f(x) = \tanh(x)$, $f(x) = \frac{2}{\pi}\arctan(x)$ and

$$f(x) = \begin{cases} -1 & \text{for } x \leq -1 \\ x & \text{for } |x| \leq 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

Furthermore if $f, g \in \mathcal{F}$ we have for all $\alpha, \beta, \lambda, \mu > 0$, $0 < \alpha + \beta \leq 1$:

$$\alpha f(\lambda x) + \beta g(\mu x) \in \mathcal{F}$$

Some other properties of $f \in \mathcal{F}$ are:

$$\begin{aligned} f(0) &= 0 \\ x f(x) &> 0 \quad \forall x \neq 0 \\ F(0) &= 0 \\ F(x) &> 0 \quad \forall x \neq 0 \end{aligned}$$

where $F(x) = \int_0^x f(\zeta) d\zeta$ denotes the primitive of $f(x)$.
This class is a subset of the class of saturating functions proposed by Sontag [5].

2.4.2 An adaptive controller

Consider the system

$$\ddot{x} + \sum_{i=1}^{m} \vartheta_i f_i(x,\dot{x},t) + \sum_{i=m+1}^{n} \theta_i f_i(x,\dot{x},t) = u \tag{2.5}$$

with $f_i(x,\dot{x},t)$ $i = 1,\dots,n$ denoting some known functions, ϑ_i $i = 1,\dots,m$ are known constants, and θ_i $i = m+1,\dots,n$ are unknown constants.
We consider the problem of controlling this system towards any desired trajectory $x_d \in \mathcal{C}^2$.
The control law

$$u = \ddot{x}_d - K_d\dot{e} - K_p e + \sum_{i=1}^{m} \vartheta_i f_i(x,\dot{x},t) + \sum_{i=m+1}^{n} \hat{\theta}_i f_i(x,\dot{x},t) \tag{2.6}$$

where $e \equiv x - x_d$ denotes the tracking error, $K_p > 0$ and $K_d > 0$ are some constants and $\hat{\theta}_i$ estimates for θ_i given by the adaption law:

$$\begin{pmatrix} \dot{\hat{\theta}}_{m+1} \\ \vdots \\ \dot{\hat{\theta}}_n \end{pmatrix} = -\Gamma \begin{pmatrix} f_{m+1}(x,\dot{x},t) \\ \vdots \\ f_n(x,\dot{x},t) \end{pmatrix} (\dot{e} + \lambda e) \tag{2.7}$$

where Γ is a $n-m \times n-m$ positive definite symmetric matrix and $0 < \lambda < K_d$ a constant, results in the closed-loop system

$$\begin{aligned} \ddot{e} + K_d\dot{e} + K_p e + \sum_{i=m+1}^{n} \tilde{\theta}_i f_i(x,\dot{x},t) &= 0 \\ \begin{pmatrix} \dot{\tilde{\theta}}_{m+1} \\ \vdots \\ \dot{\tilde{\theta}}_n \end{pmatrix} &= -\Gamma \begin{pmatrix} f_{m+1}(x,\dot{x},t) \\ \vdots \\ f_n(x,\dot{x},t) \end{pmatrix} (\dot{e} + \lambda e) \end{aligned} \tag{2.8}$$

where $\tilde{\theta}_i \equiv \theta_i - \hat{\theta}_i$ denotes the estimation error.

Proposition 2.4.1 *Consider the system (2.5) with the controller (2.6) and parameter estimation update law (2.7). Then the closed-loop system (2.8) is globally asymptotically stable with respect to e i.e.*

$$\lim_{t\to\infty} e(t) = 0$$

In case all f_i are bounded, we also obtain:

$$\lim_{t\to\infty} \dot{e}(t) = 0$$

Proof For simplicity we define

$$\tilde{\Theta} = \begin{pmatrix} \tilde{\theta}_{m+1} \\ \vdots \\ \tilde{\theta}_n \end{pmatrix}$$

Consider the radially unbounded Lyapunov function candidate

$$V(e,\dot{e},\tilde{\Theta}) = \frac{1}{2}(\dot{e} + \lambda e)^2 + \frac{1}{2}[\lambda(K_d - \lambda) + K_p]e^2 + \frac{1}{2}\tilde{\Theta}^T\Gamma^{-1}\tilde{\Theta}$$

which is positive definite. Differentiating along solutions of (2.8) results in

$$\dot{V}(e,\dot{e},\tilde{\Theta}) = -(K_d - \lambda)\dot{e}^2 - K_p\lambda e^2$$

which is negative semidefinite in the error state $(e, \dot{e}, \tilde{\Theta})$. To prove that $\lim_{t\to\infty} e(t) = 0$ we may not use LaSalle's theorem, since the closed-loop system (2.8) is time-varying. We can use Barbalat's Lemma instead. Since $V(t)$ is bounded we have that both $e(t)$ and $\dot{e}(t)$ are bounded, and therefore $e(t)$ is uniformly continuous. Furthermore for all $t \geq 0$:

$$0 \leq \int_0^t e^2(\zeta)d\zeta \leq \int_0^t e^2(\zeta) + \frac{K_d - \lambda}{K_p \lambda}\dot{e}^2(\zeta)d\zeta = -\int_0^t \frac{1}{K_p\lambda}\dot{V}(\zeta)d\zeta = \frac{1}{K_p\lambda}(V(0) - V(t)) \leq \frac{1}{K_p\lambda}V(0)$$

Using Barbalat's Lemma, if follows that $\lim_{t\to\infty} e(t) = 0$.
In case all f_i are bounded, we obtain, using (2.8), $\ddot{e}(t)$ is bounded, and therefore $\dot{e}(t)$ is uniformly continuous. Using

$$0 \leq \int_0^t \dot{e}^2(\zeta)d\zeta \leq \frac{1}{K_d - \lambda}V(0)$$

and Barbalat's Lemma, we obtain $\lim_{t\to\infty} \dot{e}(t) = 0$. ■

Chapter 3

Duffing's equation

In this chapter we study the controlled version of the forced Duffing equation (1.2):

$$\ddot{x} + p\dot{x} + p_1 x + x^3 = u + q\cos(\omega t), \tag{3.1}$$

where $p > 0$, p_1, q, and ω are constant system-parameters and $u(\cdot)$ is a control function or input. In [10], Nijmeijer and Berghuis studied the control of (3.1), and the results include a stabilizing observer-controller combination as well as a practically stabilizing system-parameter independent output feedback controller. In this chapter we want to extend those results dealing with input limitations, where we restrict ourselves to the case $p > 0$.

$$|u(t)| \leq u_{max} \qquad t \geq 0 \tag{3.2}$$

In order to achieve globally stabilizing results dealing with input limitations we introduce the concept of a composite feedback control. Furthermore, adaptive controllers will be introduced in case one or more system-parameters (except ω) are unknown, and compared to the robust controllers of Nijmeijer and Berghuis.

3.1 On Lyapunov control of the Duffing equation (I)

This section contains the results presented in [10].

Proposition 3.1.1 *Consider the system (3.1) together with the control law*

$$u = \ddot{x}_d + p\dot{x}_d + p_1 x_d + x_d^3 - q\cos(\omega t) - K_d\dot{e} - K_p e + 3xx_d e \tag{3.3}$$

where $e \equiv x - x_d$ denotes the tracking error, $K_d > -p$ and $K_p > -p_1$. Then the resulting closed-loop system is asymptotically stable.

Proposition 3.1.2 *Consider the system (3.1) together with the control law*

$$u = \ddot{x}_d + p\dot{x}_d + p_1 x_d + x_d^3 - q\cos(\omega t) - K_d\dot{\hat{e}} - K_p\hat{e} + 3xx_d e \tag{3.4}$$

where $\hat{e}$ is generated from the auxiliary system

$$\begin{array}{rcl} \dot{\hat{e}} & = & w + 2K_d(e - \hat{e}) - pe \\ \dot{w} & = & 2K_p(e - \hat{e}) - p_1 e - e^3 \end{array} \tag{3.5}$$

Then the resulting closed-loop system is asymptotically stable under the condition that $K_p = \lambda K_d$ and $0 < \lambda < \max\{K_d, p + K_d\}$.

Proposition 3.1.3 *Consider the system (3.1) under robust PD-feedback*

$$u = -K_d\dot{e} - K_p e \tag{3.6}$$

Then the resulting closed-loop dynamics are locally uniformly ultimately bounded for K_d sufficiently large.

Proposition 3.1.4 *Consider the system (3.1) under robust PD-feedback*

$$u = -K_d\dot{\hat{e}} - K_p\hat{e} \tag{3.7}$$

where $\hat{e}$ is generated from the auxiliary system

$$\begin{array}{rcl} \dot{\hat{e}} & = & w + 2K_d(e - \hat{e}) \\ \dot{w} & = & 2K_p(e - \hat{e}) \end{array} \tag{3.8}$$

Then the resulting closed-loop dynamics are locally uniformly ultimately bounded for K_d sufficiently large.

3.2 On Lyapunov control of the Duffing equation (II)

Although Nijmeijer and Berghuis already considered the problem of controlling Duffing's equation to any desired trajectory $x_d(t) \in \mathcal{C}^2$, even in case $p \leq 0$, we propose a slightly different control law, with a view to the extension of considering input limitations. Furthermore, their second order observer can be simplified to a first order auxiliary system.
Consider the control law

$$u = \ddot{x}_d + p\dot{x}_d + p_1 x + x_d^3 + 3xx_d e - K_p f_1(e) - K_d f_2(\dot{e}) - q\cos(\omega t) \tag{3.9}$$

where $e \equiv x - x_d$ denotes the tracking error, $K_p \geq 0$ and $K_d \geq 0$ are constants and $f_1, f_2 \in \mathcal{F}$. This controller results in the closed-loop system

$$\ddot{e} + p\dot{e} + e^3 + K_p f_1(e) + K_d f_2(\dot{e}) = 0 \tag{3.10}$$

Proposition 3.2.1 *Consider the system (3.1) together with the control law (3.9). Then the closed-loop system (3.10) is globally asymptotically stable.*

Proof Consider the radially unbounded Lyapunov function candidate

$$V(e, \dot{e}) = \frac{1}{2}\dot{e}^2 + \frac{1}{4}e^4 + K_p F_1(e) \tag{3.11}$$

where

$$F_1(e) = \int_0^e f_1(\zeta)d\zeta$$

which is positive definite. Along solutions of (3.10) the time-derivative of (3.11) is:

$$\dot{V}(e, \dot{e}) = \dot{e}[-p\dot{e} - e^3 - K_p f_1(e) - K_d f_2(\dot{e})] + e^3\dot{e} + K_p f_1(e)\dot{e} = -p\dot{e}^2 - K_d\dot{e}f_2(\dot{e})$$

which is negative semidefinite in the error state $(e, \dot{e})$.
To demonstrate global asymptotic stability, we use LaSalle's theorem (Theorem 2.2.10). To this end, let us define the set Ω as

$$\Omega = \{(e, \dot{e}) \in I\!R^2 | \dot{V}(e, \dot{e}) = 0\} = \{(e, \dot{e}) \in I\!R^2 | \dot{e} = 0\}$$

We now have to determine the largest invariant set in Ω with respect to (3.10). For $(e, \dot{e}) \in \Omega$ we obtain for (3.10):

$$e^3 + K_p f_1(e) = 0$$

from which it follows that the only solution is $e = 0$. Therefore, the origin is the largest invariant set in Ω with respect to (3.10) and from LaSalle's theorem it follows that the origin is globally asymptotically stable. ■

In case we are unable to measure the velocity $\dot{x}$ we have to develop an auxiliary system, to correct for the lack of velocity measurement. In the light of the previous proposition we consider the control law

$$\ddot{x}_d + p\dot{x}_d + p_1 x + x_d^3 + 3xx_d e - K_p f_1(e) - K_d f_2(w) - q\cos(\omega t) \tag{3.12}$$

where $K_p \geq 0$ and $K_d > 0$ are constants, $f_1, f_2 \in \mathcal{F}$ and w generated from the auxiliary system

$$\dot{w} = \dot{e} - L_p w \tag{3.13}$$

where $L_p > 0$ is a constant. This control law results in the closed-loop system

$$\begin{aligned} \ddot{e} + p\dot{e} + e^3 + K_p f_1(e) + K_d f_2(w) = 0 \\ \dot{w} = \dot{e} - L_p w \end{aligned} \tag{3.14}$$

Lemma 3.2.2 *Consider the system (3.1) together with the control law (3.12) and auxiliary system (3.13). Then the closed-loop system (3.14) is globally asymptotically stable.*

Proof Consider the radially unbounded Lyapunov function candidate

$$V(e, \dot{e}, w) = \frac{1}{2}\dot{e}^2 + \frac{1}{4}e^4 + K_p F_1(e) + K_d F_2(w) \tag{3.15}$$

where

$$F_1(e) = \int_0^e f_1(\zeta)d\zeta \text{ and } F_2(w) = \int_0^w f_2(\zeta)d\zeta$$

which is positive definite. Along solutions of (3.14) its time-derivative is:

$$\dot{V}(e, \dot{e}, w) = -p\dot{e}^2 - K_d L_p w f_2(w)$$

which is negative semidefinite in the state $(e, \dot{e}, w)$.
To demonstrate global asymptotic stability, we again use LaSalle's theorem. To this end, let us define the set Ω as

$$\Omega = \{(e, \dot{e}, w) \in I\!R^2 | \dot{V}(e, \dot{e}, w) = 0\} = \{(e, \dot{e}, w) \in I\!R^2 | \dot{e} = 0\}$$

We now have to determine the largest invariant set in Ω with respect to (3.14). For $(e, \dot{e}, w) \in \Omega$ we obtain for (3.14):

$$e^3 + K_p f_1(e) = 0$$

from which it follows that the only solution is $e = 0$. Therefore, the origin is the largest invariant set in Ω with respect to (3.14) and from LaSalle's theorem it follows that the origin is globally asymptotically stable. ■

We developed the control law (3.12) with auxiliary system (3.13) in order to control the system (3.1) without using measurements of $\dot{x}$ but the auxiliary system (3.13) contains $\dot{e}$ instead. How to overcome this problem?
Define the signal z via $w = e - L_p z$, where z is generated from an auxiliary system. From (3.13) we deduce:

$$\dot{z} = e - L_p z$$

Therefore

Proposition 3.2.3 *Consider the system (3.1) together with the control law*

$$\ddot{x}_d + p\dot{x}_d + p_1 x + x_d^3 + 3xx_d e - K_p f_1(e) - K_d f_2(e - L_p z) - q\cos(\omega t) \tag{3.16}$$

where $K_p \geq 0$, $K_d > 0$ and $L_p > 0$ are constants, $f_1, f_2 \in \mathcal{F}$ and z generated from the auxiliary system

$$\dot{z} = e - L_p z \tag{3.17}$$

Then the resulting closed-loop system is globally asymptotically stable.

3.3 Composite control

As already mentioned in the introduction, the first results on controlling chaos appeared in 1990, presented by Ott, Grebogi and Yorke [12]. They introduced a method to convert a chaotic attractor to any one of a large number of possible attracting time-periodic motions by making only small time-dependent perturbations of an available system parameter. This is now often referred to as the OGY-method.
Consider the system

$$\dot{x}(t) = F(x(t), p)$$

where p is a system parameter available for external adjustment within a small range about some nominal value p_0, say $p_0 - p_* < p < p_0 + p_*$.
By introducing a transversal surface of section for system trajectories we describe the system by the map

$$\xi_{n+1} = P(\xi_n, p)$$

Assume that $\xi_F = P(\xi_F, p_0)$ is an unstable fixed point of the map P to which we want to stabilize the system. A first order approximation of P near ξ_F and p_0 is given by

$$\xi'_{n+1} = p_n g + (\lambda_u e_u f_u + \lambda_s e_s f_s)(\xi'_n - p_n g)$$

where $g = \partial \xi_F(p)/\partial p|_{p=p_0}$, λ_s and λ_u denote the stable and unstable eigenvalues of the surface of section map at the chosen fixed point, e_s and e_u denote its eigenvectors and f_s and f_u are its contravariant basisvectors defined by $f_s \cdot e_s = f_u \cdot e_u = 1, f_s \cdot e_u = f_u \cdot e_s = 0$. Since ξ'_{n+1} should fall on the stable manifold of ξ_F, we choose p_n such that $f_u \xi'_{n+1} = 0$, i.e.

$$p_n = \frac{\lambda_u \xi'_n f_u}{(\lambda_u - 1) g f_u}$$

only when this results in an allowable p_n. Otherwise we choose $p_n = p_0$.
In other words, we wait until the chaotic system approaches the desired orbit. Since the system is chaotic, it will eventually come as close to the desired orbit as desired, and that is when we change one of the accessible parameters for one period of the chosen orbit, to achieve that the system settles onto the desired orbit. However, when we do not apply the parameter change, the system will stray from the desired orbit, and we have to wait, until is comes close enough again.
Although this method has been applied in many practical systems as laser systems, electrical circuits, chemical reactions, etc. it does have some drawbacks.
For instance, the time at which the parameter change is applied is very crucial. Being a second late can lead to the opposite result: instead of coming closer to the desired orbit one could move away from it.
Secondly, the assumption is made that a certain parameter can be changed within a small range: $p_0 - p_* < p < p_0 + p_*$. In most systems parameters can not be changed that easily, since they represent material specific properties like masses, spring constants, capacities, resistances, etc. This does not imply the idea is not useful, since a way to achieve a certain 'mass-change' is to apply a force proportional to the current acceleration. But dealing with input limitations on the maximum force implies that the range in which one is able to 'change the mass' is dependent on the current acceleration (time-dependent).
A third drawback is that one has to wait until the chaotic system approaches the desired orbit. The chaotic behaviour of the system guarantees that the system eventually will come close enough but one has no idea when. It is for instance not guaranteed that the system will come closer and closer as time evolves. Furthermore, as soon as the system is close enough, we have to act quickly, since before we know, the system has strayed from the desired orbit.
A way to overcome these problems is using a composite feedback controller. During the waiting phase one applies a feedback control law in order to ensure the system will come closer and closer to a specific orbit. This orbit is not neccessarily the desired trajectory to track but at least close

to it. Once the system is close to this orbit, i.e. close enough to the desired trajectory to track, one switches to a second control law, that ensures the system will follow the desired trajectory.
To track a desired trajectory, feedforward control is needed. Since we consider the problem of following a desired trajectory using a bounded control in this thesis, it is reasonable to assume that the desired trajectory satisfies:

$$|x_d(t)| \leq B_0,\ |\dot{x}_d(t)| \leq B_1,\ |\ddot{x}_d(t)| \leq B_2 \qquad t \geq 0 \tag{3.18}$$

with B_0, B_1, and B_2 denoting some known bounds.
As mentioned before we will develop two control laws. The first controller, which is used in the waiting phase, has to make sure that the state comes close enough to a specific orbit, which is close enough to the desired trajectory to track. The freedom in choosing the specific orbit, close to the desired orbit, to which the first phase controller converges, is enormous. In this thesis we choose for that specific orbit mostly the origin, since it minimizes the tracking error $e = x - x_d$ in case we only know (3.18)
The second controller, which is used in the tracking phase, should ensure the system converges to the desired trajectory.
Looking at the control laws of the previous section, for example at (3.9), we see from the proof of convergence, in order to verify whether $u(t)$ satisfies a prescribed bound, it suffices we have a beforehand known bound on x and e, whereas only one suffices, since $e \equiv x - x_d$ and we already have a bound on x_d. From the Lyapunov function (3.11), using the knowledge that it is a decreasing function, we are able to deduce bounds on $e(t)$ (and therefore $x(t)$). The only disadvantage is that those bounds depend on the values of e and $\dot{e}$ at the moment the controller is initiated. However, if we are able to construct a globally ultimately uniformly bounded (GUUB) controller that satisfies the input limitations, we can use it as a first phase controller. This waiting phase controller ensures the system finally reaches a state in which both e and $\dot{e}$ are within prescibed bounds. As soon as we have reached that situation, we can switch to the second controller. We can now use our tracking controller to establish asymptotically tracking.
This is the key idea on which most results in this thesis have been based. The specific orbit to which the first phase controller steers the system will in most cases be the origin (a fixed point), because then it is easy to deduce the tracking errors.

3.4 Bounded controllers, using state measurements

In this section we consider the problem of tracking a desired trajectory $x_d(t) \in \mathcal{C}^2$ under input limitations

$$|u(t)| \leq u_{max} \qquad t \geq 0$$

presuming we are able to measure the full state $(x, \dot{x})$.
Since tracking any desired trajectory will be accomplished by (partially) feeding forward the behaviour of the desired trajectory we have to assume that

$$|x_d(t)| \leq B_0,\ |\dot{x}_d(t)| \leq B_1,\ |\ddot{x}_d(t)| \leq B_2 \qquad t \geq 0 \tag{3.19}$$

3.4.1 Trajectory tracking

In order to achieve global error-convergence under input limitations, we use a composite controller. In the waiting phase we control the state towards the origin. As the state is close enough to the origin, we switch to the tracking phase, using a second controller to achieve tracking.

Lemma 3.4.1 *Consider the system (3.1), together with the control law*

$$u = -q\cos\omega t - K_p f_1(x) - K_d f_2(\dot{x}) \tag{3.20}$$

where $K_p > 0$, $K_d \geq 0$ and $f_1, f_2 \in \mathcal{F}$.
Then for all $C_0 > \gamma$ and $C_1 > 0$ there exists a time $t_s \geq 0$ such that $|x(t)| \leq C_0$ and $|\dot{x}(t)| \leq C_1$ for all $t \geq t_s$ where $\gamma = 0$ for $p_1 \geq 0$ and $\gamma = \sqrt{-p_1}$ otherwise.

Proof The control law (3.20) results in the closed-loop system

$$\ddot{x} + p\dot{x} + p_1 x + x^3 + K_p f_1(x) + K_d f_2(\dot{x}) = 0 \tag{3.21}$$

Consider the radially unbounded Lyapunov-like function

$$V(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + \frac{1}{4}(x^2 + p_1)^2 + K_p F_1(x)$$

where

$$F_1(x) = \int_0^x f_1(\zeta) d\zeta$$

Differentiating along solutions of (3.21) yields

$$\dot{V}(x, \dot{x}) = -p\dot{x}^2 - K_d \dot{x} f_2(\dot{x})$$

which is negative semi-definite in the state $(x, \dot{x})$. Next, we have to determine the largest invariant set in $\{(x, \dot{x}) \in I\!R^2 | \dot{V}(x, \dot{x}) = 0\}$, which is $\{(x, \dot{x}) \in I\!R^2 | p_1 x + x^3 + K_p f_1(x) = 0\}$. Application of Theorem 2.2.10 completes the proof. ■

Corollary 3.4.2 *Consider the system (3.1). If*

$$u_{max} > |q|$$

then there exist K_p and K_d such that the control law (3.20) satisfies the constraint (3.2).

Proposition 3.4.3 *Consider the system (3.1). For all $C_0 > \gamma$ and $C_1 > 0$ there exists a switching time $t_s \geq 0$ such that the composite control law*

$$u = \begin{cases} -K_{p,1} f_1(x) - K_{d,1} f_2(\dot{x}) - q\cos(\omega t) & t < t_s \\ \ddot{x}_d + p\dot{x}_d + p_1 x + x_d^3 + 3x x_d e - K_{p,2} f_3(e) - K_{d,2} f_4(\dot{e}) - q\cos(\omega t) & t \geq t_s \end{cases} \tag{3.22}$$

where $\gamma = 0$ for $p_1 \geq 0$ and $\gamma = \sqrt{-p_1}$ otherwise, $K_{p,1} \geq 0$, $K_{d,1} \geq 0$, $K_{p,2} \geq 0$, and $K_{d,2} \geq 0$ are constants, and $f_1, f_2, f_3, f_4 \in \mathcal{F}$ results in a globally asymptotically stable closed-loop system

Proof Let t_s be a moment that both $|x(t_s)| \leq C_0$ and $|\dot{x}(t_s)| \leq C_1$. Lemma 3.4.1 showed the existence of t_s. In Proposition 3.2.1 we already showed that the tracking phase controller results in a globally asymptotically closed-loop system. ■

Corollary 3.4.4 *For all $C_0 > \gamma$ and $C_1 > 0$ there exists a $\beta > 0$ such that the composite controller (3.22) satisfies*

$$|u(t)| \leq \beta \qquad t \geq 0$$

where $\gamma = 0$ for $p_1 \geq 0$ and $\gamma = \sqrt{-p_1}$ otherwise.

Proof Proposition 3.4.3 gives us a switching time $t_s \geq 0$ such that $|x(t_s)| \leq C_0$ and $|\dot{x}(t_s)| \leq C_1$. Using (3.19), we obtain $|e(t_s)| \leq B_0 + C_0$ and $|\dot{e}(t_s)| \leq B_1 + C_1$. From the proof of Proposition 3.2.1 we know that the Lyapunov function (3.11) is a decreasing function for $t \geq t_s$, i.e.

$$\frac{1}{2}\dot{e}(t)^2 + \frac{1}{4}e(t)^4 + K_{p,2}F_3(e(t)) \leq \frac{1}{2}\dot{e}(t_s)^2 + \frac{1}{4}e(t_s)^4 + K_{p,2}F_3(e(t_s)) \qquad \forall t \geq t_s$$

which implies that for all $t \geq t_s$:

$$\begin{aligned} |e(t)| &\leq \sqrt[4]{2(B_1 + C_1)^2 + (B_0 + C_0)^4 + 2K_p F_1(B_0 + C_0)} \\ |\dot{e}(t)| &\leq \sqrt{(B_1 + C_1)^2 + \frac{1}{2}(B_0 + C_0)^4 + K_p F_1(B_0 + C_0)} \end{aligned}$$

Therefore

$$|u(t)| \leq \phi(p, p_1, q, B_0, B_1, B_2, C_0, C_1, K_{p,2}, K_{d,2}) \qquad \forall t \geq t_s$$

where

$$\begin{aligned}\phi(\cdot) &= B_2 + pB_1 + |p_1| \left(B_0 + \sqrt[4]{2(B_1 + C_1)^2 + (B_0 + C_0)^4 + 2K_{p,2}F_3(B_0 + C_0)} \right) + \\ & B_0^3 + 3 \left(B_0 + \sqrt[4]{2(B_1 + C_1)^2 + (B_0 + C_0)^4 + 2K_{p,2}F_3(B_0 + C_0)} \right) B_0 \cdot \\ & \cdot \left(\sqrt[4]{2(B_1 + C_1)^2 + (B_0 + C_0)^4 + 2K_{p,2}F_3(B_0 + C_0)} \right) + K_{p,2} + K_{d,2} + |q|\end{aligned}$$

When we consider the first phase controller, we obtain

$$|u(t)| \leq K_{p,1} + K_{d,1} + |q| \qquad \forall t < t_s$$

from which it is obvious that

$$\beta = \max\{K_{p,1} + K_{d,1} + |q|, \phi(p, p_1, q, B_0, B_1, B_2, C_0, C_1, K_{p,2}, K_{d,2})\}$$

suffices. ■

Corollary 3.4.5 *Consider the system (3.1). If*

$$\begin{aligned}u_{max} &> B_2 + pB_1 + |p_1| \left(B_0 + \sqrt[4]{2B_1^2 + (B_0 + \gamma)^4} \right) + B_0^3 + 3 \left(B_0 + \sqrt[4]{2B_1^2 + (B_0 + \gamma)^4} \right) \cdot \\ & \cdot B_0 \left(\sqrt[4]{2B_1^2 + (B_0 + \gamma)^4} \right) + |q|\end{aligned}$$

where $\gamma = 0$ if $p_1 \geq 0$ and $\gamma = \sqrt{-p_1}$ otherwise, then there exists a composite control law of the form (3.22) that satisfies the constraint (3.2).

Remark 3.4.6 In Corollary 3.4.4 we use a rather weak estimate in determining the value of β. Therefore, this estimate can be much more weakend in many cases. For example in case $p_1 \geq 0$ and the desired trajectory x_d is an uncontrolled trajectory (stable, or unstable!)

$$\ddot{x}_d + p\dot{x}_d + p_1 x_d + x_d^3 = q\cos(\omega t)$$

it is easy to see we are able to design a composite feedback controller that satisfies $|u(t)| \leq u_{max}$ for all $u_{max} > \max\{p_1 B_1, |q|\}$.
In practice β will only have to be a little greater than $|q|$ plus the effort needed to feedforward the desired trajectory, since, by choosing C_0, C_1, K_p and $K_{d,2}$ as small as possible, the contribution of $-p_1 e + 3xx_d e - K_p f_2(e) - K_{d,2} f_3(\dot{e})$ to the control effort reduces to about zero.

3.4.2 Adaptive trajectory tracking

In case ω is known, we know from Proposition 2.4.1 the control law

$$u = \ddot{x}_d - K_d\dot{e} - K_p e + x^3 + \hat{p}\dot{x} + \hat{p}_1 x + \hat{q}\cos(\omega t) \tag{3.23}$$

where $K_d > 0$, $K_p > 0$ are constants and $\hat{p}$, $\hat{p}_1$ and $\hat{q}$ estimates for p, p_1 and $-q$ given by:

$$\begin{pmatrix} \dot{\hat{p}} \\ \dot{\hat{p}}_1 \\ \dot{\hat{q}} \end{pmatrix} = -\Gamma \begin{pmatrix} \dot{x} \\ x \\ \cos(\omega t) \end{pmatrix} (\dot{e} + \lambda e)$$

where $0 < \lambda < K_d$ is a constant and Γ is a 3×3 positive definite symmetric matrix, results in a globally asymptotically stable closed-loop system.
Therefore, this control law is a good candidate for the tracking phase controller of a composite control law in case some of the parameters are unknown. We only have to find a suitable waiting phase controller. Furthermore, in order to assure the controller satisfies the constraint (3.2) we have to assume that we have bounds on the initial estimate errors of the unknown variables, i.e.

$$\begin{array}{rcl} \hat{p}(0) - p & \leq & E_p \\ \hat{p}_1(0) - p_1 & \leq & E_{p_1} \\ \hat{q}(0) - (-q) & \leq & E_q \end{array}$$

with E_p, E_{p_1}, and E_q some bounds.
Notice the control law (3.20) is independent of p and p_1. In case only p and p_1 are unknown this is a suitable waiting phase controller.

Proposition 3.4.7 *Consider the system (3.1). For all $C_0 > \gamma$ and $C_1 > 0$ there exists a switching time $t_s \geq 0$ such that the composite adaptive control law*

$$u = \left\{ \begin{array}{cc} -K_{p,1} f_1(x) - K_{d,1} f_2(\dot{x}) - q\cos(\omega t) & t < t_s \\ \ddot{x}_d - K_{p,2} e - K_{d,2}\dot{e} + x^3 + \hat{p}\dot{x} + \hat{p}_1 x - q\cos(\omega t) & t \geq t_s \end{array} \right. \tag{3.24}$$

where $\gamma = 0$ for $p_1 \geq 0$ and $\gamma = \sqrt{-p_1}$ otherwise, $K_{p,1} \geq 0$, $K_{d,1} \geq 0$, $K_{p,2} > 0$, and $K_{d,2} > 0$ are constants, $f \in \mathcal{F}$, and $\hat{p}$ and $\hat{p}_1$ estimates for p and p_1 given by:

$$\begin{pmatrix} \dot{\hat{p}} \\ \dot{\hat{p}}_1 \end{pmatrix} = -\Gamma \begin{pmatrix} \dot{x} \\ x \end{pmatrix} (\dot{e} + \lambda e)$$

where $0 < \lambda < K_{d,2}$ is a constant and Γ a 2×2 positive definite symmetric matrix with eigenvalues $\lambda_{min} \leq \lambda_{max}$, results in a globally asymptotically stable closed-loop system with respect to e and $\dot{e}$, i.e.

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} \dot{e}(t) = 0$$

Proof Let t_s be a moment that both $|x(t_s)| \leq C_0$ and $|\dot{x}(t)| \leq C_1$. Lemma 3.4.1 showed the existence of t_s. In Proposition 2.4.1 we already showed that the tracking phase controller results in a globally asymptotically closed-loop system. ■

Corollary 3.4.8 *For all $C_0 > \gamma$ and $C_1 > 0$ there exists a $\beta > 0$ such that the composite controller (3.24) satisfies*

$$|u(t)| \leq \beta \qquad t \geq 0$$

where $\gamma = 0$ for $p_1 \geq 0$ and $\gamma = \sqrt{-p_1}$ otherwise.

Proof Proposition 3.4.7 gives us a switching time $t_s \geq 0$ such that $|x(t_s)| \leq C_0$ and $|\dot{x}(t_s)| \leq C_1$. Using (3.19), we obtain $|e(t_s)| \leq B_0 + C_0$ and $|\dot{e}(t_s)| \leq B_1 + C_1$. From the proof of Proposition 2.4.1 we know

$$\frac{1}{2}(\dot{e}(t) + \lambda e(t))^2 + \frac{1}{2}[K_p + \lambda(K_d - \lambda)]e(t)^2 + \frac{1}{2}\begin{pmatrix} \tilde{p}(t) \\ \tilde{p}_1(t) \end{pmatrix}^T \Gamma^{-1} \begin{pmatrix} \tilde{p}(t) \\ \tilde{p}_1(t) \end{pmatrix}$$

is a non-increasing function for $t \geq t_s$. When we initiate our adaptive second phase controller with $\tilde{p}(t_s) \leq E_p$ and $\tilde{p}_1(t_s) \leq E_{p_1}$ we are able to determine a bound on the tracking phase controller. Using

$$(\dot{e} + \lambda e)^2 + \alpha e^2 = \dot{e}^2 + 2\lambda \dot{e} e + (\lambda^2 + \alpha)e^2 = (\frac{\lambda}{\sqrt{\lambda^2 + \alpha}}\,\dot{e} + \sqrt{\lambda^2 + \alpha}\; e)^2 + \frac{\alpha}{\lambda^2 + \alpha}\dot{e}^2$$

we obtain for all $t \geq t_s$:

$$\begin{aligned}
|e(t)| &\leq \sqrt{\frac{1}{\alpha}[(B_1+C_1)+\lambda(B_0+C_0)]^2+(B_0+C_0)^2+\frac{\lambda_{max}}{\alpha}(E_p^2+E_{p_1}^2)} \\
|\dot{e}(t)| &\leq \sqrt{\frac{1}{\alpha}[\lambda(B_1+C_1)+(\lambda^2+\alpha)(B_0+C_0)]^2+(B_1+C_1)^2+\frac{\lambda_{max}(\lambda^2+\alpha)}{\alpha}(E_p^2+E_{p_1}^2)} \\
|\tilde{p}(t)| &\leq \sqrt{\frac{1}{\lambda_{min}}[(B_1+C_1)+\lambda(B_0+C_0)]^2+\frac{\alpha}{\lambda_{min}}(B_0+C_0)^2+\frac{\lambda_{max}}{\lambda_{min}}(E_p^2+E_{p_1}^2)} \\
|\tilde{p}_1(t)| &\leq \sqrt{\frac{1}{\lambda_{min}}[(B_1+C_1)+\lambda(B_0+C_0)]^2+\frac{\alpha}{\lambda_{min}}(B_0+C_0)^2+\frac{\lambda_{max}}{\lambda_{min}}(E_p^2+E_{p_1}^2)}
\end{aligned}$$

where $\alpha = K_{p,2}+\lambda(K_{d,2}-\lambda)$. Notice, the tracking phase controller satisfies

$$|u(t)| \leq B_2+K_{d,2}+K_{p,2}+(E_p+|\tilde{p}(t)|)(B_1+|\dot{e}(t)|)+(E_{p_1}+|\tilde{p}_1(t)|)(B_0+|e(t)|)+|q|$$

So $|u(t)| \leq \phi(q, B_0, B_1, B_2, C_0, C_1, K_{p,2}, K_{d,2}, E_{p_1}, E_p)$ where

$$\begin{aligned}
\phi(\cdot) \;=\; & B_2+K_{d,2}+K_{p,2}+\Bigg(E_p+ \\
& +\sqrt{\frac{1}{\lambda_{min}}[(B_1+C_1)+\lambda(B_0+C_0)]^2+\frac{\alpha}{\lambda_{min}}(B_0+C_0)^2+\frac{\lambda_{max}}{\lambda_{min}}(E_p^2+E_{p_1}^2)}\Bigg)\Bigg(B_1+ \\
& +\sqrt{\frac{[\lambda(B_1+C_1)+(\lambda^2+\alpha)(B_0+C_0)]^2+\alpha(B_1+C_1)^2+\lambda_{max}(\lambda^2+\alpha)(E_p^2+E_{p_1}^2)}{\alpha}}\Bigg)+ \\
& +\Bigg(E_{p_1}+\sqrt{\frac{1}{\lambda_{min}}[(B_1+C_1)+\lambda(B_0+C_0)]^2+\frac{\alpha}{\lambda_{min}}(B_0+C_0)^2+\frac{\lambda_{max}}{\lambda_{min}}(E_p^2+E_{p_1}^2)}\Bigg)\cdot \\
& \cdot\Bigg(B_0+\sqrt{\frac{1}{\alpha}[(B_1+C_1)+\lambda(B_0+C_0)]^2+(B_0+C_0)^2+\frac{\lambda_{max}}{\alpha}(E_p^2+E_{p_1}^2)}\Bigg)+|q|
\end{aligned}$$

where $\alpha = K_{p,2}+\lambda(K_{d,2}-\lambda)$.
When we consider the waiting phase controller we obtain

$$|u(t)| \leq K_{p,1}+K_{d,1}+|q| \qquad t < t_s$$

from which it is obvious that

$$\beta = \max\{K_{p,1}+K_{d,1}+|q|, \phi(q, B_0, B_1, B_2, C_0, C_1, K_{p,2}, K_{d,2}, E_{p_1}, E_p)\}$$

suffices. ∎

In case q is also unknown, the controller (3.23) will still be a suitable tracking phase controller. We only need a proper first phase controller, i.e. one that assures there exists a time $t_s \geq 0$ such that $|x(t_s)| \leq C_0$ and $|\dot{x}(t_s)| \leq C_1$ for any C_0 and C_1. Unfortunately we have not been able to derive a suitable first phase control law that results in a globally uniformly ultimately bounded closed-loop system.

3.5 Bounded controllers, using a velocity observer

In this section we also consider the problem of tracking a desired trajectory $x_d(t) \in \mathcal{C}^2$ under input limitations

$$|u(t)| \leq u_{max} \qquad t \geq 0$$

All controllers in the previous section require knowledge of the velocity $\dot{x}$. In case we are only able to measure x we still want to establish global asymptotical stability of the closed-loop system. Therefore, in this section we assume that we only know x and not $\dot{x}$. We develop auxiliary systems, to compensate the lack of knowledge of the velocity and use that auxiliary system in our control laws.
To deal with input limitations, we again develop composite controllers.

3.5.1 Trajectory tracking

In anology to Lemma 3.4.1 we first want to develop a waiting phase controller to control the state towards the origin. As the state is close enough, we switch to the tracking phase, using a second controller to achieve tracking.
Consider the control law

$$u = -q\cos\omega t - K_p f_1(x) - K_d f_2(w) \tag{3.25}$$

where $K_p \geq 0$, $K_d > 0$, $f_1, f_2 \in \mathcal{F}$ and w generated from the auxiliary system

$$\dot{w} = \dot{x} - L_p w \tag{3.26}$$

where $L_p > 0$ is a constant. This controller results in the closed-loop system

$$\begin{aligned} \ddot{x} + p\dot{x} + p_1 x + x^3 + K_p f_1(x) + K_d f_2(w) = 0 \\ \dot{w} = \dot{x} - L_p w \end{aligned} \tag{3.27}$$

Lemma 3.5.1 *Consider the system (3.1), together with the control law (3.25) and auxiliary system (3.26).*
Then for all $C_0 > \gamma$ and $C_1 > 0$ there exists a time $t_s \geq 0$ such that $|x(t)| \leq C_0$ and $|\dot{x}(t)| \leq C_1$ for all $t \geq t_s$ where $\gamma = 0$ for $p_1 \geq 0$ and $\gamma = \sqrt{-p_1}$ otherwise.

Proof Consider the radially unbounded Lyapunov-like function

$$V(x, \dot{x}, w) = \frac{1}{2}\dot{x}^2 + \frac{1}{4}(x^2 + p_1)^2 + K_p F_1(x) + K_d F_2(w)$$

where

$$F_1(x) = \int_0^x f_1(\zeta)d\zeta \text{ and } F_2(w) = \int_0^w f_2(\zeta)d\zeta$$

Differentiating along solutions of (3.27) yields

$$\dot{V}(x, \dot{x}, w) = -p\dot{x}^2 - K_d L_p w f_2(w)$$

which is negative semi-definite in the state $(x, \dot{x}, w)$. We next have have to determine the largest invariant set in $\{(x, \dot{x}, w) \in I\!R^3 | \dot{V}(x, \dot{x}, w) = 0\}$, which is $\{(x, \dot{x}, w) \in I\!R^3 | p_1 x + x^3 + K_p f_1(x) = 0\}$. Application of Theorem 2.2.10 completes the proof. ■

We developed the control law (3.25) with auxiliary system (3.26) in order to control the system (3.1) without using measurements of $\dot{x}$ but as before the auxiliary system (3.26) contains $\dot{e}$ instead. We can overcome this problem analogously.
We can again write the signal $w = e - L_p z$, where z is generated from an auxiliary system. From (3.26) we deduce:

$$\dot{z} = e - L_p z$$

Therefore

Lemma 3.5.2 *Consider the system (3.1), together with the control law*

$$u = -q\cos\omega t - K_p f_1(x) - K_d f_2(e - L_p z) \tag{3.28}$$

where $K_p \geq 0$, $K_d > 0$, $L_p > 0$, $f_1, f_2 \in \mathcal{F}$ and z generated from the auxiliary system

$$\dot{z} = x - L_p z \tag{3.29}$$

Then the resulting closed-loop system is globally asymptotically stable

Corollary 3.5.3 *Consider the system (3.1). If*

$$u_{max} > |q|$$

then there exist K_p and K_d such that the control law (3.28) satisfies the constraint (3.2).

Proposition 3.5.4 *Consider the system (3.1). For all $C_0 > \gamma$ and $C_1 > 0$ there exists a switching time $t_s \geq 0$ such that the composite control law*

$$u = \begin{cases} -K_{p,1}f_1(x) - K_{d,1}f_2(x - L_{p,1}z_1) - q\cos(\omega t) & t < t_s \\ \ddot{x}_d + p\dot{x}_d + p_1 x + x_d^3 + 3xx_d e - K_{p,2}f_3(e) - K_{d,2}f_4(e - L_{p,2}z_2) - q\cos(\omega t) & t \geq t_s \end{cases} \tag{3.30}$$

where $\gamma = 0$ for $p_1 \geq 0$ and $\gamma = \sqrt{-p_1}$ otherwise, $K_{p,1} \geq 0$, $K_{d,1} > 0$, $K_{p,2} \geq 0$,$K_{d,2} > 0$, $L_{p,1} > 0$, and $L_{p,2} > 0$ are constants, $f_1, f_2, f_3, f_4 \in \mathcal{F}$, and z_1 and z_2 generated from the auxiliary systems

$$\begin{array}{ll} \dot{z}_1 = x - L_{p,1}z_1 & t < t_s \\ \dot{z}_2 = e - L_{p,2}z_2 & t \geq t_s \end{array} \tag{3.31}$$

results in a globally asymptotically stable closed-loop system.

Proof Let t_s be a moment that both $|x(t_s)| \leq C_0$ and $|\dot{x}(t)| \leq C_1$. Lemma 3.5.2 showed the existence of t_s. In Proposition 3.2.3 we already showed that the tracking phase controller results in a globally asymptotically closed-loop system. ■

Corollary 3.5.5 *For all $C_0 > \gamma$ and $C_1 > 0$ there exists a $\beta > 0$ such that the composite controller (3.30) satisfies*

$$|u(t)| \leq \beta \qquad t \geq 0$$

where $\gamma = 0$ for $p_1 \geq 0$ and $\gamma = \sqrt{-p_1}$ otherwise.

Proof Proposition 3.5.4 gives us a switching time $t_s \geq 0$ such that $|x(t_s)| \leq C_0$ and $|\dot{x}(t_s)| \leq C_1$. Using (3.19), we obtain $|e(t_s)| \leq B_0 + C_0$ and $|\dot{e}(t_s)| \leq B_1 + C_1$. From the proof of Proposition 3.2.3 we know that the Lyapunov function (3.15) is a decreasing function for $t \geq t_s$. Therefore, $\forall t \geq t_s$:

$$\frac{1}{2}\dot{e}(t)^2 + \frac{1}{4}e(t)^4 + K_{p,2}F_3(e(t)) + K_{d,2}F_4(\dot{z}(t)) \leq \frac{1}{2}\dot{e}(t_s)^2 + \frac{1}{4}e(t_s)^4 + K_{p,2}F_3(e(t_s)) + K_{d,2}F_4(\dot{z}(t_s))$$

In case we initialize our second auxiliary system as $z(t_s) = 0$, this implies that for all $t \geq t_s$:

$$\begin{aligned} |e(t)| &\leq \sqrt[4]{2(B_1 + C_1)^2 + (B_0 + C_0)^4 + 2K_{p,2}F_3(B_0 + C_0)} \\ |\dot{e}(t)| &\leq \sqrt{(B_1 + C_1)^2 + \frac{1}{2}(B_0 + C_0)^4 + K_{p,2}F_3(B_0 + C_0)} \end{aligned}$$

resulting in

$$|u(t)| \leq \phi(p, p_1, q, B_0, B_1, B_2, C_0, C_1, K_{p,2}, K_{d,2}) \qquad \forall t \geq t_s$$

where

$$\begin{aligned} \phi(\cdot) = \; & B_2 + pB_1 + |p_1|\left(B_0 + \sqrt[4]{2(B_1 + C_1)^2 + (B_0 + C_0)^4 + 2K_{p,2}F_3(B_0 + C_0)}\right) + \\ & B_0^3 + 3\left(B_0 + \sqrt[4]{2(B_1 + C_1)^2 + (B_0 + C_0)^4 + 2K_{p,2}F_3(B_0 + C_0)}\right)B_0 \cdot \\ & \cdot \left(\sqrt[4]{2(B_1 + C_1)^2 + (B_0 + C_0)^4 + 2K_{p,2}F_3(B_0 + C_0)}\right) + K_{p,2} + K_{d,2} + |q| \end{aligned}$$

When we consider the first phase controller, we obtain

$$|u(t)| \leq K_{p,1} + K_{d,1} + |q| \qquad \forall t < t_s$$

from which it is obvious that

$$\beta = \max\{K_{p,1} + K_{d,1} + |q|, \phi(p, p_1, q, B_0, B_1, B_2, C_0, C_1, K_p, K_{d,2})\}$$

suffices. ■

Corollary 3.5.6 *Consider the system (3.1). If*

$$\begin{aligned} u_{max} \quad > \quad & B_2 + pB_1 + |p_1| \left(B_0 + \sqrt[4]{2B_1^2 + (B_0 + \gamma)^4} \right) + B_0^3 + 3 \left(B_0 + \sqrt[4]{2B_1^2 + (B_0 + \gamma)^4} \right) \cdot \\ & \cdot B_0 \left(\sqrt[4]{2B_1^2 + (B_0 + \gamma)^4} \right) + |q| \end{aligned}$$

where $\gamma = 0$ if $p_1 \geq 0$ and $\gamma = \sqrt{-p_1}$ otherwise, then there exists a composite control law of the form (3.30) that satisfies the constraint (3.2).

Remark 3.5.7 Notice the upperbounds in Corollaries 3.5.5 and 3.5.6 are the same as the ones found in Corollaries 3.4.4 and 3.4.5. Therefore, the lack of velocity measurements does not require a larger control effort.

Remark 3.5.8 Notice we switch controllers at a moment $|\dot{x}(t)| \leq C_1$. But we are not able to observe $\dot{x}$! How do we know when to switch?
Recall for the first phase controller when we have $|x(t_s)| \leq C_0$ and $|\dot{x}(t_s)| \leq C_1$ for certain time t_s, then $|x(t)| \leq C_0$ and $|\dot{x}(t)| \leq C_1$ for all $t \geq t_s$. Therefore it does not matter if we switch from the first controller to the second one later than suggested. Since we have $x(t)$ available, we can use numerical differentiation techniques to approximate $\dot{x}(t)$. Although it is not preferable to use those estimates in the controllers of the previous section we can nevertheless use those estimates to determine whether $|\dot{x}(t)| \leq C_1$ or not. We only know for sure that $|\dot{x}(t_s)| \leq C_1$ a little later.

3.6 Simulations

To support our results, we simulated with SIMNON^{TM} the controlled Duffing's equation. In our first simulation we compare the bounded composite controller (3.22) to (3.3) of Nijmeijer and Berghuis. In our second simulation we compare the analogous controller-observer/auxiliary-system combinations. Our last simulation compares the Nijmeijer-Berghuis robust controller (3.6) to our adaptive controller (3.23).
In all simulations we use the same parameters: $p = 0.4$, $p_1 = -1.1$, $q = 2.1$, and $\omega = 1.8$, in which case the uncontrolled forced Duffing equation displays chaotic behaviour. We define the reference motion as

$$x_d(t) = \sin(t) \qquad t \geq 0$$

In order to demonstrate the suggested controllers deal with input limitations, we choose the initial state $(x(0), \dot{x}(0)) = (-2, 3)$.
For the Nijmeijer-Berghuis controllers, the gains were choosen as in [10], i.e. $K_d = 12.5$ and $K_p = 50$. The resulting performance for (3.3) is depicted in figure 3.1.
Figure 3.1(a) shows the time-trajectories of $e(t)$ and $\dot{e}(t)$, 3.1(b) shows the control $u(t)$ needed, 3.1(c) shows the error trajectory $(e, \dot{e})$ and finally 3.1(d) is the same as 3.1(b), only using an other scale.
We see the control effort at the beginning of this simulation is more than 70, then rapidly decreases to about -20 and in about a second the errors are small enough such that the control-effort reduces to the needed feedforward control.
For the bounded composite controller (3.22) we chose all gains to be 1, chose $C_0 = 1.1$ and $C_1 = 0.1$ and used $f_1(x) = f_2(x) = \tanh(x)$. The resulting performance for (3.22) is depicted in figure 3.2.

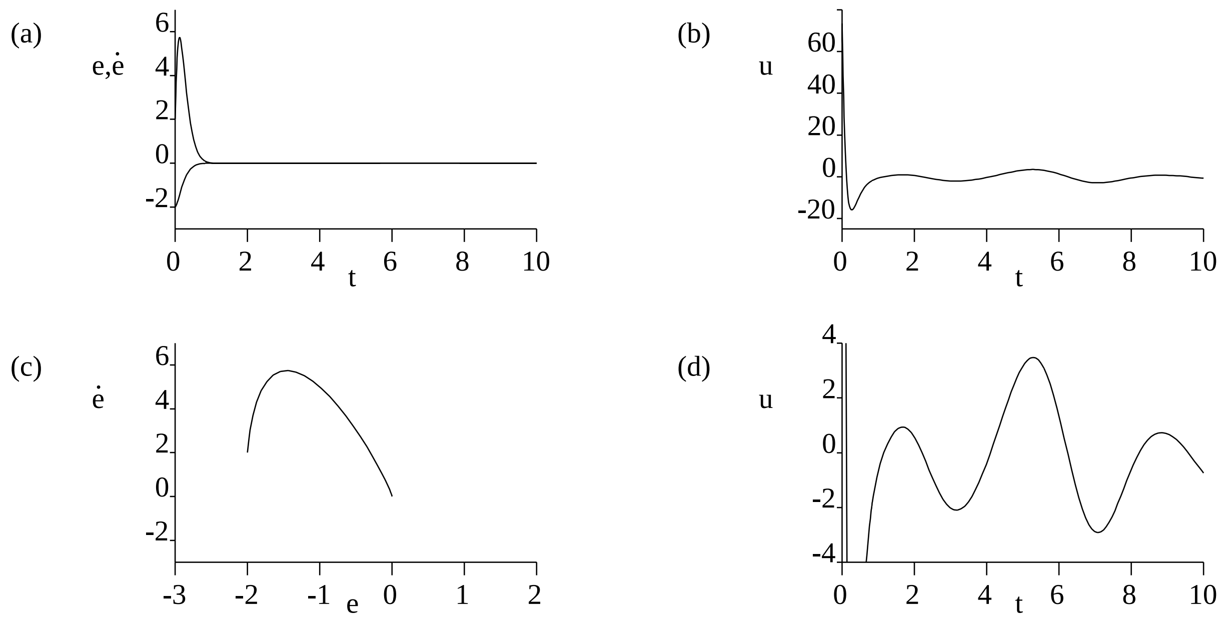

Figure 3.1: The performance of Nijmeijer-Berghuis controller (3.3).

Analogously figure 3.2(a) shows the time-trajectories of $e(t)$ and $\dot{e}(t)$, 3.2(b) shows the control $u(t)$ needed, 3.2(c) shows the error trajectory $(e, \dot{e})$ and finally 3.2(d) is the same as 3.2(b), only using an other scale.
We see the control-efforts remain small: we have $|u(t)| \leq 2$ for the waiting phase controller, whereas for the tracking controller we still have $|u(t)| \leq 4$. This is in clear contrast to the Nijmeijer-Berghuis controller. The only price we have to pay is the time needed to reduce the errors, although the maximum magnitude of the errors is less. From 3.2(d) we see the first phase controller is needed until $t = 1.76$. For $t \geq 1.76$ the tracking phase controller is used.
In order to compare both controllers, we define some performance measures. Since the bounded composite controller needs a smaller control effort in exchange for a worse error-convergence, we define the overall tracking error and control effort as

$$\begin{aligned} \text{Tracking error} &= \int_0^{t_f} \left(e(\zeta)^2 + \dot{e}(\zeta)^2\right) d\zeta \\ \text{Control effort} &= \int_0^{t_f} u(\zeta)^2 d\zeta \end{aligned}$$

where t_f is a final time, which in case of the tracking error might be chosen to equal infinity but in case of the control effort can not, since we always need a feedforward control in order to follow the desired trajectory.
When we choose our final time $t_f = 10$ seconds, we obtain:

Controller	Tracking error	Control effort
Nijmeijer-Berghuis	8.627	245.232
Bounded composite	6.001	31.260

From which it is obvious the bounded composite controller (3.22) performs much better.

In our first simulation both controllers used state measurements. In our secound simulation, we consider the Nijmeijer-Berghuis controller-observer combination (3.4,3.5) where the gains are again

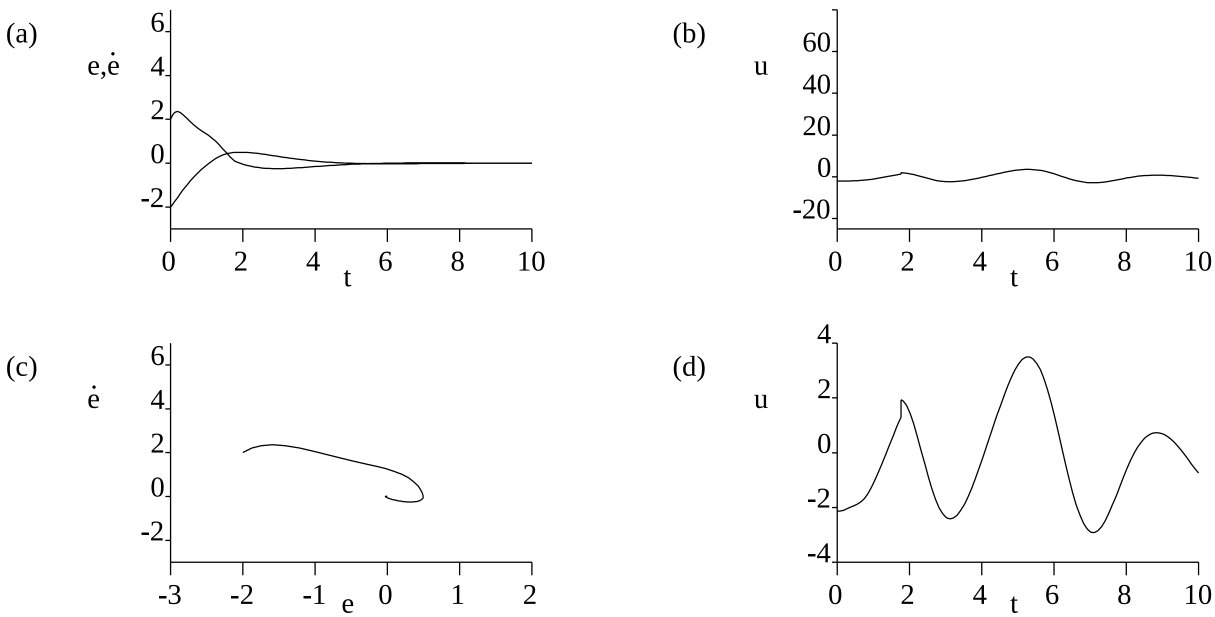

Figure 3.2: The performance of the bounded composite controller (3.22).

as in [10] $K_d = 12.5$ and $K_p = 50$ and we initialized the observer at the origin $(\hat{e}(0), w(0)) = (0, 0)$. The resulting performance for (3.4, 3.5) is depicted in figure 3.3.
Figure 3.3(a) again shows the time-trajectories of $e(t)$ and $\dot{e}(t)$, 3.3(b) shows the control $u(t)$ needed, 3.3(c) shows the error trajectory $(e, \dot{e})$ and finally 3.3(d) is the same as 3.3(b), only using an other scale.
We see the control efforts in this simulation are much larger. The initial control effort is more than 700, then rapidly decreases to about -200 and in about a second the errors again are small enough such that the control effort reduces to the needed feedforward control.
For the bounded composite controller (3.30) with auxiliary system (3.31) we again chose all gains to be 1 , $C_0 = 1.1$, $C_1 = 0.1$ and $f_1(x) = f_2(x) = \tanh(x)$. We initialized both auxiliary systems at the origin i.e. $z_1(0) = 0$ and $z_2(t_s) = 0$. The resulting performance for (3.30,3.31) is depicted in figure 3.4.
Again figure 3.4(a) shows the time-trajectories of $e(t)$ and $\dot{e}(t)$, 3.4(b) shows the control $u(t)$ needed, 3.4(c) shows the error trajectory $(e, \dot{e})$ and finally 3.4(d) is the same as 3.4(b), only using an other scale.
We again see the control-efforts remain much smalller: $|u(t)| \leq 2$ for the waiting phase controller and $|u(t)| \leq 4$ for the tracking phase controller, in clear contrast to the Nijmeijer-Berghuis controller-observer. Also the time needed to reduce the errors is again larger, and the maximum magnitude of the errors is less. From 3.4(d) we see the first phase controller is needed until $t = 1.40$. For $t \geq 1.40$ the tracking phase controller is used.
In order to compare both controllers, we again choose $t_f = 10$ seconds, resulting in:

Controller	Tracking error	Control effort
Nijmeijer-Berghuis	41.458	9301.65
Bounded composite	9.643	28.23

From which it is obvious the bounded composite controller (3.30) with auxiliary system (3.31) performs much better.

In our third, and last, simulation we compare the robust parameter-independent controller (3.6)

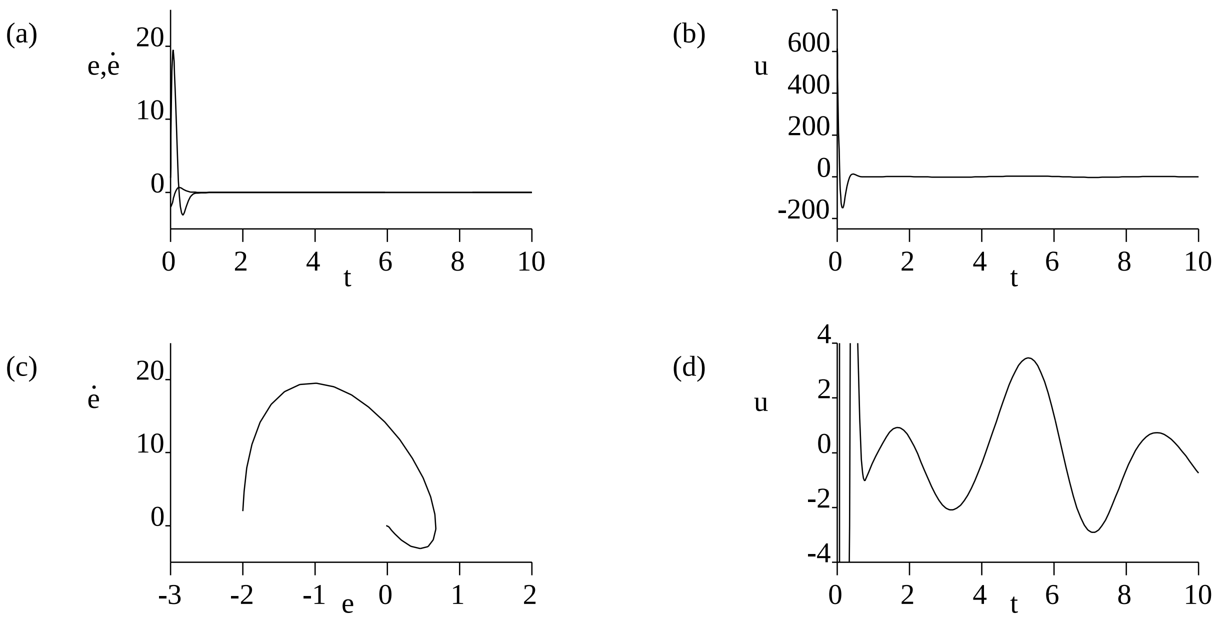

Figure 3.3: The performance of Nijmeijer-Berghuis controller-observer (3.4,3.5).

of Nijmeijer and Berghuis to our adaptive controller (3.23). The results are depicted in the figures 3.5, which contains the results of (3.6), and 3.6, which contains the results of (3.23).
In both figures figure (a) shows the time-trajectories of $e(t)$ and $\dot{e}(t)$, figure (b) shows the control effort needed, figure (c) shows the state error trajectory $(e, \dot{e})$, just as figure (d), whereas in figure (d) an other scale has been used.
For both controllers we chose the gains $K_d = 12.5$ and $K_p = 50$. In the adaptive controller (3.23) we used $\lambda = 5$, $\Gamma = I_3$, the identity-matrix, and initial estimate $(\hat{p}(0), \hat{p}_1(0), \hat{q}(0)) = (0, 0, 0)$. At first eye, both state-error-trajectories seem to behave the same. However, when we zoom in to the origin, we see the robust Nijmeijer-Berghuis controller, also suitable in case w and q are unknown, is ultimately uniformly bounded (practically stable), whereas the adaptive controller converges towards the origin.

3.7 Summary

In this chapter we considered the problem of controlling the forced Duffing equation to any desired trajectory using bounded controllers.
In order to solve this problem, the concept of composite controllers has been introduced.
Suitable first phase controllers have been developed, that established global convergence to an area around the origin. The first phase controllers consist of a controller using velocity measurements, one using an auxiliary system in case of no velocity measurements, which both could also be used in case p and/or p_1 are unknown.
Using the concept of composite controllers, these first phase controllers have been combined with tracking controllers. Those tracking phase controllers also consist of a controller using velocity measurements, one using an auxiliary system in case of no velocity measurements and a adaptive controller.
The results have been supported with some simulations, and are in clear contrast with the controllers of Nijmeijer and Berghuis in [10].

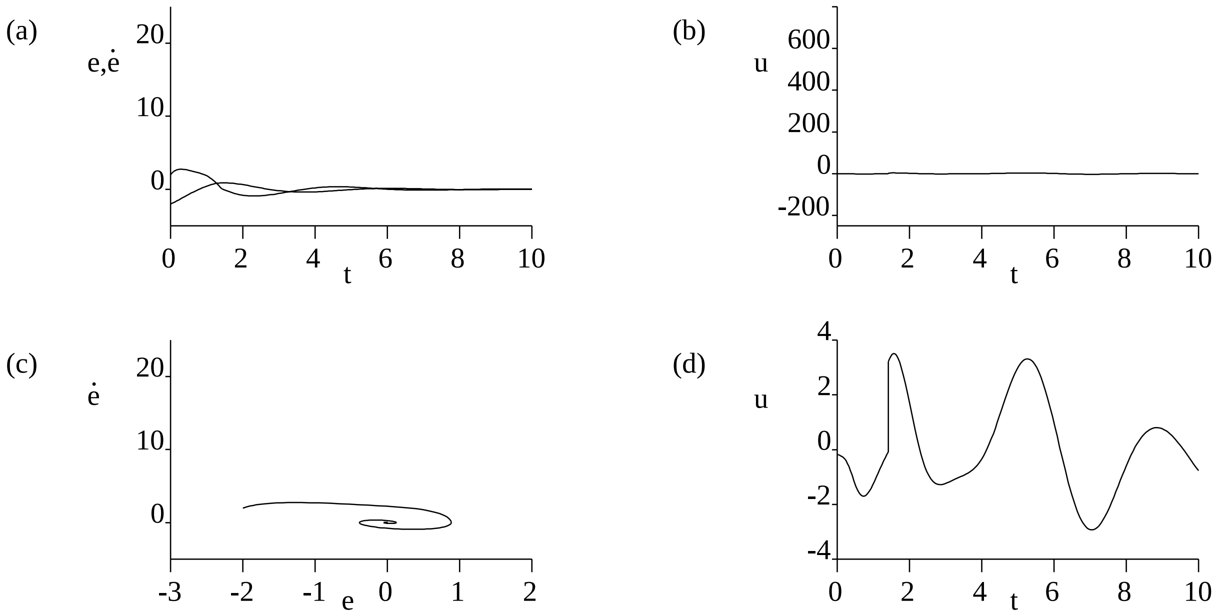

Figure 3.4: The performance of the bounded composite controller (3.30) with auxiliary system (3.31).

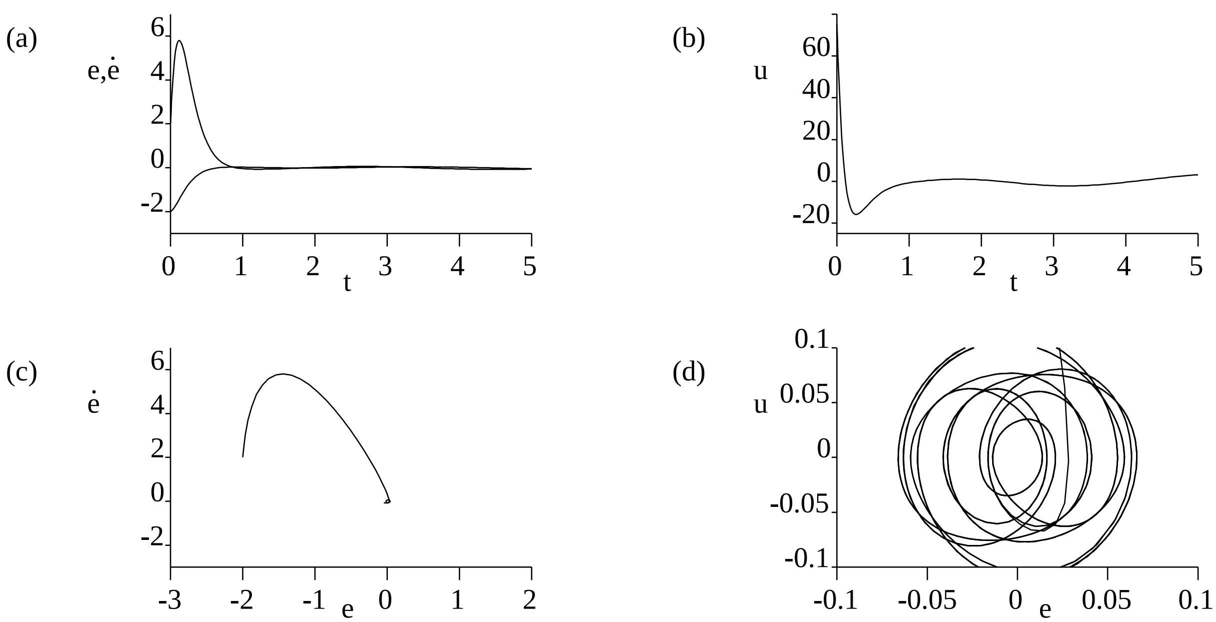

Figure 3.5: The performance of the robust Nijmeijer-Berghuis controller (3.6).

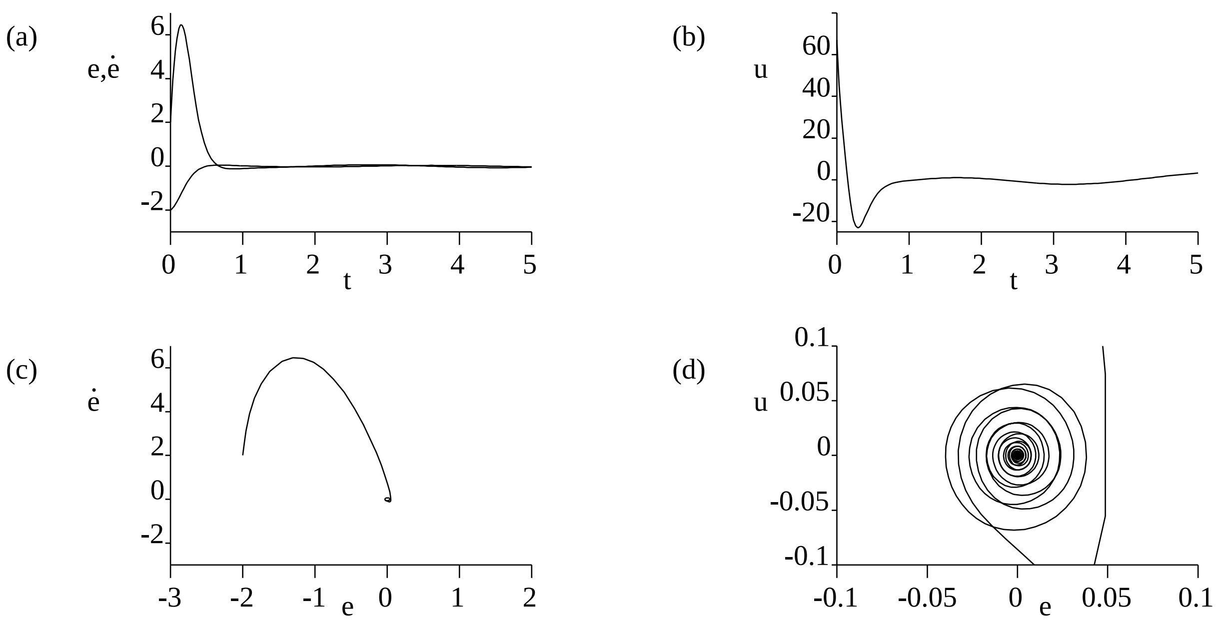

Figure 3.6: The performance of the adaptive controller (3.22).

Chapter 4

Van der Pol's equation

In this chapter we study the controlled version of the forced Van der Pol equation (1.3):

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = u + q\cos\omega t \tag{4.1}$$

where $\mu > 0$, q, and ω are (un)known constants and $u(\cdot)$ is the, physically realisable, control input. We consider the problem of controlling this system to any desired trajectory $x_d(t) \in \mathcal{C}^2$, satisfying

$$|x_d(t)| \leq B_0,\ |\dot{x}_d(t)| \leq B_1,\ |\ddot{x}_d(t)| \leq B_2 \qquad t \geq 0 \tag{4.2}$$

under input limitations:

$$|u(t)| \leq u_{max} \qquad t \geq 0 \tag{4.3}$$

In order to achieve globally stabilizing results dealing with input limitations we again use the concept of a composite feedback control. Furthermore, adaptive controllers will be introduced in case one or more system-parameters (except ω) are unknown, and these will be compared to earlier developed robust controllers.

4.1 Some useful results

In a related project Stuckings [15] did part of his MSc in Twente. His thesis contains three interesting propositions anologous to the work of Nijmeijer and Berghuis in [10] but for Van der Pols' equation instead of Duffing's equation (Propositions 4.1.1, 4.1.2 and 4.1.3). A last useful result is shown in Proposition 4.1.4.

Proposition 4.1.1 *Consider the system (4.1) together with the control law*

$$u = \ddot{x}_d - \mu\dot{x}_d + x_d - q\cos(\omega t) - K_d\dot{\hat{e}} - K_p\hat{e} + \mu x^2\dot{x}_d \tag{4.4}$$

where $e \equiv x - x_d$ *denotes the tracking error,* $\lambda K_d = K_p > 0$ *and* $K_d > \mu + \lambda + \frac{9}{2}\mu B_0^2$ *are constants and* $\hat{e}$ *is generated from the observer*

$$\begin{aligned} \dot{\hat{e}} &= w + 2K_d(e - \hat{e}) + \mu e - \frac{1}{3}\mu e^3 - \mu x_d e^2 \\ \dot{w} &= 2K_p(e - \hat{e}) - e + \mu\dot{x}_d e^2 \end{aligned} \tag{4.5}$$

Then the resulting closed-loop system is globally asymptotically stable.

Proposition 4.1.2 *Consider the system (4.1) under robust PD-feedback*

$$u = -K_d\dot{e} - K_p e \tag{4.6}$$

Then the resulting closed-loop dynamics are locally uniformly ultimately bounded for K_d *sufficiently large*

Proposition 4.1.3 *Consider the system (4.1) under the robust output-feedback control*

$$u = -K_d\dot{\hat{e}} - K_p\hat{e} \tag{4.7}$$

where $\hat{e}$ is generated by the observer

$$\begin{aligned} \dot{\hat{e}} &= w + 2K_d(e - \hat{e}) \\ \dot{w} &= 2K_p(e - \hat{e}) \end{aligned} \tag{4.8}$$

Then the resulting closed-loop dynamics are locally uniformly ultimately bounded for K_d sufficiently large.

Notice the controller of Proposition 4.1.1 is a suitable candidate to be used as a tracking phase controller in case we have no state measurements at our disposal.

A last useful result is a suitable candidate to be used as a tracking phase controller in case we are able to measure the entire state $(x, \dot{x})$. Consider the control law

$$u = \ddot{x}_d - \mu(1 - x^2)\dot{x} + x - K_d f_1(\dot{e}) - K_p f_2(e) - q\cos(\omega t) \tag{4.9}$$

where $K_p > 0$ and $K_d > 0$ are constant and $f_1, f_2 \in \mathcal{F}$. This control law results in the closed-loop system

$$\ddot{e} + K_d f_1(\dot{e}) + K_p f_2(e) = 0 \tag{4.10}$$

Proposition 4.1.4 *Consider the system (4.1) together with the control law (4.9). Then the closed-loop system (4.10) is globally asymptotically stable.*

Proof Consider the radially unbounded Lyapunov function

$$V(e, \dot{e}) = \frac{1}{2}\dot{e}^2 + K_p F_1(e) \tag{4.11}$$

where

$$F_1(e) = \int_0^e f_1(\zeta)d\zeta$$

which is positive definite. Differentiating along solutions of (4.10) results in

$$\dot{V}(e, \dot{e}) = -K_d\dot{e}f_2(\dot{e})$$

which is negative semidefinite in the error state $(e, \dot{e})$. Using LaSalle shows global asymptotic stability of the origin. ∎

4.2 Bounded control, using state measurements

In this section we consider the problem of tracking a desired trajectory $x_d(t) \in \mathcal{C}^2$ under input limitations

$$|u(t)| \leq u_{max} \qquad t \geq 0$$

assuming we are able to measure the full state $(x, \dot{x})$. In order to deal with input limitations, we again propose composite controllers.

4.2.1 Controlling towards a fixed point

In anology to the previous chapter, we want to develop waiting phase controllers that control the system towards the origin. Afterwards we consider controllers that control the state towards a fixed point x_f. We distinguish two cases, that is $|x_f| > 2$ and $|x_f| \leq 2$.

The case $|x_f| > 2$

Consider the control law

$$u = x_f - q\cos(\omega t) - K_p f(e) \tag{4.12}$$

where $e \equiv x - x_f$ denotes the error, $K_p \geq 0$ is a constant and $f \in \mathcal{F}$. This results in the closed-loop system

$$\ddot{e} - \mu(1 - x^2)\dot{e} + e + K_p f(e) = 0 \tag{4.13}$$

Proposition 4.2.1 *Consider for* $|x_f| > 2$ *the system (4.1) together with control law (4.12). Then, the closed-loop system (4.13) is globally asymptotically stable.*

Proof Consider the radially unbounded candidate Lyapunov function

$$V(e, \dot{e}) = \frac{1}{2}(\dot{e} + p(e))^2 + \frac{1}{2}e^2 + K_p F(e) \text{ where } p(e) = \mu[\frac{1}{3}e^3 + x_f e^2 + (x_f^2 - 1)e]$$

It is easy to see that this Lyapunov function candidate is positive definite. Along the closed-loop error dynamics (4.13), the time derivative of $V(e, \dot{e})$ becomes:

$$\begin{aligned}
\dot{V}(e, \dot{e}) &= [\dot{e} + p(e)][\mu(1 - x^2)\dot{e} - e - K_p f(e) + \mu(e^2\dot{e} + 2x_f e\dot{e} + x_f^2\dot{e} - \dot{e})] + e\dot{e} + K_p f(e)\dot{e} \\
&= -p(e)[e + K_p f(e)] \\
&\quad -\mu[e^2 + K_p e f(e)][\frac{1}{3}e^2 + x_f e + x_f^2 - 1]
\end{aligned}$$

This is negative semidefinite in the error state $(e, \dot{e})$, since $\forall e$: $\frac{1}{3}e^2 + x_f e + x_f^2 - 1 > 0$, which easily follows from the observation that $\forall x$: $ax^2 + bx + c > 0$ iff $a > 0$ and $b^2 - 4ac < 0$:

$$x_f^2 - \frac{4}{3}(x_f^2 - 1) = -\frac{1}{3}x_f^2 + \frac{4}{3} < 0.$$

To demonstrate global asymptotic stability, we can use LaSalle. Define the set Ω as:

$$\Omega = \{e \in I\!R, \dot{e} \in I\!R | \dot{V}(e, \dot{e}) = 0\} = \{e = 0, \dot{e} \in I\!R\}$$

The largest invariant set in Ω can only be the origin, since $e \equiv 0$ implies that $\dot{e} = 0$, thus from LaSalle's theorem we conclude that the origin is globally asymptotically stable. ■

Corollary 4.2.2 *Consider the system (4.1). If*

$$u_{max} \geq |q| + |x_f|$$

then there exists a K_p *such that the control law (4.12) satisfies the constraint (3.2).*

The case $|x_f| \leq 2$

We use a composite control strategy to solve this problem. We firstly control the state towards an $\tilde{x}_f$ for which $|\tilde{x}_f| > 2$ using (4.12). This assures $x(t_s)$ and $\dot{x}(t_s)$ are within prescribed bounds. Secondly we apply a second controller, to achieve convergence of the state towards x_f.

Proposition 4.2.3 *Consider for* $|x_f| \leq 2$ *the system (4.1). Then for all* $C_0 > 2$ *and* $C_1 > 0$ *there exists a switching time* $t_s \geq 0$ *and a constant* $\tilde{x}_f$ *such that the composite control law*

$$u = \begin{cases} \tilde{x}_f - q\cos(\omega t) - K_{p,1} f_1(x - \tilde{x}_f) & t < t_s \\ x_f - q\cos(\omega t) - K_{p,2} f_2(e) - K_d f_3(\dot{e}) - \mu\dot{e} & t \geq t_s \end{cases} \tag{4.14}$$

where $K_{p,1} \geq 0$, $K_{p,2} \geq 0$ *and* $K_d > 0$ *are constants and* $f_1, f_2, f_3 \in \mathcal{F}$, *results in a globally asymptotically stable closed-loop system.*

Proof Let t_s be a moment for which both $|x(t_s)| \leq C_0$ and $|\dot{x}(t_s)| \leq C_1$. We can choose $\tilde{x}_f$ such that $2 < |\tilde{x}_f| < C_0$. Then Proposition 4.2.1 gives the existence of t_s. For $t \geq t_s$ we have as closed-loop system:

$$\ddot{e} + \mu x^2 \dot{e} + e + K_{p,2} f_2(e) + K_d f_3(\dot{e}) = \ddot{e} + \mu(e + x_f)^2 \dot{e} + e + K_{p,2} f_2(e) + K_d f_3(\dot{e}) = 0 \quad (4.15)$$

Using the radially unbounded candidate Lyapunov function

$$V(e, \dot{e}) = \frac{1}{2}\dot{e}^2 + \frac{1}{2}e^2 + K_{p,2} F_2(e) \quad (4.16)$$

where

$$F_2(e) = \int_0^e f_2(z)dz$$

which obviously is positive definite, we get differentiating along solutions of (4.15)

$$\dot{V}(e, \dot{e}) = -\mu x^2 \dot{e}^2 - K_d f_3(\dot{e})\dot{e} = -\mu(e + x_f)^2 \dot{e}^2 - K_d f_3(\dot{e})\dot{e}$$

which is negative semidefinite in the error state $(e, \dot{e})$. To demonstrate global asymptotic stability we can again use LaSalle. To this end, define the set Ω as

$$\Omega = \{e \in I\!R, \dot{e} \in I\!R | \dot{V}(e, \dot{e}) = 0\} = \{e \in I\!R, \dot{e} = 0\}$$

The largest invariant set in Ω with respect to (4.15) is the origin, thus from LaSalle's theorem we conclude that the origin is globally asymptotically stable. ■

Corollary 4.2.4 *For all $C_0 > 2$ and $C_1 > 0$ there exists a $\beta > 0$ such that the composite controller (4.14) satisfies*

$$|u(t)| \leq \beta \qquad t \geq 0$$

Proof Proposition 4.2.3 gives us $e(t_s) \leq C_0 - |x_f|$ and $\dot{e}(t_s) \leq C_1$. From the proof of Proposition 4.2.3 we know that the Lyapunov function (4.16) is a non-increasing function for $t \geq t_s$, i.e.

$$\frac{1}{2}\dot{e}(t)^2 + \frac{1}{2}e(t)^2 + K_{p,2}F_2(e(t)) \leq \frac{1}{2}\dot{e}(t_s)^2 + \frac{1}{2}e(t_s)^2 + K_{p,2}F_2(e(t_s)) \qquad \forall t \geq t_s$$

This implies that for all $t \geq t_s$:

$$\begin{aligned} |e(t)| &\leq \sqrt{(C_0 - |x_f|)^2 + C_1^2 + 2K_{p,2}F_2(C_0 - |x_f|)} \\ |\dot{e}(t)| &\leq \sqrt{(C_0 - |x_f|)^2 + C_1^2 + 2K_{p,2}F_2(C_0 - |x_f|)} \end{aligned}$$

resulting in

$$|u(t)| \leq \phi(x_f, \mu, q, C_0, C_1, K_{p,2}, K_d)$$

where

$$\phi(x_f, \mu, q, C_0, C_1, K_{p,2}, K_d) = |x_f| + |q| + K_{p,2} + K_d\mu\sqrt{(C_0 - |x_f|)^2 + C_1^2 + 2K_{p,2}F_2(C_0 - |x_f|)}$$

When we consider the first phase controller, we obtain

$$|u(t)| \leq C_1 + K_{p,1} + |q|$$

from which it is obvious that

$$\beta = \max\{C_1 + K_{p,1} + |q|, \phi(x_f, \mu, q, C_0, C_1, K_{p,2})\}$$

suffices. ■

Corollary 4.2.5 *Consider the system (4.1). If*

$$u_{max} > |q| + \max\{2, |x_f| + \mu(2 - |x_f|)\}$$

then there exists a composite control law of the form (4.14) that satisfies the constraint (4.3).

4.2.2 Trajectory tracking

In the previous subsection we have developed two suitable first phase controllers, that both guarantee global convergence of the closed-loop system to any desired point, satisfying input limitations. In this section we combine those results with the ones in section 4.1, in anology to the previous chapter, resulting in global trajectory tracking under input limitations.

Proposition 4.2.6 *Consider the system (4.1). For all $C_0 > 2$ and $C_1 > 0$ there exists a switching time $t_s \geq 0$ and a constant $\tilde{x}_f$ such that the composite control law*

$$u = \begin{cases} \tilde{x}_f - q\cos(\omega t) - K_{p,1}f_1(x - \tilde{x}_f) & t < t_s \\ \ddot{x}_d - \mu(1-x^2)\dot{x} + x - K_d f_2(\dot{e}) - K_{p,2}f_3(e) - q\cos(\omega t) & t \geq t_s \end{cases} \tag{4.17}$$

where $K_d \geq 0$, $K_{p,1} \geq 0$, and $K_{p,2} > 0$ are constants, and $f_1, f_2, f_3 \in \mathcal{F}$, results in a globally asymptotically stable closed-loop system.

Proof Let t_s be a moment for which both $|x(t_s)| \leq C_0$ and $|\dot{x}(t_s)| \leq C_1$. We can choose $\tilde{x}_f$ such that $2 < |\tilde{x}_f| < C_0$. Then Proposition 4.2.1 gives the existence of t_s. In Proposition 4.1.4 we already proved that the tracking phase controller results in a globally asymptotically closed-loop system. ■

Corollary 4.2.7 *For all $C_0 > 2$ and $C_1 > 0$ there exists a time $t_s \geq 0$ and a $\beta > 0$ such that the composite controller (4.17) satisfies*

$$|u(t)| \leq \beta \qquad t \geq 0$$

Proof Proposition 4.2.3 gives us $e(t_s) \leq C_0 - |x_f|$ and $\dot{e}(t_s) \leq C_1$. From the proof of Proposition 4.1.4 we know that the Lyapunov function (4.11) is a non-increasing function for $t \geq t_s$, i.e.

$$\frac{1}{2}\dot{e}(t)^2 + K_{p,2}F_3(e(t)) \leq \frac{1}{2}\dot{e}(t_s)^2 + K_{p,2}F_3(e(t_s)) \qquad \forall t \geq t_s$$

This implies that for all $t \geq t_s$:

$$\begin{aligned} |e(t)| &\leq \frac{1}{K_{p,2}}F_3^{-1}(\frac{1}{2}[B_1 + C_1]^2 + K_{p,2}F_3[B_0 + C_0]) \\ |\dot{e}(t)| &\leq \sqrt{(B_1 + C_1)^2 + 2K_{p,2}F_3(B_0 + C_0)} \end{aligned}$$

Therefore

$$|u(t)| \leq \phi(\mu, q, B_0, B_1, B_2, C_0, C_1, K_d, K_{p,2}) \qquad \forall t \geq t_s$$

$$\begin{aligned} \phi(\cdot) \;=\; & B_2 + \mu\left([\frac{1}{K_{p,2}}F_3^{-1}(\frac{1}{2}[B_1 + C_1]^2 + K_{p,2}F_3[B_0 + C_0])]^2 - 1\right)\cdot \\ & \cdot(B_1 + \sqrt{(B_1 + C_1)^2 + 2K_{p,2}F_3(B_0 + C_0)}) + \\ & + \frac{1}{K_{p,2}}F_3^{-1}(\frac{1}{2}[B_1 + C_1]^2 + K_{p,2}F_3[B_0 + C_0]) + B_0 + K_d + K_{p,2} + |q| \end{aligned}$$

When we consider the first phase controller, we obtain

$$|u(t)| \leq |\tilde{x}_f| + |q| + |K_{p,1}| \qquad \forall t < t_s$$

from which it is obvious

$$\beta = \max\{|\tilde{x}_f| + |q| + |K_{p,1}|, \phi(\mu, q, B_0, B_1, B_2, C_0, C_1, K_d, K_{p,2})\}$$

suffices. ■

We can also use (4.14) as first phase controller instead of (4.12). This enables us to have a smaller position-error at the moment we switch from waiting phase to tracking phase, resulting in a lower upperbound.

Proposition 4.2.8 *Consider the system (4.1). For all $\tilde{C}_0 > 2$, $\tilde{C}_1 > 0$, $C_0 > 0$ and $C_1 > 0$ there exist switching times $t_{s,1} \geq 0$ and $t_{s,2} \geq t_{s,1}$ and a constand $\tilde{x}_f$ such that the composite control law*

$$u = \begin{cases} \tilde{x}_f - q\cos(\omega t) - K_{p,1}f_1(x - \tilde{x}_f) & t < t_{s,1} \\ -q\cos(\omega t) - K_{p,2}f_2(x) - K_{d,1}f_3(\dot{x}) - \mu\dot{x} & t_{s,1} \leq t < t_{s,2} \\ \ddot{x}_d - \mu(1-x^2)\dot{x} + x - K_{d,2}f_4(\dot{e}) - K_{p,3}f_5(e) - q\cos(\omega t) & t \geq t_{s,2} \end{cases} \tag{4.18}$$

where $f_1, f_2, f_3, f_4, f_5 \in \mathcal{F}$, and $K_{p,1} \geq 0$, $K_{d,1} > 0$, $K_{p,2} \geq 0$, $K_{d,2} \geq 0$, and $K_{p,3} > 0$ are constants, results in a globally asymptotically stable closed-loop system.

Proof Let $t_{s,1}$ be a moment for which both $|x(t_{s,1})| \leq \tilde{C}_0$ and $|\dot{x}(t_{s,1})| \leq \tilde{C}_1$ and let $t_{s,2}$ be a moment for which both $|x(t_{s,1})| \leq C_0$ and $|\dot{x}(t_{s,2})| \leq C_1$. By choosing $\tilde{x}_f$ such that $2 < |\tilde{x}_f| < C_1$ Proposition 4.2.3 gives us the existence of both $t_{s,1}$ and $t_{s,2}$. In Proposition 4.1.4 we already proved that the tracking phase controller results in a globally asymptotically closed-loop system. ∎

Corollary 4.2.9 *For all $\tilde{C}_0 > 2$, $\tilde{C}_1 > 0$, $C_0 > 0$ and $C_1 > 0$ there exists a $\beta > 0$ such that the composite controller (4.18) satisfies*

$$|u(t)| \leq \beta \qquad t \geq 0$$

Proof From the proof of Corollaries 4.3.2 and 4.2.7 it is obvious that

$$\beta = \max\{|\tilde{x}_f| + |q| + |K_{p,1}|, \phi_1(x_f, \mu, q, \tilde{C}_0, \tilde{C}_1, K_{p,2}, K_{d,1}), \phi_2(\mu, q, B_0, B_1, B_2, C_0, C_1, K_{d,2}, K_{p,3})\}$$

suffices, where

$$\begin{aligned} \phi_1(\cdot) &= |x_f| + |q| + K_{p,2} + K_{d,1}\mu\sqrt{(\tilde{C}_0 - |x_f|)^2 + \tilde{C}_1^2 + 2K_{p,2}F_2(\tilde{C}_0 - |x_f|)} \\ \phi_2(\cdot) &= B_2 + \mu\left([\frac{1}{K_{p,3}}F_5^{-1}(\frac{1}{2}[B_1 + C_1]^2 + K_{p,3}F_5[B_0 + C_0])]^2 - 1\right)\cdot \\ &\quad \cdot(B_1 + \sqrt{(B_1 + C_1)^2 + 2K_{p,3}F_5(B_0 + C_0)}) + \\ &\quad + \frac{1}{K_{p,3}}F_5^{-1}(\frac{1}{2}[B_1 + C_1]^2 + K_{p,3}F_5[B_0 + C_0]) + B_0 + K_d + K_{p,2} + |q| \end{aligned}$$

∎

4.2.3 Adaptive trajectory tracking

In case ω is known, we know from Proposition 2.4.1 the control law

$$u = \ddot{x}_d - K_d\dot{e} - K_p e + \hat{\mu}(x^2 - 1)\dot{x} + \hat{q}\cos(\omega t) \tag{4.19}$$

where $K_p > 0$, $K_d > 0$ are constants and $\hat{\mu}$ and $\hat{q}$ estimates for μ and $-q$ given by:

$$\begin{pmatrix} \dot{\hat{\mu}} \\ \dot{\hat{q}} \end{pmatrix} = -\Gamma \begin{pmatrix} (x^2 - 1)\dot{x} \\ \cos(\omega t) \end{pmatrix} (\dot{e} + \lambda e)$$

where $0 < \lambda < K_d$ is a constant and Γ a 2×2 positive definite symmetric matrix, results in a globally asymptotically stable closed-loop system.
Therefore, this control law is a good candidate for the tracking phase controller of a composite control law, in case some parameters are unknown. We only have to find a suitable waiting phase

controller. Furthermore, in order to assure the controller satisfies the constraint (4.3) we have to assume that we have bounds on the initial estimate errors of the unknown variables, i.e.

$$\begin{aligned} \hat{\mu}(0) &\leq E_\mu \\ \hat{q}(0) &\leq E_q \end{aligned}$$

with E_μ and E_q some bounds.
Notice the control law (4.12) is independent of μ. In case only μ is unknown this is a suitable first phase controller.

Proposition 4.2.10 *Consider the system (3.1). For all $C_0 > 2$ and $C_1 > 0$ there exists a switching time $t_s \geq 0$ and a constant $\tilde{x}_f$ such that the composite adaptive control law*

$$\begin{cases} \tilde{x}_f - q\cos(\omega t) - K_{p,1} f(x - \tilde{x}_f) & t < t_s \\ \ddot{x}_d - K_d \dot{e} - K_{p,2} e + \hat{\mu}(x^2 - 1)\dot{x} - q\cos(\omega t) & t \geq t_s \end{cases} \tag{4.20}$$

where $K_d > 0$, $K_{p,1} \geq 0$, and $K_{p,2} > 0$ are constants, $f \in \mathcal{F}$, and $\hat{\mu}$ an estimate for μ given by:

$$\dot{\hat{\mu}} = -\Gamma(\dot{e} + \lambda e)(1 - x^2)\dot{x}$$

where $0 < \lambda < K_d$ and $\Gamma > 0$ are constants, results in a globally asymptotically stable closed loop system with respect to e and $\dot{e}$, i.e.

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} \dot{e}(t) = 0$$

Proof Let t_s be a moment for which both $|x(t_s)| \leq C_0$ and $|\dot{x}(t_s)| \leq C_1$. In Proposition 4.2.1 we showed the existence of t_s and in Proposition 2.4.1 we already proved that the tracking phase controller results in a globally asymptotically stable closed-loop system. ■

Corollary 4.2.11 *For all $C_0 > 2$ and $C_1 > 0$ there exists a $\beta > 0$ such that the composite controller (4.20) satisfies*

$$|u(t)| \leq \beta \qquad t \geq 0$$

Proof Proposition 4.2.1 gives us a switching time $t_s \geq 0$ such that $|x(t_s)| \leq C_0$ and $|\dot{x}(t_s)| \leq C_1$. Analogously to Corollary 3.4.8 we obtain for all $t \geq t_s$:

$$\begin{aligned} |e(t)| &\leq \sqrt{\frac{1}{\alpha}[(B_1 + C_1) + \lambda(B_0 + C_0)]^2 + (B_0 + C_0)^2 + \frac{\lambda_{max}}{\alpha} E_\mu^2} \\ |\dot{e}(t)| &\leq \sqrt{\frac{1}{\alpha}[\lambda(B_1 + C_1) + (\lambda^2 + \alpha)(B_0 + C_0)]^2 + (B_1 + C_1)^2 + \frac{\lambda_{max}(\lambda^2 + \alpha)}{\alpha} E_\mu^2} \\ |\tilde{\mu}(t)| &\leq \sqrt{\frac{1}{\lambda_{min}}[(B_1 + C_1) + \lambda(B_0 + C_0)]^2 + \frac{\alpha}{\lambda_{min}}(B_0 + C_0)^2 + \frac{\lambda_{max}}{\lambda_{min}} E_\mu^2} \end{aligned}$$

where $\alpha = K_{p,2} + \lambda(K_{d,2} - \lambda)$. Notice, the tracking phase controller satisfies

$$|u(t)| \leq B_2 + K_d|\dot{e}(t)| + K_{p,2}|e(t)| + |E_\mu + \tilde{\mu}(t)||(B_0 + |e(t)|)^2 - 1||B_1 + \dot{e}(t)| + |q|$$

So $|u(t)| \leq \phi(q, B_0, B_1, B_2, C_0, C_1, K_{p,2}, K_{d,2})$ where

$$\begin{aligned} \phi(\cdot) &= B_2 + K_d\sqrt{\frac{1}{\alpha}[\lambda(B_1 + C_1) + (\lambda^2 + \alpha)(B_0 + C_0)]^2 + (B_1 + C_1)^2 + \frac{\lambda_{max}(\lambda^2 + \alpha)}{\alpha} E_\mu^2} \\ &\quad + K_{p,2}\sqrt{\frac{1}{\alpha}[(B_1 + C_1) + \lambda(B_0 + C_0)]^2 + (B_0 + C_0)^2 + \frac{\lambda_{max}}{\alpha} E_\mu^2} \end{aligned}$$

$$+\left|E_\mu + \sqrt{\frac{1}{\lambda_{min}}[(B_1+C_1)+\lambda(B_0+C_0)]^2 + \frac{\alpha}{\lambda_{min}}(B_0+C_0)^2 + \frac{\lambda_{max}}{\lambda_{min}}E_\mu^2}\right| \cdot$$

$$\cdot \left|\left(B_0 + \sqrt{\frac{1}{\alpha}[(B_1+C_1)+\lambda(B_0+C_0)]^2 + (B_0+C_0)^2 + \frac{\lambda_{max}}{\alpha}E_\mu^2}\right)^2 - 1\right| \cdot$$

$$\cdot \left|B_1 + \sqrt{\frac{1}{\alpha}[\lambda(B_1+C_1)+(\lambda^2+\alpha)(B_0+C_0)]^2 + (B_1+C_1)^2 + \frac{\lambda_{max}(\lambda^2+\alpha)}{\alpha}E_\mu^2}\right| + |q|$$

where $\alpha = K_{p,2} + \lambda(K_{d,2} - \lambda)$.
When we consider the waiting phase controller we obtain

$$|u(t)| \le K_{p,1} + K_{d,1} + |q| \qquad t < t_s$$

from which it is obvious that

$$\beta = \max\{K_{p,1} + K_{d,1} + |q|, \phi(q, B_0, B_1, B_2, C_0, C_1, K_{p,2}, K_{d,2}, E_{p_1}, E_p)\}$$

suffices. ∎

In case also q is unknown, the controller (4.19) will still be a suitable tracking phase controller. We only need a proper first phase controller, i.e. one that assures there exists a time $t_s \ge 0$ such that $|x(t_s)| \le C_0$ and $|\dot{x}(t_s)| \le C_1$ for any C_0 and C_1. Unfortunately we have not been able to derive a suitable first phase control law that results in a globally uniformly ultimately bounded closed-loop system.

4.3 Bounded control, using a velocity observer

In this section we consider the problem of tracking a desired trajectory $x_d(t) \in \mathcal{C}^2$ under input limitations

$$|u(t)| \le u_{max} \qquad t \ge 0$$

The controllers in the previous section require knowledge of the velocity $\dot{x}$. In case we are only able to measure x, we still want to establish global asymptotical stability of the closed-loop system. Therefore, in this section we assume that we only know x and not $\dot{x}$. We develop observers for the velocity and use the velocity estimates in our control laws.
In order to deal with input limitations, we again propose composite controllers.

4.3.1 Controlling towards a fixed point

In anology to the previous section, we want to develop waiting phase controllers that control the system towards the origin. Therefore we consider controller-observer combinations that control the state towards a fixed point x_f. We again distinguish two cases, that is $|x_f| > 2$ and $|x_f| \le 2$.

The case $|x_f| > 2$

Notice that the control law (4.12) does not use $\dot{x}$. Therefore, in case $|x_f| > 2$ we can simply use (4.12):

$$u = x_f - q\cos(\omega t) - K_p f(e) \tag{4.21}$$

where $K_p > 0$ is a constant and $f \in \mathcal{F}$.

The case $|x_f| \leq 2$

We again use a composite control strategy to solve this problem. We firstly control the state towards an $\tilde{x}_f$ for which $|\tilde{x}_f| > 2$ using (4.21). This assures $x(t_s)$ and $\dot{x}(t_s)$ are within prescribed bounds. Secondly we apply a second controller, to achieve convergence of the state towards x_f.

Proposition 4.3.1 *Consider for $|x_f| \leq 2$ the system (4.1). Then for all $C_0 > 2$ and $C_1 > 0$ there exists a switching time $t_s \geq 0$ and a constant $\tilde{x}_f$ such that the composite control law*

$$u = \begin{cases} \tilde{x}_f - q\cos(\omega t) - K_{p,1} f_1(x - \tilde{x}_f) & t < t_s \\ x_f - q\cos(\omega t) - K_{p,2} f_2(e) + [\mu + K_d][w + p(e) - (\mu + K_p + L_p)(e - z)] & t \geq t_s \end{cases} \tag{4.22}$$

where $e \equiv x - \tilde{x}_f$ denotes the error, $K_{p,1} \geq 0$, $K_{p,2} \geq 0$, $K_d > 0$ and $L_p > 0$ are constants, $f_1, f_2 \in \mathcal{F}$, and z and w generated from the observer

$$\begin{aligned} \dot{z} &= -w - p(e) + (\mu + K_d + L_p)(e - z) \\ \dot{w} &= z + K_{p,2} f_2(e) \end{aligned} \tag{4.23}$$

and $p(e) = \mu[\frac{1}{3}e^3 + x_d e^2 + (x_d^2 - 1)e]$, results in a globally asymptotically stable closed-loop system.

Proof Let t_s be a moment for which both $|x(t_s)| \leq C_0$ and $|\dot{x}(t_s)| \leq C_1$. We can choose $\tilde{x}(f)$ such that $2 < |\tilde{x}_f| < C_0$. In Proposition 4.2.1 the existence of t_s has been shown. For $t \geq t_s$ we can write the closed-loop system as:

$$\begin{gathered} \ddot{e} - \mu(1 - x^2)\dot{e} + e + (\mu + K_d)\dot{z} + K_{p,2} f_2(e) = 0 \\ \ddot{z} - \mu(1 - x^2)\dot{e} + z - (\mu + K_d)\dot{e} + (\mu + K_d)\dot{z} + K_{p,2} f_2(e) - L_p(\dot{e} - \dot{z}) = 0 \end{gathered} \tag{4.24}$$

From (4.24) it is easy to see that

$$\ddot{\tilde{e}} + \tilde{e} + (\mu + K_d)\dot{e} + L_p \dot{\tilde{e}} = 0$$

where $\tilde{e} \equiv e - z$ denotes the estimate error.
Consider the radially unbounded candidate Lyapunov function

$$V(e, \dot{e}, \tilde{e}, \dot{\tilde{e}}) = \frac{1}{2}\dot{e}^2 + \frac{1}{2}e^2 + \frac{1}{2}\dot{\tilde{e}}^2 + \frac{1}{2}\tilde{e}^2 + K_{p,2} F_2(e) \tag{4.25}$$

where

$$F_2(e) = \int_0^e f_2(\zeta) d\zeta$$

which is obviously positive definite. Differentiating along the closed-loop system (4.24) results in

$$\dot{V}(e, \dot{e}, \tilde{e}, \dot{\tilde{e}}) = -(\mu x^2 + K_d)\dot{e}^2 - L_p \dot{\tilde{e}}^2$$

which is negative semi-definite in the state $(e, \dot{e}, \tilde{e}, \dot{\tilde{e}})$. LaSalle's theorem again completes the proof. ∎

Corollary 4.3.2 *For all $C_0 > 2$ and $C_1 > 0$ there exists a $\beta > 0$ such that the composite controller (4.22) satisfies*

$$|u(t)| \leq \beta \qquad t \geq 0$$

Proof Proposition 4.2.1 gives us a switching time $t_s \geq 0$ such that $|x(t_s)| \leq C_0$ and $|\dot{x}(t_s)| \leq C_1$, so $|e(t_s)| \leq B_0 + C_0$ and $|\dot{e}(t_s)| \leq B_1 + C_1$. From the proof op Proposition 4.3.1 we know that the Lyapunov function (4.25) is a non-increasing function for all $t \geq t_s$. In case we initialize our observer as $z(t_s) = \dot{z}(t_s) = 0$, we have for all $t \geq t_s$:

$$\begin{aligned} |\dot{e}(t)| &\leq \sqrt{2(B_0 + C_0)^2 + 2(B_1 + C_1)^2 + K_{p,2} F_2(B_0 + C_0)} \\ |\dot{\tilde{e}}(t)| &\leq \sqrt{2(B_0 + C_0)^2 + 2(B_1 + C_1)^2 + K_{p,2} F_2(B_0 + C_0)} \end{aligned}$$

resulting in

$$|u(t)| \leq \phi(\mu, q, B_0, B_1, C_0, C_1, K_d, K_{p,2})$$

where

$$\phi(\cdot) = |x_f| + |q| + K_{p,2} + 2(\mu + K_d)\sqrt{2(B_0 + C_0)^2 + 2(B_1 + C_1)^2 + K_{p,2}F_2(B_0 + C_0)}$$

When we consider the first phase controller we obtain

$$|u(t)| \leq |\tilde{x}_f| + |q| + K_{p,1} \qquad \forall t < t_s$$

from which it is obvious that

$$\beta = \max\{|\tilde{x}_f| + |q| + K_{p,1}, \phi(\mu, q, B_0, B_1, C_0, C_1, K_d, K_{p,2})\}$$

suffices. ∎

Corollary 4.3.3 *Consider the system (4.1). If*

$$u_{max} > |q| + \max\{2, |x_f| + 2\mu\sqrt{2(B_0 + 2)^2 + 2B_1^2}\}$$

then there exists a composite control law of the form (4.22) that satisfies the constraint (4.3).

4.3.2 Trajectory tracking

In the previous subsection we have developed two suitable first phase controllers, that both guarantee global convergence of the closed-loop system to any desired point, satisfying input limitations. In this section we combine those results with the ones in section 4.1, in anology to the previous section, resulting in global trajectory tracking under input limitations.

Proposition 4.3.4 *Consider the system (4.1). For all $C_0 > 2$ and $C_1 > 0$ there exists a switching time $t_s \geq 0$ and a constant $\tilde{x}_f$ such that the composite control law*

$$u = \begin{cases} \tilde{x}_f - q\cos(\omega t) - K_{p,1}f(x - \tilde{x}_f) & t < t_s \\ \ddot{x}_d - \mu\dot{x}_d + x_d - q\cos(\omega t) - K_d\dot{\hat{e}} - K_{p,2}\hat{e} + \mu x^2\dot{x}_d & t \geq t_s \end{cases} \tag{4.26}$$

where $K_{p,1} \geq 0$, $\lambda K_d = K_{p,2} \geq 0$ and $K_d > \mu + \lambda + \frac{9}{2}\mu B_0^2$ are constants, $f \in \mathcal{F}$, and $\hat{e}$ generated by the observer

$$\begin{aligned} \dot{\hat{e}} &= w + 2K_d(e - \hat{e}) + \mu e - \frac{1}{3}\mu e^3 - \mu x_d e^2 \\ \dot{w} &= 2K_p(e - \hat{e}) - e + \mu\dot{x}_d e^2 \end{aligned} \tag{4.27}$$

results in a globally asymptotically stable closed-loop system.

Corollary 4.3.5 *For all $C_0 > 2$ and $C_1 > 0$ there exists a $\beta > 0$ such that the composite controller (4.26) satisfies*

$$|u(t)| \leq \beta \qquad t \geq 0$$

Proof In his proof of Proposition 4.1.1 Stucking showed the Lyapunov function

$$V = \frac{1}{2}(\dot{e} + \lambda e)^2 + \frac{1}{2}((K_p + 1) + \lambda(K_d - \mu) - \lambda^2)e^2 + \frac{1}{4}\mu\lambda e^4 + \frac{1}{2}(\dot{\tilde{e}} + \lambda\tilde{e})^2 + \frac{1}{2}(K_p + \lambda K_d - \lambda^2)\tilde{e}^2$$

is non-increasing along solutions of the closed-loop system (4.1,4.4,4.5), where $\tilde{e} \equiv e - \hat{e}$ denotes the observer-error. In case we initialize our observer as $\dot{\hat{e}} = \hat{e} = 0$ we obtain for $t \geq t_s$:

$$\begin{aligned}
|e(t_s)| &\leq \sqrt{\frac{2V(t_s)}{\alpha_1}} \\
|\dot{e}(t_s)| &\leq \sqrt{\frac{2(\lambda^2+\alpha_1)V(t_s)}{\alpha_1}} \\
|\tilde{e}(t_s)| &\leq \sqrt{\frac{2V(t_s)}{\alpha_2}} \\
|\dot{\tilde{e}}(t_s)| &\leq \sqrt{\frac{2(\lambda^2+\alpha_2)V(t_s)}{\alpha_2}}
\end{aligned}$$

where $\alpha_1 = (K_p + 1) + \lambda(K_d - \mu) - \lambda^2$, $\alpha_2 = K_p + \lambda K_d - \lambda^2$ and

$$\begin{aligned}
V(t_s) &= \frac{1}{2}((B_1 + C_1) + \lambda(B_0 + C_0))^2 + \frac{1}{2}((K_p + 1) + \lambda(K_d - \mu) - \lambda^2)(B_0 + C_0)^2 + \\
&\quad + \frac{1}{4}\mu\lambda(B_0 + C_0)^4 + \frac{1}{2}((B_1 + C_1) + \lambda(B_0 + C_0))^2 + \frac{1}{2}(K_p + \lambda K_d - \lambda^2)(B_0 + C_0)^2
\end{aligned}$$

where we used

$$(\dot{e} + \lambda e)^2 + \alpha e^2 = \dot{e}^2 + 2\lambda \dot{e} e + (\lambda^2 + \alpha)e^2 = (\frac{\lambda}{\sqrt{\lambda^2+\alpha}}\,\dot{e} + \sqrt{\lambda^2+\alpha}\;e)^2 + \frac{\alpha}{\lambda^2+\alpha}\dot{e}^2$$

So $|u(t)| \leq \phi(\mu, q, B_0, B_1, B_2, C_0, C_1, K_p, K_d)$ where

$$\begin{aligned}
\phi(\cdot) &= B_2 + \mu B_1 + B_0 + |q| + K_d\left(\sqrt{\frac{2(\lambda^2+\alpha_1)V(t_s)}{\alpha_1}} + \sqrt{\frac{2(\lambda^2+\alpha_2)V(t_s)}{\alpha_2}}\right) \\
&\quad + K_{p,2}\left(\sqrt{\frac{2V(t_s)}{\alpha_1}} + \sqrt{\frac{2V(t_s)}{\alpha_2}}\right) + \mu B_1\left(C_0 + \sqrt{\frac{2V(t_s)}{\alpha_1}}\right)^2
\end{aligned}$$

When we consider the waiting phase controller we obtain

$$|u(t)| \leq |\tilde{x}_f| + |q| + K_{p,1} \qquad t < t_s$$

from which it is obvious

$$\beta = \max\{|\tilde{x}_f| + |q| + K_{p,1}, \phi(\mu, q, B_0, B_1, B_2, C_0, C_1, K_{p,3}, K_{d,2})\}$$

suffices. ∎

Remark 4.3.6 We can also use (4.22) as first phase controller instead of (4.21). This enables us to have a smaller position-error at the moment we switch from waiting phase to tracking phase, resulting in a lower upperbound.

Remark 4.3.7 Notice we switch controllers at a moment $|\dot{x}(t)| \leq C_1$. But we are not able to observe $\dot{x}$! How do we know when to switch? As already mentioned in Remark 3.5.8, we can use numerical differentiation techniques to approximate $\dot{x}(t)$. Although it is not preferable to use those estimates in the controllers of the previous section we can still use those estimates to determine whether $|\dot{x}(t)| \leq C_1$ or not. We only know for sure that $|\dot{x}(t_s)| \leq C_1$ a little later.

Remark 4.3.8 In anology to the previous chapter, we achieved rather weak estimates in determining the value of β. In case the desired trajectory x_d is an uncontrolled trajectory (stable, or unstable)

$$\ddot{x}_d - \mu\left(1 - x_d^2\right)\dot{x}_d + x_d = q\cos(\omega t)$$

the bound on the control input can be less restrictive:

$$\begin{aligned}
\beta &= \mu|e(t)|(x(t) + x_d(t))B_1 + K_d|\dot{\hat{e}}(t)| + K_{p,2}|\hat{e}(t)| \\
&= \mu\sqrt{\frac{2V(t_s)}{\alpha_1}}\left(2B_1 + \sqrt{\frac{2V(t_s)}{\alpha_1}}\right)B_1 + K_d\left(\sqrt{\frac{2(\lambda^2+\alpha_1)V(t_s)}{\alpha_1}} + \sqrt{\frac{2(\lambda^2+\alpha_2)V(t_s)}{\alpha_2}}\right) + \\
&\quad + K_{p,2}\left(\sqrt{\frac{2V(t_s)}{\alpha_1}} + \sqrt{\frac{2V(t_s)}{\alpha_2}}\right)
\end{aligned}$$

Proposition 4.3.9 *Consider the system (4.1). For all $\tilde{C}_0 > 2$, $\tilde{C}_1 > 0$, $C_0 > 0$ and $C_1 > 0$ there exist switching times $t_{s,1} \geq 0$ and $t_{s,2} \geq t_{s,1}$ such that the composite control law*

$$u = \begin{cases} \tilde{x}_f - q\cos(\omega t) - K_{p,1}f(x - \tilde{x}_f) & t < t_{s,1} \\ -q\cos(\omega t) - K_{p,2}f_2(x) + [\mu + K_{d,1}][w_1 + p(x) - (\mu + K_{d,1} + L_p)(x - z)] & t_{s,1} \leq t < t_{s,2} \\ \ddot{x}_d - \mu\dot{x}_d + x_d - q\cos(\omega t) - K_{d,2}\dot{\hat{e}} - K_{p,3}\hat{e} + \mu x^2\dot{x}_d & t \geq t_{s,2} \end{cases} \tag{4.28}$$

where $|\tilde{x}_f| > 2$, $K_{p,1} \geq 0$, $K_{p,2} \geq 0$, $K_{d,1} > 0$ and $K_{d,2} \geq 0$ are constants, $f \in \mathcal{F}$, and w_1, z and $\hat{e}$ respectively generated by the observers

$$\begin{array}{lrcll} & \dot{z} &=& -w_1 - p(x) + (\mu + K_{d,1} + L_p)(x - z) & \multirow{2}{*}{$t_{s,1} \leq t < t_{s,2}$} \\ & \dot{w}_1 &=& z + K_{p,2}f(x) & \\ \textit{and} & & & & \\ & \dot{\hat{e}} &=& w_2 + 2K_d(e - \hat{e}) - p(e) + \mu x_d^2 e & \multirow{2}{*}{$t \geq t_{s,2}$} \\ & \dot{w}_2 &=& 2K_p(e - \hat{e}) - e + \mu\dot{x}_d e^2 & \end{array} \tag{4.29}$$

and $p(e) = \mu[\frac{1}{3}e^3 + x_d e^2 + (x_d^2 - 1)e]$, results in a globally asymptotically stable closed-loop system.

Corollary 4.3.10 *For all $\tilde{C}_0 > 2$, $\tilde{C}_1 > 0$, $C_0 > 0$ and $C_1 > 0$ there exist a time $t_s \geq 0$, an $\tilde{x}_f > 2$ and a $\beta > 0$ such that the composite controller (4.28) satisfies*

$$|u(t)| \leq \beta \qquad t \geq 0$$

Proof From the proof of Corollaries 4.3.2 and 4.3.5 is is obvious that

$$\beta = \max\{|\tilde{x}_f| + |q| + K_{p,1}, \phi_1(\mu, q, B_0, B_1, \tilde{C}_0, \tilde{C}_1, K_d, K_{p,2}), \phi_2(\mu, q, B_0, B_1, B_2, C_0, C_1, K_{p,3}, K_{d,2})\}$$

suffices, where

$$\begin{aligned}
\phi_1(\cdot) &= |x_f| + |q| + K_{p,2} + 2(\mu + K_d)\sqrt{2(B_0 + \tilde{C}_0)^2 + 2(B_1 + \tilde{C}_1)^2 + K_{p,2}F_2(B_0 + \tilde{C}_0)} \\
\phi_2(\cdot) &= B_2 + \mu B_1 + B_0 + |q| + K_{d,2}\left(\sqrt{\frac{2(\lambda^2+\alpha_1)V(t_s)}{\alpha_1}} + \sqrt{\frac{2(\lambda^2+\alpha_2)V(t_s)}{\alpha_2}}\right) \\
&\quad + K_{p,3}\left(\sqrt{\frac{2V(t_s)}{\alpha_1}} + \sqrt{\frac{2V(t_s)}{\alpha_2}}\right) + \mu B_1\left(C_0 + \sqrt{\frac{2V(t_s)}{\alpha_1}}\right)^2
\end{aligned}$$

where $\alpha_1 = (K_{p,2} + 1) + \lambda(K_{d,1} - \mu) - \lambda^2$, $\alpha_2 = K_{p,3} + \lambda K_{d,2} - \lambda^2$ and

$$V(t_s) = \frac{1}{2}((B_1 + C_1) + \lambda(B_0 + C_0))^2 + \frac{1}{2}((K_{p,3} + 1) + \lambda(K_{d,2} - \mu) - \lambda^2)(B_0 + C_0)^2 +$$

$$+\frac{1}{4}\mu\lambda(B_0+C_0)^4+\frac{1}{2}((B_1+C_1)+\lambda(B_0+C_0))^2+\frac{1}{2}(K_{p,3}+\lambda K_{d,2}-\lambda^2)(B_0+C_0)^2$$
$$\lambda = \frac{K_{p,3}}{K_{d,2}}$$

■

4.4 Simulations

To support our results, we simulated with SIMNON^{TM} the controlled van der Pol's equation. In our first simulation we compare the bounded composite controller (4.18) with the controller

$$u = \ddot{x}_d - \mu(1-x^2)\dot{x} + x - K_d\dot{e} - K_p e - q\cos(\omega t) \tag{4.30}$$

of Stuckings. In our second simulation we compare the analogous controller-observer combinations. In all simulations we use the same parameters $\mu = 5$, $q = 5$, and $\omega = 2.463$ in which case the uncontrolled forced van der Pol equation displays chaotic behaviour [13]. We define the reference motion as

$$x_d(t) = \sin t \qquad t \geq 0$$

In order to demonstrate the suggested controllers deal with input limitations, we choose the initial state $(x(0), \dot{x}(0)) = (3, -2)$.
For both controllers mentioned in [15], i.e. (4.30) and the one of proposition 4.1.1 the gains were chosen analogously to Stuckings thesis as $K_d = 15.5$ and $K_p = 46.5$. The resulting performance is depicted in figure 4.1.

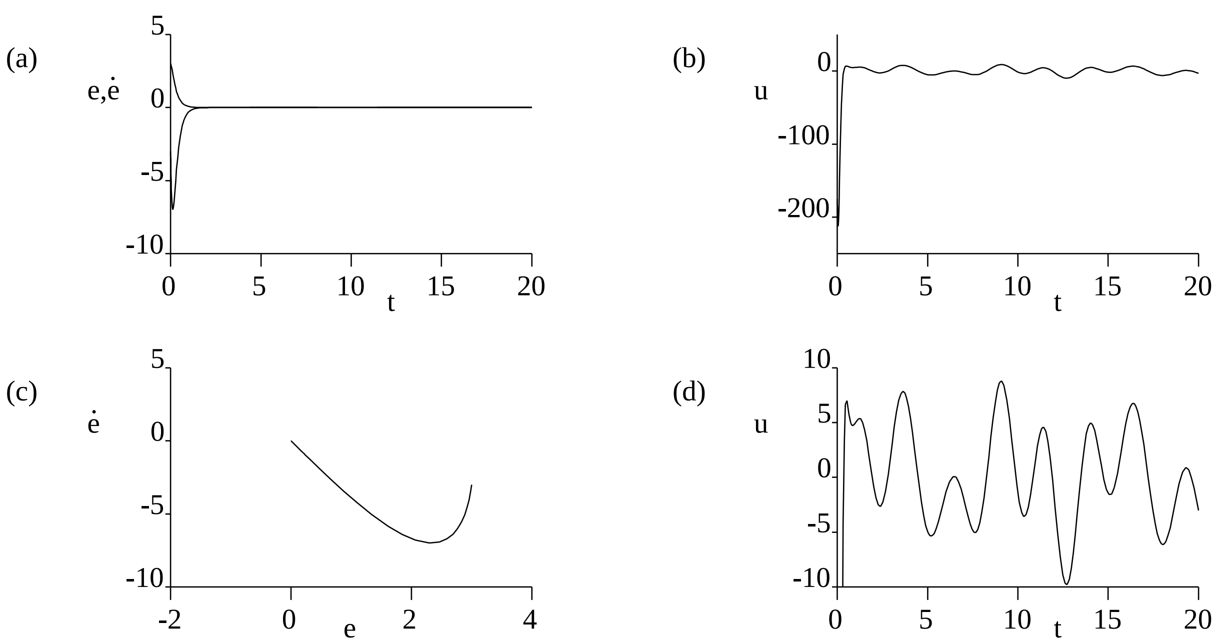

Figure 4.1: The performance of Stucking's controller (4.30)

Figure 4.1(a) shows the time-trajectories of $e(t)$ and $\dot{e}(t)$, 4.1(b) shows the control $u(t)$ needed, 4.1(c) shows the error trajectory $(e, \dot{e})$ and finally 4.1(d) is the same as 4.1(b), only using an other scale.

We can see the control effort is more than 200 (in absolute value), resulting in a quick error convergence.
For the bounded composite controller (4.18) we chose the gains $K_{p,1} = 9$, $K_{p,2} = 4$, $K_{d,2} = 1$, $K_{p,3} = 4$, $K_{d,3} = 4$, $x_f = 2.1$, $C_0 = 2.2$ $C_1 = 0.1$, and used $f_1(x) = f_2(x) = f_3(x) = f_4(x) = f_5(x) = \tanh(x)$. The resulting performance is depicted in figure 4.2.

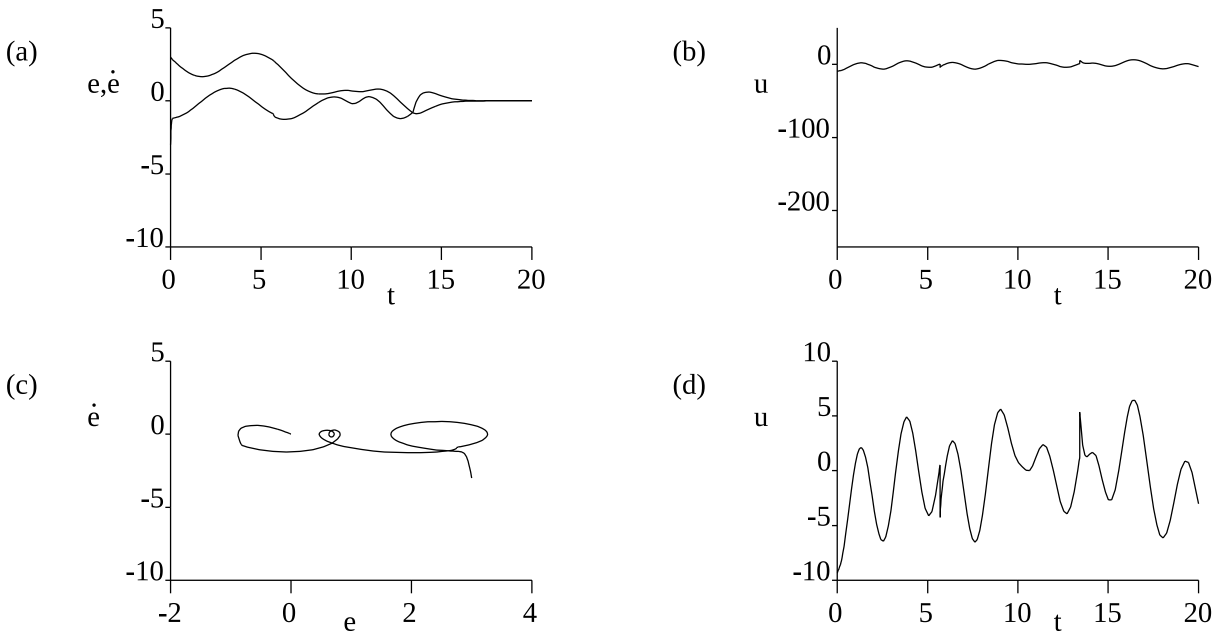

Figure 4.2: The performance of the bounded composite controller (4.18)

Analogously figure 4.2(a) shows the time-trajectories of $e(t)$ and $\dot{e}(t)$, 4.2(b) shows the control $u(t)$ needed, 4.2(c) shows the error trajectory $(e, \dot{e})$ and finally 4.2(d) is the same as 4.2(b), only using an other scale.
We can see the control efforts remain small: we have $|u(t)| \leq 10$. Again the only price we have to pay is the time needed to reduce the errors. In figure 4.2(d) we can also see the two moments we switched controllers.
When we look at the overall tracking error and control effort as defined in the previous chapter, we obtain, using $t_f = 20$:

Controller	Tracking error	Control effort
Stuckings	15.391	5787.86
Bounded composite	52.930	242.05

From which we might conclude the bounded composite controller (4.18) performs better. When we increase the gains of the composite controller, it is possible to obtain both a smaller tracking error and control effort than for the Stuckings controller.

In our second simulation, we compare the controller-observer variants of the previously mentioned controllers. For the Stuckings-controller-observer, we used the same gains and we initiated the observer in $(\hat{e}(0), w(0)) = (0, 0)$. The resulting performance is depicted in figure 4.3.
Figure 4.3(a) shows the time-trajectories of $e(t)$ and $\dot{e}(t)$, 4.3(b) shows the control $u(t)$ needed, 4.3(c) shows the error trajectory $(e, \dot{e})$ and finally 4.3(d) is the same as 4.3(b), only using an other scale.
Most striking are the large control efforts between -1000 and 1100 and the relatively large time needed before the error converges to zero.

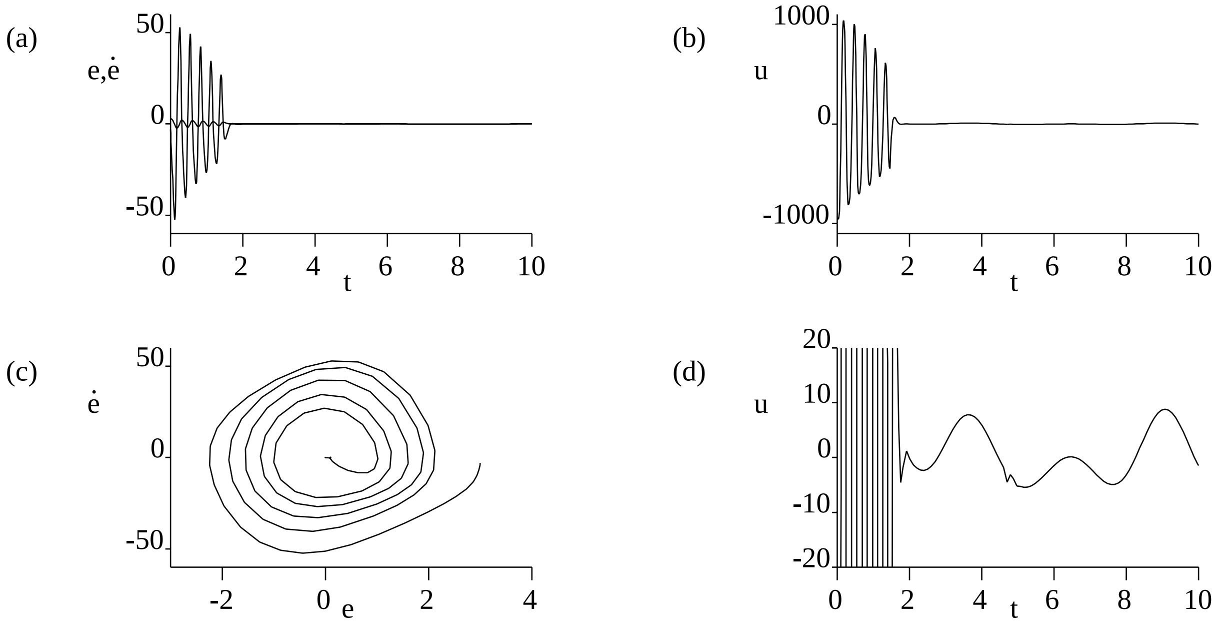

Figure 4.3: The performance of the Stuckings controller-observer (4.4,4.5).

If we replace this controller-observer with its composite analogy, i.e. first using the same first phase controller as in the previous simulation, and then switching to the Stukings controller-observer to obtain tracking, results in the behaviour depicted in figure 4.4.

We can see both the errors and control effort are much less! Most remarkable at the control, is the switching moment. Whereas in the waiting phase the control input satisfies $|u(t)| \leq 10$, we shortly need, when we switch to the Stuckings controller-observer, an effort of about 500! What should we think of this. Is all the effort of using a bounded composite control worthless? No, when we only use Stucking's controller we have completely no idea of the largest control effort needed, whearas in the composite controller, we still have a prescribed bound the input will not exceed. The peek can be much more reduced by choosing a better first phase controller. For instance if we use the composite controller-observer (4.22, 4.23) to steer the system towards $(x, \dot{x}) = (1, 0)$ and switch to the Stuckings controller-observer at the first moment both e and $\dot{e}$ are very small, the prescribed input bound at the tracking phase controller can be much more reduced.

The contrast between both controllers is demonstrated by the tracking error and control effort as defined in the previous chapter:

$$
\begin{aligned}
\text{Tracking error} &= \int_0^{t_f} \left(e(\zeta)^2 + \dot{e}(\zeta)^2\right) d\zeta \\
\text{Control effort} &= \int_0^{t_f} u(\zeta)^2 d\zeta
\end{aligned}
$$

where we used $t_f = 10$:

Controller	Tracking error	Control effort
Stuckings	1052.68	523860
Bounded composite	47.79	5606

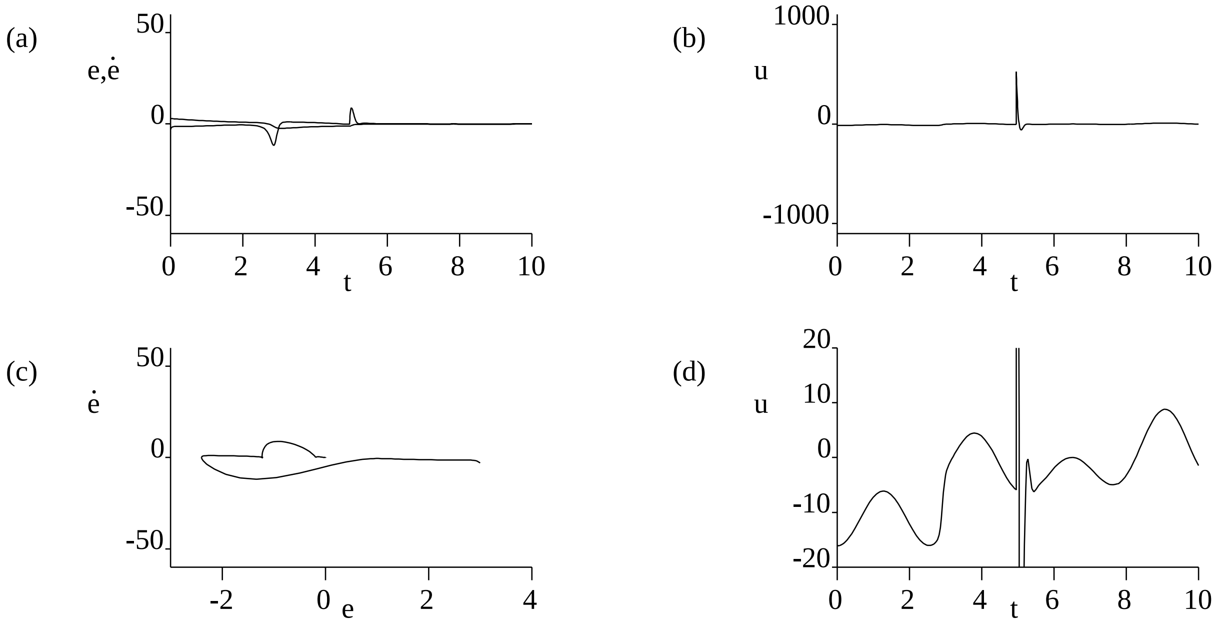

Figure 4.4: The performance of the bounded composite controller-observer (4.26,4.27).

4.5 Summary

In this chapter we considered the problem of controlling the forced van der Pol equation to any desired trajectory using bounded controllers.
We also solved this problem using the concept of composite controllers.
Suitable first phase controllers have been developed, that established global convergence to the origin. The first phase controllers consist of one using state measurements and one using an observer in case velocity measurements are missing. Both controllers established global convergence to a fixed point $|x_f| > 2$, and can also be used in case μ is unknown. In case we want to control the system towards the origin, both first phase controllers can be extended to a composite version.
Those first phase controllers have all been combined with tracking phase controllers, to establish global tracking-error-convergence using bounded composite controllers.
Furthermore, a bounded composite adaptive controller has been developed.
The results have been suported with some simulations, showing a clear contrast with existing controllers.

Chapter 5

Rigid robot systems

In this section we will study the dynamics of a serial n-link rigid robot manipulator (2.4)

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau \tag{5.1}$$

where q is the $n \times 1$ vector of joint displacements, τ is the $n \times 1$ vector of applied joint torques, $M(q)$ is the $n \times n$ symmetric positive definite manipulator inertia matrix, $C(q,\dot{q})\dot{q}$ is the $n \times 1$ vector of centripetal and Coriolis torques, and $g(q)$ is the $n \times 1$ vector of gravitational torques obtained as the gradient of the robot potential energy $\mathcal{U}(q)$. We assume that the links are connected with revolute joints.
Some properties of this system have been described in section 2.3.
We consider the problem of controlling this system to any desired trajectory $q_d(t) \in \mathcal{C}^2$, satisfying

$$|q_d(t)| \leq B_0, |\dot{q}_d(t)| \leq B_1, |\ddot{q}_d(t)| \leq B_2 \qquad t \geq 0 \tag{5.2}$$

under input limitations:

$$|\tau(t)| \leq \tau_{max} \qquad t \geq 0$$

In order to achieve globally stabilizing results dealing with input limitations we again use the concept of a composite feedback controller. We develop some bounded first phase controllers and combine those with already known controllers of Berghuis [1] and Loria and Nijmeijer [9] to obtain bounded globally asymptotically stable tracking controllers.

5.1 Bounded first phase controllers

This section contains an overview of the developed first phase controllers to be combined with the controllers in [1, 9]. We consider the problem of controlling the system (5.1) towards a desired fixed point q_d.

Remark 5.1.1 In this chapter we have to deal with vectors q instead of scalars x. To make the proofs easier to read, we abuse notation to simplify the equations. These are

$$f_i(\varsigma) = \begin{pmatrix} f_{i,1}(\varsigma_1) \\ f_{i,2}(\varsigma_2) \\ \vdots \\ f_{i,n}(\varsigma_n) \end{pmatrix} \text{ and } \sqrt{F_i(\varsigma)} = \begin{pmatrix} \sqrt{\int_0^{\varsigma_1} f_{i,1}(\zeta_1)d\zeta_1} \\ \sqrt{\int_0^{\varsigma_2} f_{i,2}(\zeta_2)d\zeta_2} \\ \vdots \\ \sqrt{\int_0^{\varsigma_n} f_{i,n}(\zeta_n)d\zeta_n} \end{pmatrix}$$

with $f_{i,j} \in \mathcal{F}$.

5.1.1 Using state measurements

Consider the control law

$$\tau = -K_p f_1(e) - K_d f_2(\dot{e}) + g(q) \tag{5.3}$$

where $e \equiv q - q_d$ denotes the position error, K_p and K_d are diagonal positive definite matrices and $f_i(\cdot)$ as mentioned in Remark 5.1.1.
This control law results in the closed-loop system

$$M(q)\ddot{e} + C(q,\dot{q})\dot{e} + K_p f_1(e) + K_d f_2(\dot{e}) = 0 \tag{5.4}$$

Proposition 5.1.2 *Consider the system (5.1) with the control law (5.3). Then the resulting closed-loop system (5.4) is globally asymptotically stable.*

Proof Consider the radially unbounded Lyapunov function candidate

$$V(e,\dot{e}) = \frac{1}{2}\dot{e}^T M(e+q_d)\dot{e} + \sqrt{F_1(e)}^T K_p \sqrt{F_1(e)} \tag{5.5}$$

which is positive definite. Along solutions of (5.4) the time-derivative of (5.5) becomes, using property 2.3.1:

$$\dot{V}(e,\dot{e}) = -\dot{e}^T K_d f_2(\dot{e})$$

which is negative semi-definite in the error state $(e,\dot{e})$. Using LaSalle's Theorem, we conclude asymptotic stability. ■

Remark 5.1.3 This control law is an extension of the one presented in [7].

The controller (5.3) is a suitable first phase controller. In case some parameters are unknown, we have to use another one. A modification of (5.3) suffices.

$$\tau = -K_p f_1(e) - K_d f_2(\dot{e}) + g(q_d) \tag{5.6}$$

where $K_p > k_p^{min} I$ and $K_d > 0$ are $n \times n$ diagonal matrices.
Then we can prove in analogy with [7]:

Proposition 5.1.4 *Consider the system (5.1). Then there exists a constant k_p^{min} such that the controller (5.6) results in a globally asymptotically stable closed-loop system.*

5.1.2 Without velocity measurements

In case we have no velocity measurements at our disposal, we develop an auxiliary system to compensate the lack of knowledge of $\dot{q}$ and use that auxiliary system in our control laws.
Consider the control law

$$\tau = -K_p f_1(e) - K_d f_2(\dot{\hat{e}}) + g(q) \tag{5.7}$$

where $e \equiv q - q_d$ denotes the position error and K_p and K_d are diagonal positive definite matrices. $\hat{e}$ denotes an estimate for $\dot{e}$, and is generated from the first order auxiliary system

$$\dot{\hat{e}} = -L_p \hat{e} + K_d(e - \hat{e}) \tag{5.8}$$

where L_p is an $n \times n$ diagonal matrix such that $L_p + K_d$ is a positive definite matrix
This control law results in the closed-loop system

$$\begin{aligned} M(q)\ddot{e} + C(q,\dot{q})\dot{e} + K_p f_1(e) + K_d f_2(\dot{\hat{e}}) &= 0 \\ \dot{\hat{e}} &= -L_p \hat{e} + K_d(e - \hat{e}) \end{aligned} \tag{5.9}$$

Proposition 5.1.5 *Consider the system (5.1) with the control law (5.7) and auxiliary system (5.8). Then the resulting closed-loop system (5.9) is globally asymptotically stable.*

Proof Consider the radially unbounded Lyapunov function candidate

$$V(e,\dot{e},\hat{e}) = \frac{1}{2}\dot{e}^T M(q)\dot{e} + \sqrt{F_1(e)}^T K_p \sqrt{F_1(e)} + \sqrt{F_2(-L_p\hat{e}+K_d(e-\hat{e}))}^T \sqrt{F_2(-L_p\hat{e}+K_d(e-\hat{e}))} \tag{5.10}$$

which is positive definite. Along solutions of (5.9) the time-derivative of (5.10) becomes, using property 2.3.1:

$$\dot{V}(e,\dot{e},-L_p\hat{e}+K_d(e-\hat{e})) = -(-L_p\hat{e}+K_d(e-\hat{e}))^T(L_p+K_d)f_2(-L_p\hat{e}+K_d(e-\hat{e}))$$

which is negative semi-definite. Using LaSalle's Theorem, we conclude asymptotic stability. ■

The controller (5.7) with auxiliary system (5.8) is a suitable first phase controller, in case all parameters are known. Otherwise, we again use the modified controller:

$$\tau = -K_p f_1(e) - K_d f_2(\dot{\hat{e}}) + g(q_d) \tag{5.11}$$

where $K_p > k_p^{min} I$ and $K_d > 0$ are $n \times n$ diagonal matrices, and $\hat{e}$ generated form the auxiliary system

$$\dot{\hat{e}} = -L_p\hat{e} + K_d(e-\hat{e}) \tag{5.12}$$

resulting in:

Proposition 5.1.6 *Consider the system (5.1). Then there exists a constant k_p^{min} such that the controller (5.11) with auxiliary system (5.12) results in a globally asymptotically stable closed-loop system.*

Proof In analogy with Proposition 5.1.4 ■

5.2 Some useful results

For developing a bounded composite feedback tracking controller, we not only need a globally ultimately uniformly bounded first phase controller but also suitable tracking phase controllers. Many useful results can be found in [1]. In this section we present some results that achieve trajectory tracking to any desired trajectory $q_d(t)$.
Proposition 5.2.1 can be found in [1], as well as Proposition 5.2.3. Proposition 5.2.2 is a straightforward extension of [9].

Proposition 5.2.1 *Consider the system (5.1) with the control law*

$$\tau = M(q)\ddot{q}_d + C(q,\dot{q})\dot{q}_d + g(q) - K_p e - K_d\dot{e} \tag{5.13}$$

where K_p and K_d are $n \times n$ positive definite matrices. Then the resulting closed-loop system is globally asymptotically stable.

Proposition 5.2.2 *Consider the system (5.1) with the control law*

$$\tau = M(q)\ddot{q}_d + C(q,\dot{q}_d)\dot{q}_d + g(q) - K_p f(e) - K_d f(z) \tag{5.14}$$

where K_p and K_d are $n \times n$ positive definite matrices, z is generated from the observer

$$\begin{array}{rcl} z_i & = & w_i + b_i e_i \\ \dot{w}_i & = & -a_i f_i(w_i + b_i e_i) \end{array} \tag{5.15}$$

where this time not $f \in \mathcal{F}$ but $f \in \tilde{\mathcal{F}}$ where

$$\tilde{\mathcal{F}} = \{f \in \mathcal{F} | \forall x \in I\!R : \frac{F(x)}{f^2(x)} \geq \Gamma > 0 \quad \wedge \quad 0 < f'(x) \leq \Delta\}$$

with Γ and Δ some constants and $F(x) = \int_0^x f(\zeta)d\zeta$. Then the resulting closed-loop system is semi-globally asymptotically stable, i.e the resulting closed-loop system is locally asymptotically stable but its region of attraction can be arbitrarily enlarged by suitably selecting the observer gains a_i and b_i.

Proposition 5.2.3 *Consider the system (5.1) with the control law*

$$\tau = M(q,\hat{\theta})\ddot{q}_r + C(q,\dot{q},\hat{\theta})\dot{q}_r + g(g,\hat{\theta}) - K_d s_1 = M_0(q)\ddot{q}_r + C_0(q,\dot{q})\dot{q}_r + g_0(q) + Y(q,\dot{q},\dot{q}_r,\ddot{q}_r)\hat{\theta} - K_d s_1 \tag{5.16}$$

where $\dot{q}_r = \dot{q}_d - \Lambda e$, $s_1 = \dot{q} - \dot{q}_r = \dot{e} + \Lambda e$, and K_p and Λ are $n \times n$ positive definite matrices, and $\hat{\theta}$ generated from the parameter update-law:

$$\dot{\tilde{\theta}}(t) = -\Gamma_0 Y^T(q,\dot{q},\dot{q}_r,\ddot{q}_r)s_1$$

where Γ_0 is a positive definite diagonal matrix. Then the resulting closed-loop system is globally asymptotically stable with respect to e and $\dot{e}$, i.e.

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} \dot{e}(t) = 0$$

5.3 Composite results

In this section, the bounded first phase controllers of this chapter will be combined with the tracking controllers of the previous section, resulting in bounded tracking phase controllers that are capable of tracking any desired tracjectory $q_d(t)$.

Proposition 5.3.1 *There exists a switching time $t_s \geq 0$ such that the composite tracking phase control law*

$$\tau = \begin{cases} -K_{p,1}f_1(e) - K_{d,1}f_2(\dot{e}) + g(0) & t < t_s \\ M(q)\ddot{q}_d + C(q,\dot{q})\dot{q}_d + g(q) - K_{p,2}e - K_{d,2}\dot{e} & t \geq t_s \end{cases} \tag{5.17}$$

where $K_{p,1}$, $K_{d,1}$, $K_{p,2}$ and $K_{d,2}$ are positive definite matrices, results in a globally asymptotically stable closed-loop system.

Proof (Sketch) Proposition 5.2.1 has been established, by showing that a certain Lyapunov function of e and $\dot{e}$ is non-increasing. This gives bounds on $|e(t)|$ and $|\dot{e}(t)|$ for $t \geq t_s$, assuming $|e(t_s)| \leq B_0 + C_0$ and $|\dot{e}(t_s)| \leq B_1 + C_1$ for certain $C_0 > 0$, $C_1 > 0$. Those bounds yield a bound on $|u(t)|$ for $t \geq t_s$. We only need to achieve $|q(t_s)| \leq C_0$ and $|q(t_s)| \leq C_1$ within prescribed bounds. From Proposition 5.1.2 we know this is possible. ∎

Corollary 5.3.2 *Consider the system (5.1). If*

$$u_{max} > M_M B_2 + g_M$$

there exists a composite control law of the form (5.17) that satisfies the constraint (5.2).

Proposition 5.3.3 *There exists a switching time $t_s \geq 0$ such that the composite tracking phase control law*

$$\tau = \begin{cases} -K_{p,1}f_1(e) - K_{d,1}f_2(\dot{\hat{e}}) + g(0) & t < t_s \\ M(q)\ddot{q}_d + C(q,\dot{q}_d)\dot{q}_d + g(q) - K_{p,2}f(e) - K_{d,2}f(z) & t \geq t_s \end{cases} \tag{5.18}$$

where $K_{p,1}$, $K_{d,1}$, $K_{p,2}$ and $K_{d,2}$ are positive definite diagonal matrices, and $\hat{e}$ and z are generated form the observers

$$\begin{aligned} \dot{\hat{e}} &= -L_p\hat{e} + K_d(e - \hat{e}) \\ z_i &= w_i + b_i e_i \\ \dot{w}_i &= -a_i f_i(w_i + b_i e_i) \end{aligned} \tag{5.19}$$

where L_p and $L_p + K_d$ are positive definite diagonal matrices, and $f \in \tilde{\mathcal{F}}$, where

$$\tilde{\mathcal{F}} = \{f \in \mathcal{F} | \forall x \in I\!R : \frac{F(x)}{f^2(x)} \geq \Gamma > 0 \quad \wedge \quad 0 < f'(x) \leq \Delta\},$$

results in a globally asymptotically stable closed-loop system.

Proof (Sketch) Proposition 5.2.2 has been established, by showing that a certain Lyapunov function is non-increasing. This gives bounds on $|e(t)|$, $|\dot{e}(t)|$, and $|z(t)|$ for $t \geq t_s$, assuming $|e(t_s)| \leq B_0 + C_0$, $|\dot{e}(t_s)| \leq B_1 + C_1$, and $|z(t)| \leq C_2$ for certain $C_0 > 0$, $C_1 > 0$, and C_2. From those bounds we obtain a bound on $|u(t)|$ for $t \geq t_s$. We only need to achieve $|q(t_s)| \leq C_0$ and $|q(t_s)| \leq C_1$ within prescribed bounds. From Proposition 5.1.2 we know this is possible. Since we have $|q(t_s)| \leq C_0$ and $|q(t_s)| \leq C_1$ and since we can choose $z(t_s)$ such that $|z(t_s)| \leq C_2$, we are also able to calculate the gains a_i and b_i needed to achieve asymptotically tracking. ■

Remark 5.3.4 Notice this result is stronger than the one proved in Loria and Nijmeijer [9]. They achieved a semi-globally asymptotically stable, i.e. their resulting closed-loop system is locally asymptotically stable but its region of attraction can be arbitrarily enlarged by suitably selecting the observer gains a_i and b_i. Since our first phase controller steers the system towards the origin, we can assure that at the switching time t_s, the state is within prescribed bounds, from which we can determine the observer gains a_i and b_i that guarantee asymptotic stability. Since our first phase controller is globally asymptotically stable, we obtain a globally asymptotically stable result.

Corollary 5.3.5 *Consider the system (5.1). If*

$$u_{max} > M_M B_2 + C_M B_1^2 + g_M$$

there exists a composite control law of the form (5.18) that satisfies the constraint (5.2).

Proposition 5.3.6 *There exists a switching time $t_s \geq 0$ such that the composite tracking phase control law*

$$\tau = \begin{cases} -K_{p,1} f_1(e) - K_{d,1} f_2(\dot{e}) + g(0) & t < t_s \\ M(q, \hat{\theta})\ddot{q}_r + C(q, \dot{q}, \hat{\theta})\dot{q}_r + g(g, \hat{\theta}) - K_{d,2} s_1 & t \geq t_s \end{cases} \tag{5.20}$$

where $K_{p,1}$, $K_{d,1}$, and $K_{d,2}$ are positive definite matrices, and $\hat{\theta}$ generated from the update-law:

$$\dot{\tilde{\theta}}(t) = -\Gamma_0 Y^T(q, \dot{q}, \dot{q}_r, \ddot{q}_r)s_1$$

results in a globally asymptotically stable closed-loop system.

Proof (Sketch) Proposition 5.2.3 has been achieved, by showing that a certain Lyapunov function of e and $\dot{e}$ is non-increasing. This gives bounds on $|e(t)|$ and $|\dot{e}(t)|$ for $t \geq t_s$, assuming $|e(t_s)| \leq B_0 + C_0$ and $|\dot{e}(t_s)| \leq B_1 + C_1$ for certain $C_0 > 0$, $C_1 > 0$, from which we can deduce a bound on $|u(t)|$ for $t \geq t_s$. We only need to achieve $|q(t_s)| \leq C_0$ and $|q(t_s)| \leq C_1$ within prescribed bounds. From Proposition 5.1.4 we know this is possible. ■

Chapter 6

Conclusions and future research

6.1 Conclusions

So far, the research on controlling chaos has mainly been directed to controlling the system towards fixed points or periodic orbits of the system. In [10], Nijmeijer and Berghuis for the first time considered the problem of controlling the chaotic forced Duffing system towards *any* desired trajectory, using feedback control. This thesis extended their results in three ways:

1. The controllers for the Duffing system developed in this thesis, including controller-observer combinations, deal with natural input limitations.
2. Controllers, including controller-observer combinations, for the chaotic forced van der Pol system have been developed, also dealing with natural input limitations.
3. Furthermore adaptive controllers have been developed, to deal with parameter uncertainties (also dealing with natural input limitations).

In order to develop bounded controllers, a new concept has been introduced: composite controllers. The tracking control problem has been devided into the problem of finding a suitable waiting phase controller and a tracking phase controller. The main objective of the waiting phase controller is to reduce the tracking error to allowable proportions. Then one can switch to a second controller, the tracking phase controller, to achieve tracking of the desired trajectory.
Since the waiting phase controller assures that the tracking errors are within previously determined bounds, one is sure the tracking phase controller remains within prescribed bounds. The problem of finding a bounded globally asymptotically stable tracking phase controller, in case a (not neccessarily bounded) locally asymptotically stable tracking controller is available, has been reduced to finding a bounded globally ultimately uniformly bounded controller, whose ball in which finally all solutions will arrive is a subset of the region of attraction of the already known locally asymptotically stable tracking controller. In other words, instead of controlling the system towards the desired trajectory, we define another trajectory, close to the desired trajectory, to which we first control the system. In this thesis, we choose the origin, a fixed point, as trajectory to which we first control the system.
The concept of composite controllers has also been applied in the tracking control problem for rigid robots, resulting in bounded composite controllers, including controller-observer combinations and composite adaptive controllers.

6.2 Future research

The concept of composite controllers is very powerful. Using this concept, as soon as a locally asymptotically stable tracking controller is found, bounded controllers are available for every

system that contains enough damping, i.e. using a bounded controller assures the system will be globally ultimately bounded.
An illustration are the results presented in chapter 5. The development of parameter independent bounded first phase controllers to control the rigid robot system towards a fixed point, resulted in bounded composite tracking controllers and bounded adaptive controllers.

None of the proofs in this thesis conclude that the control laws guarantee an exponentially asymptotically stable closed-loop system. When we look at the simulations, especially the second phase controllers seem to be locally exponentially stable. Such results might be established by choosing other Lyapunov functions.
The controllers in this thesis can also be improved, e.g by choosing a better switching moment or first phase controller. All first phase controllers developed in this thesis try to steer the system towards the origin, since it is the best point to control the system to in case we only know the desired trajectory satisfies

$$|x_d(t)| \leq B_0,\ |\dot{x}_d(t)| \leq B_1,\ |\ddot{x}_d(t)| \leq B_2$$

since it minimizes $e = x - x_d$. In case we explicitely know the desired tracjectory, we could consider better choices. For example, if we know that the desired trajectory is $x_d(t) = \sin(t)$, we might consider to control the system towards the fixed point $(x, \dot{x}) = (1, 0)$, or to the signal $\frac{1}{2}\sin(t)$ in the waiting phase. Using this idea, the large control effort just after switching controllers in case of our second simulation of van der Pol's system, as displayed in figure 4.4(c), can be reduced to less than 10 percent of the current value!

However, most promising is the concept of a composite controller. Using this idea the problem of finding bounded globally asymptotically stable tracking controllers has been reduced to finding locally asymptotically stable tracking controllers (not necessarily bounded) and bounded globally ultimately bounded controllers that control the system towards a fixed point. As shown in chapter 5, this concept opens the way to a lot of new results on bounded globally asymptotically stable tracking controllers!

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