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Memorandum No. 1404

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September 1997

ISSN 0169-2690

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Keywords: Nonlinear Control, Lyapunov Stability, Global Stability, Robotics, State Feedback, Output Feedback, Adaptive Control.

AMS Subject Classification: 93C85, 93C10, 93D15

Abstract

In this paper we address the problem of designing bounded global tracking controllers for rigid robot manipulators. A solution to the problem is obtained by using composite controllers. In the first part, the system is steered 'close' to the desired trajectory, after which a local tracking controller is applied. Both controllers are bounded and yield boundedness of the composite controller. The use of bounded composite controllers is shown to be effective for state feedback controllers, output (position) feedback controllers and adaptive controllers. Simulations illustrate the developed controllers.

1 Introduction

In recent years there has been a strong interest in the development of controllers for the regulation or tracking of rigid robot manipulators. Starting with the computed torque controller several controllers have been designed, which exploit the physical nature of the robot system. Furthermore, several other aspects have been incorporated in modern robot controllers, as for instance, the construction of adaptive controllers in case parameter uncertainties are present in the manipulator model, or the development of controller-observer combinations when velocity measurements are not available for control, see e.g. [2, 4, 14] as well as references therein.
In the last few years some interest has arisen in the from a practical perspective important question of designing tracking controllers which respect actuator constraints. In particular, for position control a bounded controller was developed in [7], see also [5] for an alternative bounded PD-like controller.
Sofar the tracking problem under input constraints has only partially been solved in that a semi-global tracking controller or controller-observer has been derived, see [13].
The main purpose of the present is to develop a globally bounded tracking controller for rigid robot systems or, if desired, slightly more general Euler-Lagrange systems, see [11]. Our control scheme essentially combines a bounded regulation controller with a local asymptotically stable tracking controller, and in essence contains the earlier mentioned results on bounded regulation. The idea of using a combination of two controllers can be exploited in various cases including the situation where only position measurements are available or when an adaptation mechanism is required. In fact, although we will not pursue that here further, we believe the idea to be useful in a far more general context than only for rigid robot manipulators. For instance, in [10] we have shown that also in controlling the periodically forced Duffing equation the same idea can be successfully exploited.

As mentioned our rigid robot controllers are composite and it is worth mentioning that the classes of controllers we use is much broader than those from the literature [1, 3, 5, 6, 7, 12, 13, 15, 17]. The differences in this regard might be of great importance in practical situations.

The organisation of this paper is as follows. Section 2 contains the problem formulation, preliminaries and notation. In section 3 the key idea for the construction of a globally bounded controller is explained. In sections 4, 5 and 6 respectively the controller design with full state measurements, controller design with only position measurements and an adaptive controller design are considered in detail, and allustrated by means of a simulation example. Section 7 contains our concluding remarks.

2 Problem formulation, preliminaries and notation

2.1 Dynamics of rigid robot systems

The dynamics of an n-link rigid robot manipulator without friction or other disturbances can be written as (see e.g. [14, 18])

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau \tag{1}$$

where q is the $n \times 1$ vector of joint displacements, τ is the $n \times 1$ vector of applied torques, $M(q)$ is the $n \times n$ symmetric positive definite manipulator inertia matrix, $C(q,\dot{q})\dot{q}$ is the $n \times 1$ vector of centripetal and Coriolis torques, and $G(q)$ is the $n \times 1$ vector of gravitational torques obtained as the gradient of the robot potential energy $P(q)$, i.e. $G(q) = \frac{\partial P}{\partial q}(q)$. The robots potential energy $P(q)$ is, without loss of generality, assumed to have a global minimum at $q = 0$.

Note that, although the vector $C(q,\dot{q})\dot{q}$ is uniquely defined, several choices for the matrix $C(q,\dot{q})$ are possible. Henceforth we assume that $C(q,\dot{q})$ is defined using the Christoffel symbols. The system (1) possesses some important properties (see e.g. [2, 14, 18]):

Property 2.1 The matrix $\dot{M}(q) - 2C(q,\dot{q})$ is skew-symmetric, that is

$$x^T(\dot{M}(q) - 2C(q,\dot{q}))x = 0 \qquad \forall x \in I\!R^n.$$

Property 2.2 The matrices $M(q)$, $C(q,\dot{q})$ and $G(q)$ are bounded with respect to q, i.e. there exist positive constants M_m, M_M, C_M and G_M such that

$$\begin{array}{ll} 0 < M_m \le \|M(q)\| \le M_M & \forall q \in I\!R^n \\ \|C(q,x)\| \le C_M\|x\| & \forall q, x \in I\!R^n \\ \|G(q)\| \le G_M & \forall q \in I\!R^n. \end{array}$$

Property 2.3 There exists a reparametrization of all unknown parameters into a parameter vector $\theta \in I\!R^p$ that enters linearly in the system dynamics (1). Therefore, the following holds:

$$M(q,\theta)\ddot{q} + C(q,\dot{q},\theta)\dot{q} + G(q,\theta) = M_0(q)\ddot{q} + C_0(q,\dot{q})\dot{q} + G_0(q) + Y(q,\dot{q},\dot{q},\ddot{q})\theta.$$

2.2 Problem formulation

Consider the robotic system (1) where the actuator torques τ are constrained, i.e.

$$\|\tau(t)\| \le \tau_{max} \qquad \forall t \ge 0. \tag{2}$$

Here $\|\cdot\|$ denotes some suitable norm, e.g. $\|\cdot\|_\infty$. Suppose that measurements of the joint positions $q(t)$ and velocities $\dot{q}(t)$ are available, or in case of the output feedback problem that only measurements of the joint positions $q(t)$ are available. Let $q_d(t), t \ge 0$ be a desired trajectory for the manipulator, and assume that $q_d(t)$ is at least two times continuously differentiable in t and satisfies

$$\|q_d(t)\| \le B_0, \quad \|\dot{q}_d(t)\| \le B_1, \quad \|\ddot{q}_d(t)\| \le B_2 \qquad \forall t \ge 0. \tag{3}$$

for given positive constants B_0, B_1, B_2. Then the tracking control problem under actuator constraints consists of designing, if possible, a state feedback law respectively an output feedback law for the actuator torques $\tau(t)$ such that the joint positions $q(t)$ and velocities $\dot{q}(t)$ asymptotically tend towards the desired positions $q_d(t)$ and velocities $\dot{q}_d(t)$ while keeping the applied torques within the in advance specified bounds (2). In other words: design a controller for $\tau(t)$ such that

$$\lim_{t\to\infty} q(t) = q_d(t) \text{ and } \lim_{t\to\infty} \dot{q}(t) = \dot{q}_d(t)$$

while satisfying (2).

2.3 Mathematical preliminaries and notation

Let $\mathcal{C}^k$ denote the set of (at least) n times continuously differentiable functions.
A continuous function $f : I\!R^+ \to I\!R^+$ is said to be of class $\mathcal{K}$ if (see [8])

- $f(x)$ is strictly increasing,
- $f(0) = 0$.

Throughout we denote for any $f : I\!R^n \to I\!R$: $Df(x) \triangleq \frac{\partial f}{\partial x}(x)$.

Definition 2.4 Let $\mathcal{F}^n$ denote the class of continuous functions $f : I\!R^n \to I\!R^n$ for which there exists a radially unbounded positive definite $F : I\!R^n \to I\!R$ such that

$$f(x) = f(x_1, \dots, x_n) = \begin{bmatrix} \frac{\partial F}{\partial x_1}(x_1, \dots, x_n) \\ \vdots \\ \frac{\partial F}{\partial x_n}(x_1, \dots, x_n) \end{bmatrix} \tag{4}$$

and for which $x^T f(x)$ is a positive definite function.

Definition 2.5 Let $\mathcal{B}^n$ denote the class of $f \in \mathcal{F}^n$ that are bounded, i.e the class of $f \in \mathcal{F}^n$ for which there exists a constant $f_M \in I\!R$ such that $\|f(x)\| \leq f_M$ for all $x \in I\!R^n$.

In general it is not always easy to verify whether a given $f : I\!R^n \to I\!R^n$ can be written as the gradient of a radially unbounded $F : I\!R^n \to I\!R$. However, a necessary condition for continuously differentiable f is that its Jacobian $\frac{\partial f}{\partial x}$ is symmetric.
It is easy to see that all functions of the form

$$f(x) = \Lambda \begin{bmatrix} f_1(x_1) \\ \vdots \\ f_n(x_n) \end{bmatrix}$$

or

$$f(x) = Kx$$

are elements of $\mathcal{F}^n$, where Λ is an $n \times n$ diagonal positive definite matrix, $K = K^T$ is an $n \times n$ (not necessarily diagonal) positive definite matrix, and f_i are continuous nondecreasing functions satisfying $f_i(0) = 0$ and $f_i'(0) > 0$ $(i = 1, \dots, n)$.
By choosing $f_i(x) = \tanh(\lambda_i x)$, $f_i(x) = \text{sat}(\lambda_i x)$ or $f_i(x) = \frac{x}{\lambda_i + |x|}$ $(\lambda_i > 0)$ we obtain elements of $\mathcal{B}$, whereas $f(x) = Kx$, $K = K^T > 0$, is an element of $\mathcal{F}^n$ however not of $\mathcal{B}$. Throughout we denote for $f \in \mathcal{F}^n$ by $F(x)$ the via (4) associated function of which f is the gradient.
Two important, easy to verify, properties for any $f \in \mathcal{F}^n$ are the following:

Property 2.6 Let $f \in \mathcal{F}^n$. Then $f(x) = 0$ if and only if $x = 0$.

Property 2.7 Let $f \in \mathcal{F}^n$. Then $(\forall \epsilon > 0)(\exists \delta > 0)(\|x\| > \epsilon \Rightarrow \|f(x)\| > \delta)$.

Definition 2.8 Let $\Delta > 0$ be a positive constant. Then $s_\Delta(x) : I\!R \to I\!R$ denotes a nondecreasing continuous function that equals 0 for $x \leq 0$ and equals 1 for $x \geq \Delta$.

Examples of such $s_\Delta(x) \in \mathcal{C}^\infty$ are

$$s_\Delta(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{2}\exp\left(\dfrac{a}{x(x-b)} - \dfrac{a}{\frac{1}{2}\Delta(\frac{1}{2}\Delta - b)}\right) & 0 < x \leq \frac{1}{2}\Delta \\ 1 - \frac{1}{2}\exp\left(\dfrac{a}{(x-\Delta)(x-\Delta+b)} - \dfrac{a}{\frac{1}{2}\Delta(\frac{1}{2}\Delta - b)}\right) & \frac{1}{2}\Delta < x < \Delta \\ 1 & \Delta \leq x \end{cases} \tag{5}$$

where $a > 0$, $b \geq \Delta > 0$. In Figure 1 such a function $s_\Delta(x)$ is shown where $a = 3$, $b = \frac{3}{2}$ and $\Delta = 1$.

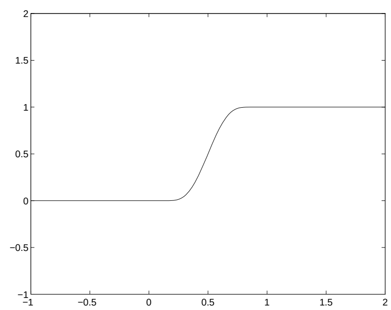

Figure 1: An example of $s_\Delta(x)$ like in (5) where $a = 3$, $b = \frac{3}{2}$ and $\Delta = 1$.

An important tool needed in the stability analysis of the next sections is given by the following theorem due to Matrosov, see [16].

Theorem 2.9 *Consider the system $\dot{x} = f(t,x)$ with $f(t,0) = 0$ for all $t \geq 0$. Assume there exist two $\mathcal{C}^1$ functions $V(t,x)$, $W(t,x)$ from $[0,\infty) \times \Omega$ into $I\!R$, with Ω an open connected region in $I\!R^n$ containing the origin, a $\mathcal{C}^0$ function $V^* : \Omega \to I\!R$; three functions a, b, c of class $\mathcal{K}$ such that, for every $(x,t) \in \Omega \times [0,\infty)$*

M1. $a(\|x\|) \leq V(t,x) \leq b(\|x\|)$,

M2. $\dot{V}(t,x) \leq V^*(x) \leq 0$,

M3. $|W(t,x)|$ *is bounded,*

M4. $\max(d(x,E), |\dot{W}(t,x)|) \geq c(\|x\|)$, *where* $E \equiv \{x \in \Omega | V^*(x) = 0\}$,

M5. $\|f(t,x)\|$ *is bounded.*

Then:

1. *For all $x_0 \in \{x \in \Omega | V(t,x) \leq a(r)\}$, with $r > 0$ such that the closed ball $B_r \subset \Omega$, $x(t)$ tends to zero uniformly in t_0 and x_0 as t tends to infinity.*

2. *The origin is uniformly asymptotically stable.*

Remark 2.10 Since M2 is a relaxation of the condition known from standard Lyapunov theory, we have to establish that no trajectory can stay identically at a point where $V^*(x) = 0$, except at the origin. That is what the conditions M3 and M4 are for. A second auxiliary function W is defined that according to M3 has to be bounded. Suppose the trajectory stays close to points x for which $V^*(x) = 0$, different from the origin. Then M4 implies that the rate of change of W is of constant sign, which contradicts the fact that W is bounded. Therefore trajectories have to converge to the origin.

Lemma 2.11 (see [15]) *Condition M4 of Theorem 2.9 is satisfied if the following conditions are satisfied:*

M4'a. *$\dot{W}(x,t)$ is continuous in both arguments and depends on time in the following way: $\dot{W}(x,t) = g(x,\beta(t))$ where g is continuous in both of its arguments, β is also continuous and its image lies in a bounded set K_1.*

M4'b. *There exists a class $\mathcal{K}$ function, k, such that $|\dot{W}(x,t)| \geq k(\|x\|)$ for all $x \in E$ and $t \geq 0$.*

Remark 2.12 Following the nomenclature used in [15], we say that $\dot{W}(x,t)$ depends on time continuously through a bounded function.

3 Composite controllers

Suppose there exists a controller that steers the system towards the origin, provided one starts within some (possibly small) region of attraction. In case we want to extend this controller to a global one, there are two ways to establish this. The first way is to modify the controller in such a way that global asymptotic stability of the error-dynamics is achieved. A second way is trying to find a global controller that steers the system into the region of attraction of the (locally) stabilizing controller. As soon as we are in its region of attraction, we can switch controllers and the resulting composite controller is a globally stabilizing controller. This second approach results in an easier problem, since we only have to find a controller that steers the system into some prescribed region, instead of to the origin. So we seek a globally ultimately uniformly bounded or practically stable controller, whereas in the first approach we have to find a globally asymptotically stable controller.

If we want to find a globally stabilizing controller we can use this idea and therefore seperate the problem into two problems that both may be easier to solve. In case we are able to find both a locally stabilizing controller with some region of attraction and a second controller that globally steers the system into that region of attraction, the composite controller will be a globally stabilizing one. Here the composition of both controllers consists of using the global controller until the system is inside the region of attraction of the stabilizing controller and then switch to this stabilizing controller.

In case we want to find *bounded* globally stabilizing controllers, the concept of composite controllers may become more important. Not only can we separate the problem into the two easier problems of finding both a locally stabilizing controller with some region of attraction and a bounded controller that globally steers the system into that region of attraction, also the stabilizing controller does not neccessarily have to be a priori bounded. From the stability analysis of the stabilizing controller we usually know that all signals will remain bounded. The only problem is that those bounds depend on the initial conditions. Since we only switch to the stabilizing controller in case we are in a prescribed region, we can determine in advance from this stability analysis an upperbound on the control input of the stabilizing controller. Therefore only the controller that steers the system into that region has to be a priori bounded.

Throughout we use this idea to derive bounded globally asymptotically stable tracking controllers in the following way: first we derive a (semi)globally asymptotically stable tracking controller (not a priori bounded). Then we derive a bounded globally regulating controller that steers the system to a fixed point. Since $q_d(t)$ and its derivatives are within the bounds (3), a regulating controller will bring the tracking error within a priori known bounds. Since we only switch to the (semi)globally tracking controller as soon as we are within those bounds, we can derive an upperbound on the control input of the (semi)globally tracking controller. Therefore, the overall controller is bounded and yields global tracking.

4 Using state measurements

In this section we consider the bounded tracking problem as formulated in section 2.2, assuming that the full state $(q, \dot{q})$ is available for measurement. We tackle the problem using the idea presented in the previous section. We split the problem of finding a bounded globally asymptotically controller into two subproblems, namely that of finding a (semi)globally asymptotically stable tracking controller and that of finding a globally bounded regulating controller which steers the system towards a fixed point. Section 4.1 contains the derivation of the bounded globally asymptotically stable tracking controller. Section 4.2 shows the performance of the controllers derived in some simulations.

4.1 Derivation of the composite controller

As mentioned before we first need both a (semi)globally asymptotically tracking controller and a bounded globally asymptotically regulating controller. We present a large class of globally tracking controllers which contains the results presented in [15]. In case we restrict the reference trajectory $q_d(t)$ to be a fixed point we can easily derive a class of globally bounded asymptotically regulating controllers. Clearly, our results includes and generalizes those of [5, 7, 17].

To solve the problem of tracking a desired reference trajectory $q_d(t) \in \mathcal{C}^2$ we propose the control law:

$$\tau = M(q)\ddot{q}_d + C(q, \dot{q})\dot{q}_d + G(q) - f_1(\dot{e}) - f_2(e) \tag{6}$$

where $e \equiv q - q_d$ denotes the tracking error and $f_1, f_2 \in \mathcal{F}^n$. This control law results in the closed-loop system

$$M(q)\ddot{e} + C(q, \dot{q})\dot{e} + f_1(\dot{e}) + f_2(e) = 0. \tag{7}$$

Proposition 4.1 *Consider the system (1) together with the control law (6). If $f_1 \in \mathcal{C}^2$, $f_2 \in \mathcal{C}^1$ and $Df_1(0) > 0$ then the resulting closed-loop system (7) is globally asymptotically stable.*

Proof This proof is a straightforward extension of the proof of Paden and Panja [15], where this proposition is proved in case $f_1(\dot{e}) = K_d\dot{e}$ and $f_2(e) = K_p e$ with $K_p = K_p^T$ and $K_d = K_d^T$ $n \times n$ diagonal positive definite matrices. To prove the proposition we use Matrosov's Theorem (Theorem 2.9).

Consider the function

$$V(t, e, \dot{e}) = \frac{1}{2}\dot{e}^T M(q)\dot{e} + F_2(e) \tag{8}$$

which satisfies condition M1 of Theorem 2.9. Calculating the time-derivative of (8) along solutions of (7) results in

$$\dot{V}(e, \dot{e}) = -\dot{e}^T f_1(\dot{e}) \tag{9}$$

where we used Property 2.1. Therefore, with

$$V^*(e, \dot{e}) = -\dot{e}^T f_1(\dot{e})$$

condition M2 has also been fulfilled.

Since $V(t, e, \dot{e})$ is a decreasing function of time, we conclude that e and $\dot{e}$ are bounded and then from (3), also q and $\dot{q}$ are bounded. Since

$$\ddot{e} = -M(q)^{-1}[C(q, \dot{q})\dot{e} + f_1(\dot{e}) + f_2(e)] \tag{10}$$

is a continuous function of e, $\dot{e}$, q and $\dot{q}$, we know that $\ddot{e}$ is bounded and thus using (3) also $\ddot{q}$ is bounded.

In anology with [15] we define $W(t,e,\dot{e}) = \ddot{V}(t,e,\dot{e})$:

$$W(t,e,\dot{e}) = -\ddot{e}^T f_1(\dot{e}) - \dot{e}^T Df_1(\dot{e})\ddot{e} \tag{11}$$

where $\ddot{e}$ is given by (10). Obviously condition M3 is satisfied. Let $g(\dot{e})$ denote $Df_1(\dot{e})\dot{e} = Df_1(\dot{e})^T\dot{e}$. Then we can rewrite (11) as:

$$W(t,e,\dot{e}) = -\ddot{e}^T f_1(\dot{e}) - \ddot{e}^T g(\dot{e}). \tag{12}$$

To verify condition M4 we use Lemma 2.11. Differentiating (12) with respect to time results in

$$\dot{W}(t,e,\dot{e}) = -(\frac{d}{dt}\ddot{e})^T f_1(\dot{e}) - \ddot{e}^T Df_1(\dot{e})\ddot{e} - (\frac{d}{dt}\ddot{e})^T g(\dot{e}) - \ddot{e}^T Dg(\dot{e})\ddot{e}.$$

Since $f_1 \in \mathcal{C}^2$ we know that all arguments, except $\frac{d}{dt}\ddot{e}$, are continuous in the tracking error and depend continuously on time through a bounded function (cf. Remark 2.12). That $\frac{d}{dt}\ddot{e}$ is continuous with respect to the tracking error and continuous with respect to time through a bounded function follows from differentiating (10) with respect to time and noticing that since $f_1, f_2 \in \mathcal{C}^1$, the functions $\frac{d}{dt}f_1(\dot{e})$, $\frac{d}{dt}f_2(e)$ and both $\frac{d}{dt}M(q)$ and $\frac{d}{dt}C(q,\dot{q})$ are continuous with with respect to the tracking error and continuous with respect to time through a bounded function. Furthermore for $(e,\dot{e}) \in \{(e,\dot{e}) | V^*(e,\dot{e}) = 0\} = \{(e,\dot{e}) | \dot{e} = 0\}$ it follows that

$$\dot{W}(t,e,\dot{e}) = -\ddot{e}^T Df_1(0)\ddot{e} - \ddot{e}^T Dg(0)\ddot{e} = -2\ddot{e}^T Df_1(0)\ddot{e}.$$

Since $Df_1(0) > 0$ it follows from Lemma 2.11 that also the condition M4 has been fulfilled.
Since all signals remain bounded and the closed-loop system (7) is a continuous function of those signals, the fifth and last condition of Theorem 2.9 has also been satisfied.
So we can conclude from Matrosov's Theorem that the origin is globally asymptotically stable. ■

Corollary 4.2 *Assume that $q_d(t)$ is a fixed point and consider the system (1) in closed-loop with the control law*

$$\tau = G(q) - f_1(\dot{e}) - f_2(e) \tag{13}$$

where $e \equiv q - q_d$ and $f_1, f_2 \in \mathcal{F}^n$. Then the equilibrium point $e = 0$, $\dot{e} = 0$ for the resulting closed-loop system (7) is globally asymptotically stable.

Proof Use the Lyapunov function candidate (8), whose derivative along the closed-loop dynamics (7) becomes (9). LaSalle's invariance principle completes the proof. ■

Remark 4.3 Note that we did not need the facts that $f_1 \in \mathcal{C}^2$, $f_2 \in \mathcal{C}^1$ and $Df_1(0) > 0$ anymore. By choosing $f_1, f_2 \in \mathcal{B}$, this corollary contains the results presented in [5, 7, 17].

From Proposition 4.1 we know that the controller

$$\tau = M(q)\ddot{q}_d + C(q,\dot{q})\dot{q}_d + G(q) - f_1(\dot{e}) - f_2(e) \tag{14}$$

results in a globally asymptotically stable closed-loop system, but can we use this control law when we have to deal with input constraints? From Property 2.2 and (3) we know that by choosing $f_1, f_2 \in \mathcal{B}$ we almost have a bounded control law. Every term of (14) is bounded, except for $C(q,\dot{q})\dot{q}_d$, since $\dot{q}$ is not a priori bounded. However, from the proof of Proposition 4.1 we know that for all $t \geq 0$:

$$\frac{1}{2}M_m\|\dot{e}(t)\|^2 \leq \frac{1}{2}\dot{e}(t)^T M(q)\dot{e}(t) \leq V(t,e(t),\dot{e}(t)) \leq V(0,e(0),\dot{e}(0))$$

and therefore

$$\|\dot{q}(t)\| \leq \|\dot{q}_d(t)\| + \|\dot{e}(t)\| \leq B_1 + \sqrt{\frac{2}{M_m}V(0,e(0),\dot{e}(0))}, \qquad \forall t \geq 0 \tag{15}$$

which gives us a bound on (14), provided we know the initial conditions. However, the control effort increases as $\|\dot{q}(0)\|$ increases, and therefore (14), although it is a globally asymptotically stable controller, it is not a globally bounded tracking controller.
How to obtain a globally bounded tracking controller? Following section 3 we now proceed to construct a bounded composite controller. That is, we seek for a globally bounded controller that steers the system into a region in which e and $\dot{e}$ are within given bounds. As soon as we are in that region, we can switch to (14). Since we switch at a time t_s at which $e(t_s)$ and $\dot{e}(t_s)$ are within bounds that we know in advance, we also have a bound on $V(t_s, e(t_s), \dot{e}(t_s))$ in advance, from which an in advance known bound on (14) follows using (15). By using a globally bounded controller to steer the system into the region in which e and $\dot{e}$ are within prescribed known bounds, the resulting composite controller is a globally bounded controller.
Along the lines of section 3 we can take for this first phase controller the regulating controller (13) where we choose $q_d(t) \equiv 0$ and $f_1, f_2 \in \mathcal{B}$ so that it is a bounded globally asymptotically stable regulating controller. Therefore, for all $\epsilon > 0$ there exists a time $t_s \geq 0$ such that $\|q(t)\| \leq \epsilon, \|\dot{q}(t)\| \leq \epsilon$ for any $t \geq t_s$, and then also $\|e(t)\| \leq B_0 + \epsilon$ and $\|\dot{e}(t)\| \leq B_1 + \epsilon$ for any $t \geq t_s$, resulting into:

Proposition 4.4 *Consider the system (1). Then there exists a switching time $t_s \geq 0$ such that given any $\tilde{t}_s \geq t_s$ the composite controller*

$$\tau = \begin{cases} G(q) - f_1(\dot{q}) - f_2(q) & t < \tilde{t}_s \\ \\ M(q)\ddot{q}_d + C(q,\dot{q})\dot{q}_d + G(q) - f_3(\dot{e}) - f_4(e) & t \geq \tilde{t}_s \end{cases} \tag{16}$$

where $e \equiv q - q_d$ and $f_1, f_2, f_3, f_4 \in \mathcal{F}^n$, $f_3 \in \mathcal{C}^2$, $f_4 \in \mathcal{C}^1$, $Df_3(0) > 0$, results in a globally asymptotically stable closed-loop system. Furthermore, if $f_1, f_2 \in \mathcal{B}$ we can determine a τ_{max} such that the controller (16) satisfies

$$\|\tau(t)\| \leq \tau_{max} \qquad \forall t \geq 0.$$

Proof From Proposition 4.1 we know that for all $\epsilon > 0$ there exists a time $t_s \geq 0$ such that for any $\tilde{t}_s \geq t_s$: $\|e(\tilde{t}_s)\| \leq B_0 + \epsilon$ and $\|\dot{e}(\tilde{t}_s)\| \leq B_1 + \epsilon$. Furthermore we know that the first phase controller is global and if $f_1, f_2 \in \mathcal{B}$ we can also determine in advance a bound $\tau_{max,1}$ within which this first phase controller will remain.
For the second phase, we know from Proposition 4.1 that the resulting closed-loop system is asymptotically stable. Since $\|e(\tilde{t}_s)\| \leq B_0 + \epsilon$ and $\|\dot{e}(\tilde{t}_s)\| \leq B_1 + \epsilon$, we know from the foregoing lines that we can determine a bound $\tau_{max,2}$ on $\tau(t)$ for the second phase.
So it is obvious that $\tau_{max} = \max\{\tau_{max,1}, \tau_{max,2}\}$ suffices. ∎

Although we have found a controller that globally steers the system towards any desired trajectory, there still exists one drawback, in that we discontinuously change the controller at $t = \tilde{t}_s$. We can overcome this problem by using a convex combination of the both controllers that varies over time. First, we use the first phase controller to reduce the tracking error within prescribed bounds. Then we smoothly change to the second phase controller using a convex combination of both the first and second phase controllers. For the smooth change of controllers, we take a period of $\Delta > 0$ seconds, guaranteeing that the tracking errors are still within a priori known bounds. After that period of Δ seconds we use the second phase controller completely, yielding asymptotic stability:

Proposition 4.5 *Consider the system (1). Then there exists a switching time $t_s \geq 0$ and a $\Delta > 0$ such that for any $\tilde{t}_s \geq t_s$ the control law*

$$\tau = G(q) - [1 - s_\Delta(t - \tilde{t}_s)][f_1(\dot{q}) + f_2(q)] + s_\Delta(t - \tilde{t}_s)[M(q)\ddot{q}_d + C(q,\dot{q})\dot{q}_d - f_3(\dot{e}) - f_4(e)] \tag{17}$$

where $e \equiv q - q_d$ and $f_1, f_2, f_3, f_4 \in \mathcal{F}^n$, $f_3 \in \mathcal{C}^2$, $f_4 \in \mathcal{C}^1$, $Df_3(0) > 0$, results in a globally asymptotically stable closed-loop system. Furthermore, if $f_1, f_2 \in \mathcal{B}$ we can determine a τ_{max} such that the controller (17) satisfies

$$\|\tau(t)\| \leq \tau_{max} \qquad \forall t \geq 0.$$

Proof From Proposition 4.1 we know that for all $\tilde{t}_s \geq t_s$: $\|q(\tilde{t}_s)\| \leq \epsilon_1$ and $\|\dot{q}(\tilde{t}_s)\| \leq \epsilon_2$. Suppose weaker bounds $\epsilon_3 > \epsilon_1$ and $\epsilon_4 > \epsilon_2$ be given. We now determine Δ such that after using the convex combination of the two controllers for that period of time, it is guaranteed that q and $\dot{q}$ will be within those weaker bounds, i.e. $\|q(\tilde{t}_s + \Delta)\| \leq \epsilon_3$ and $\|\dot{q}(\tilde{t}_s + \Delta)\| \leq \epsilon_4$.
Observe that (17) is always a convex combination of the controllers (13) and (14). Let τ_M denote the maximum possible value of (17) in the region where $\|q(t)\| \leq \epsilon_3$ and $\|\dot{q}(t)\| \leq \epsilon_4$. If we choose

$$\Delta = \min\left\{ \frac{\epsilon_3 - \epsilon_1}{\epsilon_4}, \frac{M_M(\epsilon_4 - \epsilon_2)}{\tau_M + C_M\epsilon_4^2 + G_M} \right\}$$

we claim that $\|q(\tilde{t}_s + \Delta)\| \leq \epsilon_3$ and $\|\dot{q}(\tilde{t}_s + \Delta)\| \leq \epsilon_4$. Let $T \geq 0$ be the first moment that either $\|q(\tilde{t}_s + T)\| > \epsilon_3$ or $\|\dot{q}(\tilde{t}_s + T)\| > \epsilon_4$. To prove our claim we only have to show that $T \geq \Delta$.
Assume that $T < \Delta$. From (1) and Property 2.2 we have that for all $t \leq T$:

$$\|\ddot{q}(t)\| \leq \frac{\tau_M + C_M\epsilon_4^2 + G_M}{M_M}$$

Therefore

$$\|\dot{q}(\tilde{t}_s + T)\| \leq \epsilon_2 + \frac{\tau_M + C_M\epsilon_4^2 + G_M}{M_M} T < \epsilon_2 + \frac{\tau_M + C_M\epsilon_4^2 + G_M}{M_M} \Delta \leq \epsilon_4.$$

But if $\|\dot{q}(t)\| < \epsilon_4$ for all $t \in [\tilde{t}_s, \tilde{t}_s + T]$, we also have that

$$\|q(\tilde{t}_s + \Delta)\| < \epsilon_1 + \epsilon_4 T < \epsilon_1 + \epsilon_4 \Delta \leq \epsilon_3.$$

Therefore $T \geq \Delta$, which is a contradiaction. So both $\|q(\tilde{t}_s + \Delta)\| \leq \epsilon_3$ and $\|\dot{q}(\tilde{t}_s + \Delta)\| \leq \epsilon_4$. This implies that $e(\tilde{t}_s + \Delta)$ and $\dot{e}(\tilde{t}_s + \Delta)$ are within in advance known bounds. Similar to Proposition 4.4 the proof can be completed. ∎

4.2 Simulations

To support our findings, we consider the two link robot manipulator of [2], which in case of no payload can be described by

$$\begin{aligned}
M(q) &= \begin{bmatrix} 8.77 + 1.02\cos q_2 & 0.76 + 0.51\cos q_2 \\ 0.76 + 0.51\cos q_2 & 0.62 \end{bmatrix} \\
C(q,\dot{q}) &= 0.51\sin q_2 \begin{bmatrix} -\dot{q}_2 & -(\dot{q}_1 + \dot{q}_2) \\ \dot{q}_1 & 0 \end{bmatrix} \\
G(q) &= 9.81 \begin{bmatrix} 7.6\sin q_1 + 0.63\sin(q_1 + q_2) \\ 0.63\sin(q_1 + q_2) \end{bmatrix}
\end{aligned}$$

We consider the problem of tracking the desired trajectory

$$q_d(t) = \begin{bmatrix} \sin t \\ \sin t \end{bmatrix} \tag{18}$$

under the input constraints

$$|\tau_1| \leq 120 \qquad |\tau_2| \leq 10. \tag{19}$$

We start our simulations from the initial conditions

$$q(0) = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad \dot{q}(0) = \begin{bmatrix} 10 \\ 10 \end{bmatrix}.$$

To solve the globally tracking problem, we consider the controller (6), where we use

$$f_1(\dot{e}) = \begin{bmatrix} 20\tanh(2.5\dot{e}_1) \\ 1.7\tanh(3\dot{e}_2) \end{bmatrix} \qquad f_2(e) = \begin{bmatrix} 19\tanh(4e_1) \\ 1.5\tanh(4e_2) \end{bmatrix} \tag{20}$$

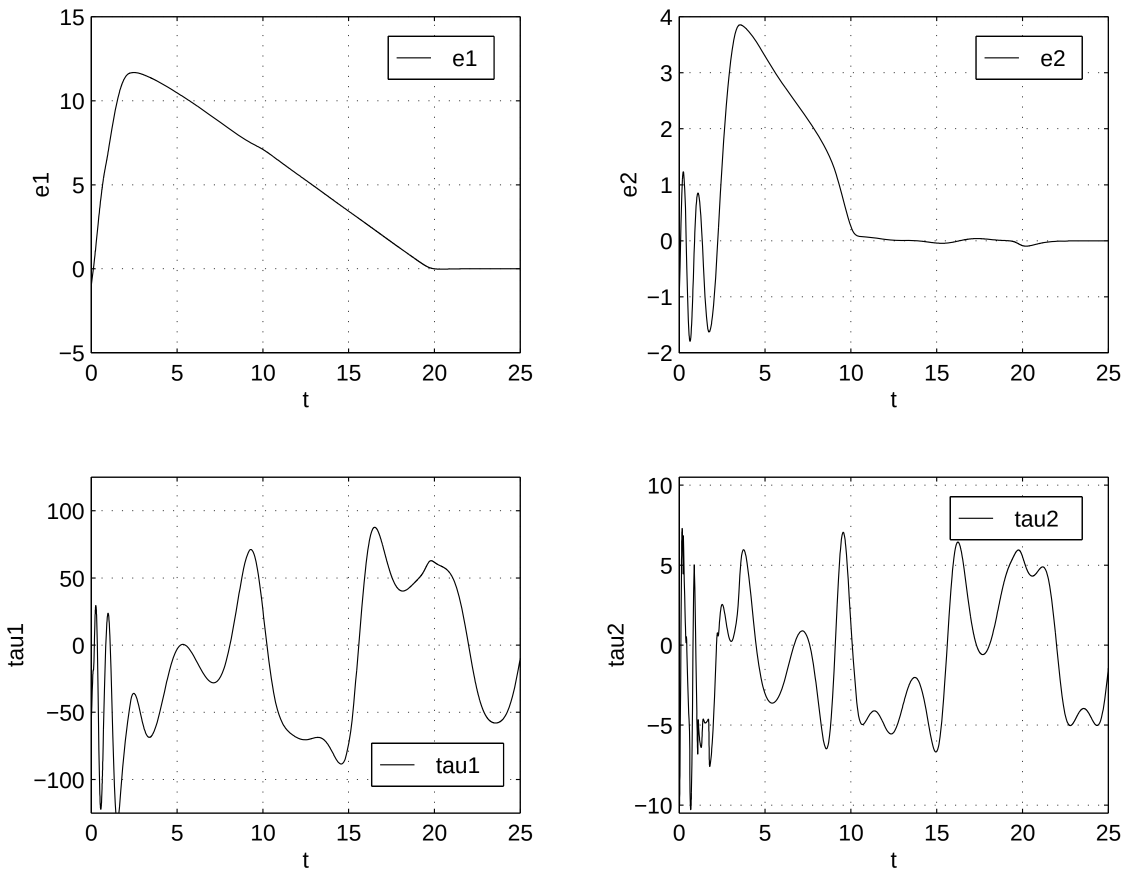

Figure 2: The tracking controller (6) using (20)

Note that this controller is not a priori bounded. The resulting tracking errors of the two joints are depicted in Figure 2, as well as the control input needed.
From this figure we can see that due to the initial velocities, the error in the first seconds increases a lot, but after a while both position errors tend to zero. Due to the saturated position feedback the convergence is slow. Nevertheless, we can also see that both τ_1 and τ_2 violate the input constraints (19). This is caused by the large values for $\dot{q}$.
In order to be able to satisfy the input constraints (19) we replace the controller with the composite controller as proposed in Proposition 4.4, i.e.

$$\tau = \begin{cases} G(q) - \begin{bmatrix} 20\tanh(2.5\dot{q}_1) \\ 1.7\tanh(3\dot{q}_2) \end{bmatrix} - \begin{bmatrix} 19\tanh(4q_1) \\ 1.5\tanh(4q_2) \end{bmatrix} & t < t_s \\ M(q)\ddot{q}_d + C(q,\dot{q})\dot{q}_d + G(q) - \begin{bmatrix} 20\tanh(2.5\dot{e}_1) \\ 1.7\tanh(3\dot{e}_2) \end{bmatrix} - \begin{bmatrix} 19\tanh(4e_1) \\ 1.5\tanh(4e_2) \end{bmatrix} & t \geq t_s \end{cases} \tag{21}$$

The only question remaining is how to determine t_s such that we meet the input constraints (19). If we define

$$B_r \triangleq \left\{ (e,\dot{e}) \in \mathbb{R}^{2\times 2} \middle| \forall t \in \mathbb{R}^+ : V(t,e,\dot{e}) \leq r \right\}$$

where $V(t,e,\dot{e})$ is given by (8), that is

$$B_r \triangleq \left\{ (e,\dot{e}) \in \mathbb{R}^{2\times 2} \middle| \forall t \in \mathbb{R}^+ : \frac{1}{2}\dot{e}^T M(e+q_d(t))\dot{e} + \frac{19}{4}\ln(\cosh(4e_1)) + \frac{3}{8}\ln\cosh(4e_2)) \leq r \right\} \tag{22}$$

we know that for all $r \in I\!R$ B_r is a positively invariant set, i.e.

$$x(t_s) \in B_r \Rightarrow x(t) \in B_r, \quad \forall t \geq t_s.$$

If we define

$$r_1 \triangleq \liminf_{t \in I\!R^+} \left[\frac{1}{2} \dot{q}_d(t)^T M(0) \dot{q}_d(t) + \frac{19}{4} \ln(\cosh(4q_{d,1}(t))) + \frac{3}{8} \ln\cosh(4q_{d,2}(t))) \right]$$

we are guaranteed to enter the region $B_{r_1+\epsilon}$ for any $\epsilon > 0$, Therefore, we can define t_s as the first moment we enter $B_{r_1+\epsilon}$, resulting in a globally asymptotically stable closedloop system. However, we are not allowed to choose arbitrary $\epsilon > 0$, since we also have to satisfy the input constraints (19). We know that B_r is a positive invariant set and therefore can define r_2 to be the largest value of r such that (21) satisfies (19) for all $t \geq t_s$ and $(e, \dot{e}) \in B_r$. If $r_1 < r_2$ we can define t_s to be the first moment we enter the region B_r where $r_1 < r \leq r_2$. Then the existance of t_s is guaranteed, as well as the satisfaction of (19).

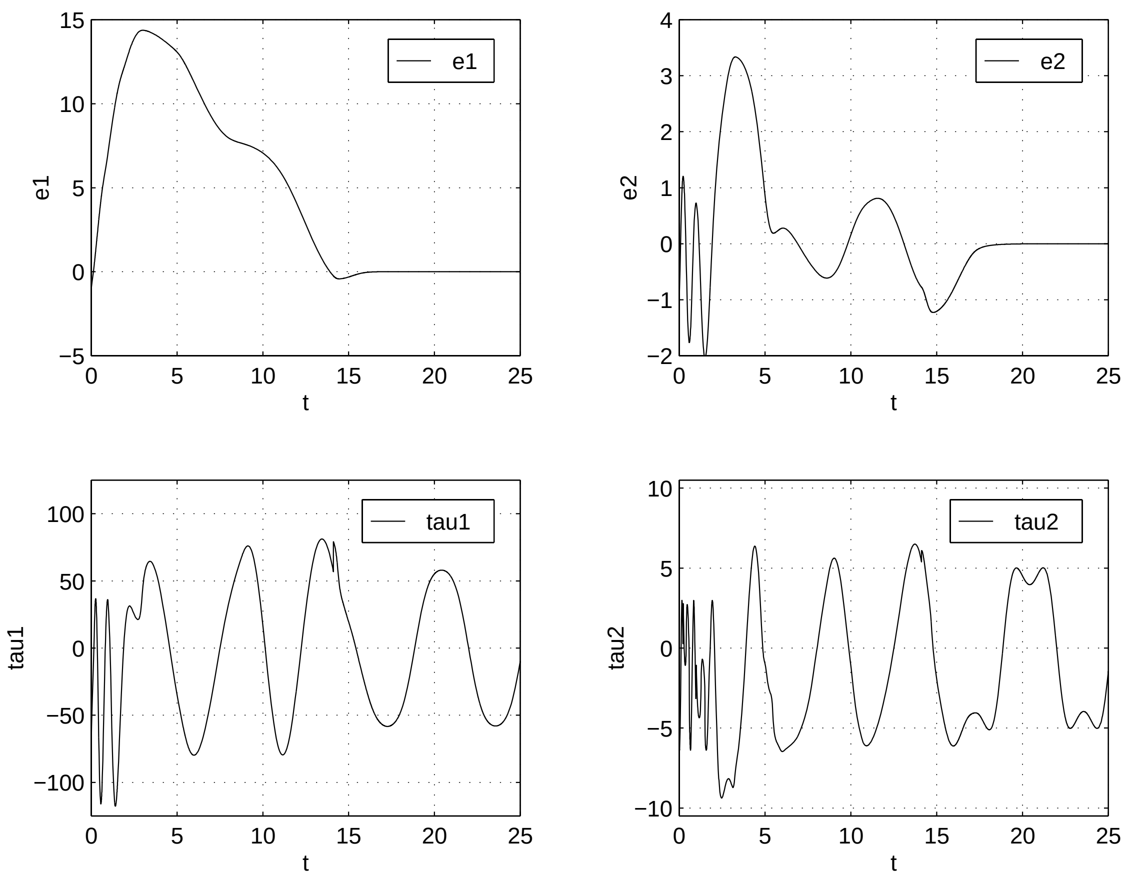

Figure 3: The composite controller (21)

For our simulations, it turned out that $r = 17$ suffices. The resulting performance is depicted in Figure 3. From this figure we can see that again the large initial velocities result in a growth of the position error in the beginning. Since we use a composite controller with in the first stage a controller that is a priori bounded, we do not violate the input constraints (19) in our attempts to reduce the errors. At $t = 14.1$ the errors in both position and velocity have become small enough, i.e. $V(t, e(t), \dot{e}(t)) \leq 17$, so we then switch to the second controller, as becomes clear from the discontinuous change of the control input applied.

So, although the use of a composite controller results in a satisfaction of the input constraints, we now have a discontinuous switch in the input, which also is undesired. We can overcome the latter by using the smoothened controller as proposed in Proposition 4.5. This controller is a convex combination of the two controllers presented in (21). The combination is determined by the function $s_\Delta(\cdot)$, which has to be chosen. Let r be given such that $r_1 < r < r_2$. We determine the largest possible Δ_M such that for all initial conditions $(e(t_s), \dot{e}(t_s)) \in B_r$ and all initial times t_s after applying the controller (17) we have that $(e(t_s + \Delta_M), \dot{e}(t_s + \Delta_M)) \in B_{r_2}$. It suffices to choose $\Delta \leq \Delta_M$.
For our third simulation we use the controller (17), with the same f_1, f_2, f_3, f_4 as in our second simulation and the same definition of t_s. This controller is a convex combination of the two controllers presented in (21). The combination is determined by the function $s_\Delta(\cdot)$ which is chosen as depicted in Figure 1, i.e. a function given by (5) where $a = 3$, $b = \frac{3}{2}$ and $\Delta = 1$. The resulting performance of this smoothened composite controller is depicted in Figure 4.

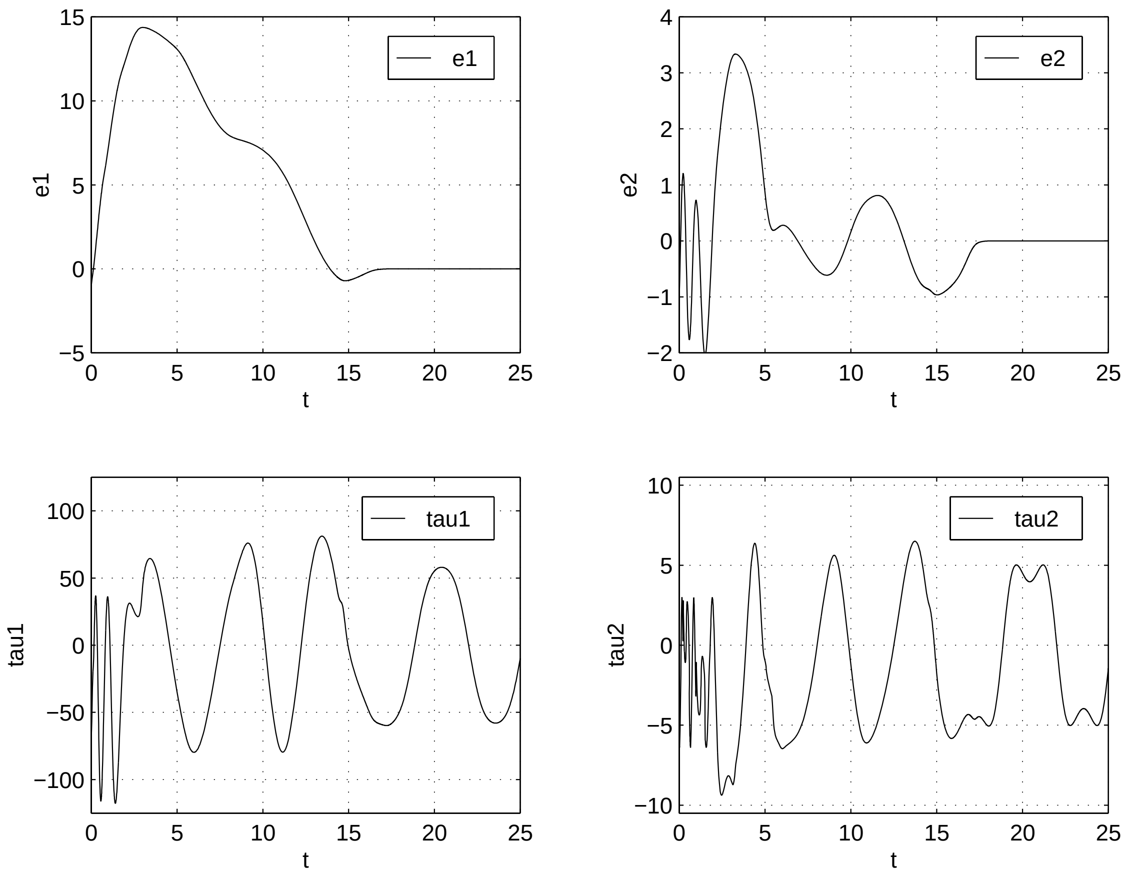

Figure 4: The smoothened composite controller (17)

We see a similar performance as in the previous simulations. However, this time we not only meet the input constraints (19), but also use a smooth controller to achieve tracking.

5 Using only position measurements

In this section we consider the bounded tracking problem as formulated in section 2.2 with the extra restriction that we now assume that the joint velocities $\dot{q}$ are not available for measurement. As in the previous section we first derive a composite controller and the show a simulation.

5.1 Derivation of the composite controller

For solving the bounded tracking problem under output feedback, we use similar to the previous section the idea of a composite controller as presented in section 3 by combining the semiglobal tracking output feedback controller presented in [13] with a globally bounded regulating output feedback controller.
In [13] Loría and Nijmeijer presented the controller

$$\tau = M(q)\ddot{q}_d + C(q, \dot{q}_d)\dot{q}_d + G(q) - K_d f(z) - K_p f(e) \tag{23}$$

where K_p and K_d are $n \times n$ diagonal positive definite matrices, z is generated from the filter

$$\begin{aligned} z &= w + Be \\ \dot{w} &= -Af(w + Be) \end{aligned} \tag{24}$$

where A and B are $n \times n$ diagonal positive definite matrices and $f(x) = [\tanh(x_1), \ldots, \tanh(x_n)]^T = \mathrm{Tanh}(x)$. The following was shown in [13]:

Proposition 5.1 *Consider the system (1) together with the control law (23) and filter (24). Then the resulting closed-loop system is semi-globally stable, i.e. the resulting closed-loop system is locally asymptotically stable but its region of attraction can be arbitrarily enlarged by suitably selecting the observer gains A and B.*

Remark 5.2 The only properties of the function $f(x) = \mathrm{Tanh}(x)$ being used in the proof given in [13] are that $f \in \mathcal{B} \cap \mathcal{C}^1$ and there exist contants $\Gamma > 0$, $\Delta > 0$ such that $\forall x \in I\!R$:

- $\dfrac{F(x)}{f(x)^T f(x)} \geq \Gamma$,
- $0 < Df(x) \leq \Delta$.

and furthermore the fact that for all $n \times n$ diagonal positive definite matrices Λ also $\Lambda f \in \mathcal{F}^n$. Therefore several choices for $f \in \mathcal{B} \cap \mathcal{C}^1$ other than $f(x) = \mathrm{Tanh}(x)$ are also possible.

Proposition 5.1 is a semi-global result. In case we want to extend this to a bounded global result we only have to find a bounded globally regulating controller the steers the system to a fixed point $(q_d, 0)$.
For this we propose the control law

$$\tau = G(q) - f_1(z) - f_2(e) \tag{25}$$

where $e = q - q_d$, $f_1, f_2 \in \mathcal{F}^n$ and z is generated from the filter

$$\begin{aligned} z &= e - w \\ \dot{w} &= e - w \end{aligned} \tag{26}$$

which results in the time-invariant closed-loop system

$$\begin{gathered} M(q)\ddot{e} + C(q, \dot{q})\dot{e} + f_1(z) + f_2(e) = 0 \\ \dot{z} = \dot{e} - z \end{gathered} \tag{27}$$

Then we claim:

Proposition 5.3 *Assume that $q_d(t)$ is a fixed point and consider the system (1) in closed-loop with the control law (25) and the filter (26). Then the resulting closed-loop system (27) is globally asymptotically stable.*

Proof Consider the Lyapunov function candidate

$$V(e, \dot{q}, z) = \dot{q}^T M(q)\dot{q} + F_1(z) + F_2(e) \tag{28}$$

which is positive definite and radially unbounded. Along solutions of (27) its time-derivative becomes

$$\dot{V}(e, \dot{q}, z) = -z^T f_1(z)$$

which is negative semidefinite in the state $(e, \dot{q}, z)$. Application of LaSalle's Theorem completes the proof. ∎

Remark 5.4 The filter (26) can be seen as a simple representative of a whole class of possible filters.

For instance if $f_1 \in \mathcal{F}^n$ satisfies the property that for any fixed $n \times n$ diagonal positive definite matrix Λ also $\Lambda f_1 \in \mathcal{F}^n$, then it can be seen that instead of (26) also the filter

$$\begin{array}{rcl} z &=& \Lambda_1 e - \Lambda_2 w \\ \dot{w} &=& \Lambda_3(\Lambda_2 e - \Lambda_2 w) \end{array} \tag{29}$$

where Λ_1, Λ_2, and Λ_3 are arbitrary $n \times n$ diagonal positive definite matrices, can be used. The filter (29) is similar to the ones presented in [3, 9]. Also the more general class of linear filters presented in [1, 6, 12] can similarly be viewed as a special case of (26).

Also a wide variety of nonlinear filters can be rewritten as (26). Here one can for instance think of the filter

$$\begin{array}{rcl} z &=& e - w \\ \dot{w} &=& f(e - w) \end{array} \tag{30}$$

with $f \in \mathcal{F}^n$. Using the same assumption used to derive (29) and the change of coordinates from w to $-w$, one obtains the filter (24).

In general one can say that the filter (26) is a representative of a whole class of filters that takes its simple form due to well chosen coordinates.

To obtain other possible filters, just apply a suitable change of coordinates in z and w (suitable in the sense that $\dot{V}$ remains negative definite, V based on (28)). Note that possibly the term $f_1(z)$ in (25) will change, and correspondingly the term $F_1(z)$ in the Lyapunov function candidate (28), as was the case in deriving (29).

When we choose $f_1, f_2 \in \mathcal{B}$ we have that (25) together with the filter (26) yields a globally bounded control law that steers the system towards the origin. Therefore:

Proposition 5.5 *Consider the system (1). Then there exists a switching time $t_s \geq 0$ such that given any $\tilde{t}_s \geq t_s$ the composite control law*

$$\tau = \begin{cases} G(q) - f_1(z_1) - f_2(q) & t < \tilde{t}_s \\ \\ M(q)\ddot{q}_d + C(q, \dot{q}_d)\dot{q}_d + G(q) - K_d f(z_2) - K_p f(e) & t \geq \tilde{t}_s \end{cases} \tag{31}$$

where $f \in \mathcal{F}^n$ satisfies the conditions mentioned in Remark 5.2 and z_1 and z_2 are given by

$$\begin{array}{rcl} z_1 &=& q - w_1 \\ z_2 &=& w_2 + Be \end{array} \tag{32}$$

and w_1 and w_2 are generated from the filters

$$\begin{array}{rcll} \dot{w}_1 &=& q - w_1 & t < \tilde{t}_s \\ \dot{w}_2 &=& -Af(w_2 + Be) & t \geq \tilde{t}_s \end{array} \tag{33}$$

results in a globally asymptotically stable closed-loop system. Furthermore, if $f_1, f_2 \in \mathcal{B}$ we can determine a τ_{max} such that the controller (31) together with the filter (32, 33) satisfies

$$\|\tau(t)\| \leq \tau_{max} \qquad \forall t \geq 0.$$

Proof Similarly to that of Proposition 4.4. ■

Remark 5.6 Notice that although the existence of t_s has been guaranteed by Proposition 5.5, it is in practice not easy or even impossible to determine when to switch, since usually one has to switch if $\dot{e}$ is within certain a priori determined bounds. In case one is unable to measure $\dot{e}$ one can not determine when $\dot{e}$ is within these bounds, so then it is impossible to determine when to switch. Therefore, the result of Proposition 5.5 is more a theoretical result than a practically useful result.

As in the previous section Proposition 5.5 can be modified to overcome the problem of a discontinuous change of the control input:

Proposition 5.7 *Consider the system (1). Then there exists a switching time $t_s \geq 0$ and a $\Delta > 0$ such that for any $\tilde{t}_s \geq t_s$ the control law*

$$\tau = G(q) - [1 - s_\Delta(t - \tilde{t}_s)][f_1(z_1) + f_2(q)] + s_\Delta(t - \tilde{t}_s)[M(q)\ddot{q}_d + C(q, \dot{q}_d)\dot{q}_d - K_d f(z_2) - K_p f(e)] \quad (34)$$

where $f \in \mathcal{F}^n$ satisfies the conditions mentioned in Remark 5.2 and z_1 and z_2 are given by

$$\begin{array}{rcl} z_1 & = & q - w_1 \\ z_2 & = & w_2 + Be \end{array} \quad (35)$$

and w_1 and w_2 are generated from the filters

$$\begin{array}{rcll} \dot{w}_1 & = & q - w_1 & t < \tilde{t}_s + \Delta \\ \dot{w}_2 & = & -Af(w_2 + Be) & t \geq \tilde{t}_s \end{array} \quad (36)$$

results in a globally asymptotically stable closed-loop system. Furthermore, if $f_1, f_2 \in \mathcal{B}$ we can determine a τ_{max} such that the controller (34) together with the filter (35, 36) satisfies

$$\|\tau(t)\| \leq \tau_{max} \qquad \forall t \geq 0.$$

Proof Similarly to that of Proposition 4.5. ■

5.2 Simulations

In this section we consider again the system described in section 4.2 under the same input constraints and using the same initial conditions. In order to track the reference trajectory

$$q_d(t) = \begin{bmatrix} \sin t \\ \sin t \end{bmatrix}$$

we use the output feedback controller

$$\begin{array}{rcl} \tau & = & [1 - s_\Delta(t - t_s)][G(q) - K_d \tanh(\Lambda_d z_1) - K_p \tanh(\Lambda_p q)] + \\ & & + s_\Delta(t - t_s)[G(q) + M(q)\ddot{q}_d + C(q, \dot{q}_d)\dot{q}_d - K_d \tanh(\Lambda_d z_2) - K_p \tanh(\Lambda_p e)] \end{array}$$

where $K_p = \mathrm{diag}(19, 1.5)$, $K_d = \mathrm{diag}(20, 1.7)$, $\Lambda_p = \mathrm{diag}(4, 4)$, $\Lambda_d = \mathrm{diag}(2.5, 3)$, $z_1 = q - w_1$, $z_2 = w_2 + Be$ and

$$\begin{array}{rcll} \dot{w}_1 & = & q - w_1 & t < t_s + \Delta \\ \dot{w}_2 & = & -Af(w_2 + Be) & t \geq t_s \end{array}$$

where $A = B = \mathrm{diag}(1000, 1000)$. For $s_\Delta(\cdot)$ we again use a function given by (5) where $a = 3$, $b = \frac{3}{2}$ and $\Delta = 1$.

As already pointed out in Remark 5.6 the remaining problem is to determine when to start the switching , i.e. how to determine t_s such that we are guaranteed to remain within our input constraints. In anology with section 4.2 we define

$$B_r \triangleq \left\{ (e, \dot{e}, z_2) \in I\!R^{2\times 2} \middle| \forall t \in I\!R^+ : V(t, e, \dot{e}, z_2) \leq r \right\} \tag{37}$$

where $V(t, e, \dot{e}, z_2)$ is the Lyapunov-function used in [13], and switch as soon as we are in a certain B_r. The only problem is that (37) contains $\dot{e}$, which we can not measure! If we would be able to give an estimate of $\dot{e}$ within a known accuracy, we could mimic the approach as sketched in section 4.2.
If we take a closer look at what we actually are doing when applying the composite controller as proposed, is that we initially try to steer the system towards the origin before we really start tracking. The moment t_s is the moment we are close enough to the origin to be guaranteed to meet the input constraint for all later time. Therefore, what we in our simulation do, is wait until we are close enough to the origin. In our eyes (based on measurements of q) this is the case at $t = 25$, so at that time we decide to switch. The resulting performance is depicted in Figure 5.

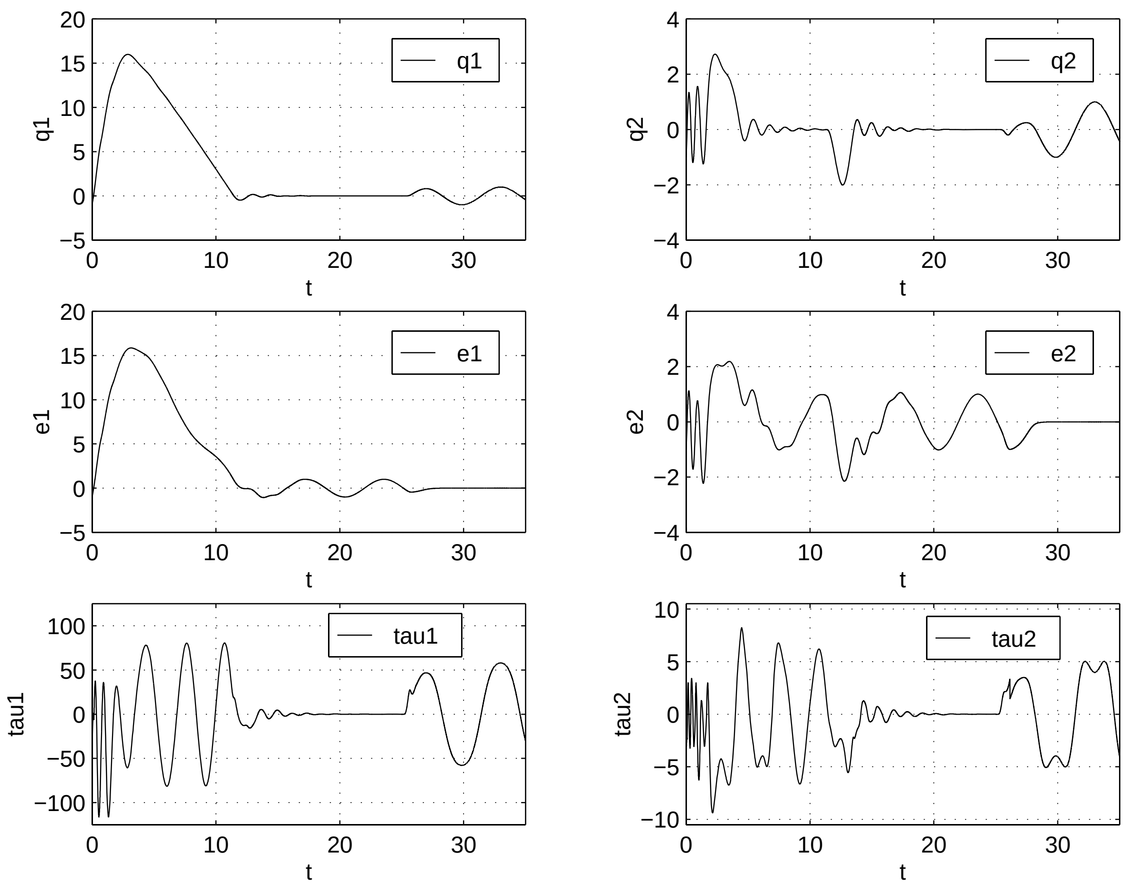

Figure 5: The composite output feedback-controller (34,35,36).

We see a similar performance as in section 4.2. Within about 20 seconds the first phase controller is able to control our system towards the origin. Although we have no measurements of $\dot{q}$ available we are almost sure at $t = 25$ that we are close enough to the drigin as to meet the input constraints (19).

6 Adaptive Controller

In this section we consider the bounded tracking problem as formulated in section 2.2 under the assumption that some of the parameters that describe the robot dynamics are unknown. We assume that the full state $(q, \dot{q})$ is available for measurement. For solving the problem we use similar to the previous sections the idea of a composite controller as presented in section 3 by combining a global adaptive controller (however not bounded) with a globally bounded *non-adaptive* controller that regulates the system towards the origin. We conclude the section with a simulation.

6.1 The derivation of the composite controller

Using Property 2.3 we know that the system

$$M(q,\theta)\ddot{q} + C(q,\dot{q},\theta)\dot{q} + G(q,\theta) = \tau$$

can also be described as

$$M_0(q)\ddot{q} + C_0(q,\dot{q})\dot{q} + G_0(q) + Y(q,\dot{q},\dot{q},\ddot{q})\theta = \tau$$

The global adaptive controller we present is an extension of the one presented in [2] and takes the following form:

$$\tau = M_0(q)\ddot{q}_r + C_0(q,\dot{q})\dot{q}_r + G_0(q) + Y(q,\dot{q},\dot{q}_r,\ddot{q}_r)\hat{\theta} - f_1(e) - f_2(s) \tag{38}$$

where $\dot{q}_r = \dot{q}_d - f_3(e)$, $e = q - q_d$, $s = \dot{q} - \dot{q}_r = \dot{e} + f_3(e)$, $f_1, f_2 \in \mathcal{F}^n$, $\hat{\theta}$ is an estimation for the vector of unknown (but constant) system parameters, and f_3 is such that $f_3(x)^T f_1(x)$ is a positive definite function (e.g. $f_3(x) = f_1(x)$ or $f_3(x) = \Lambda x$ with Λ an $n \times n$ positive definite matrix).
This controller reduces to the one presented in [2] in case $f_1(e) = K_{d,1}$, $f_2(s) = K_{d,2}s$, $f_3(e) = \Lambda e$ with $K_{d,1}$, $K_{d,2}$ and Λ $n \times n$ diagonal positive definite matrices.
The controller (38) results in the closed-loop system

$$M(q,\theta)\dot{s} + C(q,\dot{q},\theta)s + f_1(e) + f_2(s) = Y(q,\dot{q},\dot{q}_r,\ddot{q}_r)\tilde{\theta} \tag{39}$$

where $\tilde{\theta} \equiv \hat{\theta} - \theta$. When we use the parameter update law

$$\dot{\hat{\theta}} = -\Gamma Y^T(q,\dot{q},\dot{q}_r,\ddot{q}_r)s \tag{40}$$

where Γ is a positive definite symmetric matrix, we obtain

Proposition 6.1 *Consider the system (1) together with the controller (38) and the parameter update law (40). Then the equilibrium point $e = 0$, $\dot{e} = 0$ for the resulting closed-loop system (39) is globally asymptotically stable*

Proof This proof is a straightforward extension of the proof of Berghuis [2].
Consider the candidate Lyapunov function

$$V(t,s,e,\tilde{\theta}) = \frac{1}{2}s^T M(q,\theta)s + F_1(e) + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} \tag{41}$$

which is positive definite and radially unbounded. Along solutions of (39,40) its time-derivative becomes

$$\dot{V}(s,e,\tilde{\theta}) = -f_3(e)^T f_1(e) - s^T f_2(s)$$

which is negative semi-definite in the error-state $(s,e,\tilde{\theta})$. Hence, we can conclude that (41) is a nonincreasing function, resulting in s, e, and $\tilde{\theta}$ are bounded and therefore $\dot{e}$ is bounded and from (39) together with (3) and Property 2.2 also $\ddot{e}$ is bounded. Furthermore we know that $\int_0^t f_3(e)^T f_1(e)dt$ and $\int_0^t s^T f_2(s)dt$ remain bounded for all $t \geq 0$ and therefore that e and s are uniformly continuous. Barbălat's Lemma gives us that e and s and therefore also $\dot{e}$ converge to zero asymptotically. ■

Although the controller (38) is a globally asymptotically stable controller, it is not a globally bounded tracking controller. We can again construct a globally bounded tracking controller along the lines of section 3. We have to construct a globally bounded controller that steers the system (1) to the origin. Therefore we propose first a *non-adaptive* controller that steers the system towards a fixed point in case we know all the parameters that describe the manipulator-dynamics (i.e. θ is known). This controller is an extention of the one presented in [19].
Assume that $q_d(t)$ is fixed and consider the regulating control law

$$\tau = G(q_d) - f_1(\dot{e}) - \alpha f_2(e) \tag{42}$$

where $e \equiv q - q_d$, $f_1, f_2 \in \mathcal{F}^n$, $\alpha > 0$ and f_2 and α such that αf_2 satisfies for all $q \neq q_d$:

$$\|\alpha f_2(q - q_d)\| > \|G(q) - G(q_d)\|. \tag{43}$$

This controller yields the time-invariant closed-loop dynamics

$$M(q)\ddot{e} + C(q, \dot{q})\dot{e} + G(q) - G(q_d) + f_1(\dot{q}) + \alpha f_2(e) = 0 \tag{44}$$

Then we claim:

Proposition 6.2 *Assume that $q_d(t)$ is fixed and consider the system (1) in closed-loop with the regulating control law (42). Then the equilibrium $e = 0$, $\dot{e} = 0$ of the resulting closed-loop system (44) is globally asymptotically stable, provided α is chosen large enough such that (43) is satisfied.*

Before proving this Proposition we first need the following Lemma:

Lemma 6.3 *Let $f \in \mathcal{F}^n$. Then a sufficient condition for f that there exists a sufficiently large $\alpha > 0$ such that*

$$\|\alpha f(q - q_d)\| > \|G(q) - G(q_d)\|.$$

is fulfilled for all $e \neq 0$, is that there exists an $\epsilon > 0$ and a sufficiently large $\beta > 0$ such that

$$\|\beta f(e)\| > \|G(q) - G(q_d)\| \quad \forall \|e\| \leq \epsilon, e \neq 0.$$

Therefore, for $f \in \mathcal{F}^n \cap \mathcal{C}^1$ a sufficient condition for the existence of a sufficiently large $\alpha > 0$ for (43) to be fulfilled for all e is that $Df(0) > 0$.

Proof Suppose there exist a sufficiently large β such that

$$\|\beta f(e)\| > \|G(q) - G(q_d)\| \quad \forall \|e\| \leq \epsilon, e \neq 0.$$

From Property 2.2 we know that $\|G(q)\|$ is bounded, so $\|G(q) - G(q_d)\|$ also. From Property 2.7 we know that for $\|e\| > \epsilon$: $\|f(e)\| > \delta$ for certain δ. Therefore, we have to multiply f with a sufficiently large constant $\alpha \geq \beta$ such that $\alpha\delta$ is larger than the bound on $\|G(q) - G(q_d)\|$. ■

Proof (of Proposition 6.2) From (43) it follows that the closed-loop system (44) has a unique equilibrium.
Consider the Lyapunov function candidate

$$V(e, \dot{e}) = \frac{1}{2}\dot{e}^T M(q)\dot{e} + \alpha F_2(e) - G(q_d)^T(e) + P(q) - P(q_d)$$

which is radially unbounded and positive definite, provided α is chosen large enough such that (43) is satisfied. Along solutions of (44) its time-derivative becomes

$$\dot{V}(e, \dot{e}) = -\dot{e}^T f_1(\dot{e})$$

which is only negative semi-definite in the error state $(e, \dot{e})$. Using LaSalle's invariance principle completes the proof. ■

Remark 6.4 We can prove in a similar way that the controller

$$\tau = G(q_d) - f_1(z) - \alpha f_2(e)$$

where $f_1, f_2 \in \mathcal{F}^n$ and z is generated from the filter (26) results in a globally asymptotically stable closed-loop system, provided that f_2 is such that there exists an α large enough such that (43) is satisfied.

Proof (sketch) Use the Lyapunov function candidate

$$V(q - q_d, \dot{q}, z) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + F_1(z) + \alpha F_2(q - q_d) - G(q_d)^T(q - q_d) + P(q) - P(q_d)$$

and LaSalle's invariance principle. ∎

Since the robot's potential energy $P(q)$ has a global minimum at $q = 0$ for all θ, it follows that $G(0, \theta) = 0$ for all θ. Therefore, we can use the controller (42) to steer the system towards the origin parameter-independent and globally bounded. We only need a bound on θ, i.e. Θ_M such that $\|\tilde{\theta}\| \leq \Theta_M$, to be able to verify (43)

Proposition 6.5 *Consider the system (1) and assume we know Θ_M such that $\|\tilde{\theta}\| \leq \Theta_M$. Then there exists a switching time $t_s \geq 0$ such that given any $\tilde{t}_s \geq t_s$ the composite control law*

$$\tau = \begin{cases} -f_1(\dot{q}) - \alpha f_2(q) & t < \tilde{t}_s \\ \\ M(q, \hat{\theta})\ddot{q}_r + C(q, \dot{q}, \hat{\theta})\dot{q}_r + G(q, \hat{\theta}) - f_3(e) - f_4(s) & t \geq \tilde{t}_s \end{cases} \tag{45}$$

where $f_1, f_2, f_3, f_4 \in \mathcal{F}^n$ and $\hat{\theta}$ is updated by

$$\dot{\hat{\theta}} = -\Gamma Y^T(q, \dot{q}, \dot{q}_r, \ddot{q}_r)s \qquad t \geq \tilde{t}_s, \qquad \|\tilde{\theta}(\tilde{t}_s)\| \leq \Theta_M \tag{46}$$

yields global asymptotic stability of the point $e = 0$, $\dot{e} = 0$, provided that α and f_2 are chosen such that (43) is satisfied. Furthermore, if $f_1, f_2 \in \mathcal{B}$ we can determine a τ_{max} such that the controller (45) together with the parameter-update-law (46) satisfies

$$\|\tau(t)\| \leq \tau_{max} \qquad \forall t \geq 0.$$

Proof Since $\|\tilde{\theta}(\tilde{t}_s)\| \leq \Theta_M$ we have a bound on the Lyapunov function (41) from which we can derive bounds on $\tilde{\theta}$, e and $\dot{e}$. The proof can be completed similarly to that of Proposition 4.4. ∎

Also this controller can be converted into a smooth globally bounded asymptotically stable control law

Proposition 6.6 *Consider the system (1) and assume we know Θ_M such that $\|\tilde{\theta}\| \leq \Theta_M$. Then there exists a switching time $t_s \geq 0$ and a $\Delta > 0$ such that given any $\tilde{t}_s \geq t_s$ the control law*

$$\tau = -[1 - s_\Delta(t - \tilde{t}_s)][f_1(\dot{q}) + \alpha f_2(q)] + s_\Delta(t - \tilde{t}_s)[M(q, \hat{\theta})\ddot{q}_r + C(q, \dot{q}, \hat{\theta})\dot{q}_r + G(q, \hat{\theta}) - f_3(e) - f_4(s)] \tag{47}$$

where $\hat{\theta}$ is updated by (46) results in global asymptotic stability of the point $e = 0$, $\dot{e} = 0$, provided that αf_2 satisfies (43). Furthermore, if $f_1, f_2 \in \mathcal{B}$ we can determine a τ_{max} such that the controller (47) together with the parameter-update-law (46) satisfies

$$\|\tau(t)\| \leq \tau_{max} \qquad \forall t \geq 0.$$

Proof Similarly to that of Proposition 4.5. ∎

6.2 Simulations

In this section we again consider the two link robot manipulator of [2], however, this time with an unknown additional payload of $\theta = 1kg$, in contrast with the previous two sections. The dynamics with an additional payload are described by

$$
\begin{aligned}
M_0(q) &= \begin{bmatrix} 8.77 + 1.02\cos q_2 & 0.76 + 0.51\cos q_2 \\ 0.76 + 0.51\cos q_2 & 0.62 \end{bmatrix} \\
C_0(q,\dot{q}) &= 0.51\sin q_2 \begin{bmatrix} -\dot{q}_2 & -(\dot{q}_1 + \dot{q}_2) \\ \dot{q}_1 & 0 \end{bmatrix} \\
G_0(q) &= 9.81 \begin{bmatrix} 7.6\sin q_1 + 0.63\sin(q_1+q_2) \\ 0.63\sin(q_1+q_2) \end{bmatrix} \\
Y(w,x,y,z) &= \begin{bmatrix} 2 + 2\cos w_2 & 1 + \cos w_2 \\ 1 + \cos w_2 & 0.62 \end{bmatrix} z + \sin w_2 \begin{bmatrix} -x_2 & -(x_1+x_2) \\ x_1 & 0 \end{bmatrix} y + \\
& +9.81 \begin{bmatrix} \sin w_1 + \sin(w_1+w_2) \\ \sin(w_1+w_2) \end{bmatrix}
\end{aligned}
$$

Due to the additional payload, we are not able to track the desired trajectory of the previous sections, since we have to deal with the input constraints

$$
|\tau_1| \le 120 \qquad |\tau_2| \le 10. \tag{48}
$$

and the control input needed to remain on the desired trajectory (18) if we are on it already exceeds those input constraints.
Therefore, in this section we consider the problem of tracking the desired trajectory

$$
q_d(t) = \begin{bmatrix} 0.1\sin t \\ 0.1\sin t \end{bmatrix}
$$

under the input constraints (48).
We start our simulation from the same initial conditions, i.e.

$$
q(0) = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad \dot{q}(0) = \begin{bmatrix} 10 \\ 10 \end{bmatrix}.
$$

For our simulation we used the controller

$$
\begin{aligned}
\tau &= -[1 - s_\Delta(t - \tilde{t}_s)][\begin{bmatrix} 20\tanh(\dot{q}_1) \\ 2\tanh(\dot{q}_2) \end{bmatrix} + \begin{bmatrix} 100\tanh(2q_1) \\ 8\tanh(2q_2) \end{bmatrix}] + \\
& +s_\Delta(t - \tilde{t}_s)[M(q,\hat{\theta})\ddot{q}_r + C(q,\dot{q},\hat{\theta})\dot{q}_r + G(q,\hat{\theta}) - \begin{bmatrix} 19\tanh(4\dot{e}_1) \\ 1.5\tanh(4\dot{e}_2) \end{bmatrix} - \begin{bmatrix} 20\tanh(2.5\dot{s}_1) \\ 1.7\tanh(3\dot{s}_2) \end{bmatrix}]
\end{aligned} \tag{49}
$$

where $\hat{\theta}$ is updated by (46) where $\Gamma = 1$. We choose $s_\Delta(\cdot)$ to be a function given by (5) where $a = 1$, $b = \frac{3}{2}$ and $\Delta = 1$. We defined t_s to be the first time-instant t such that

$$
\frac{1}{2}\dot{s}(t)^T M(q(t))\dot{s}(t) + \frac{19}{4}\ln(\cosh(4e_1(t))) + \frac{3}{8}\ln\cosh(4e_2(t))) < 0.1.
$$

Assuming the given information that $\Theta_M = 2$, i.e. $\|\tilde{\theta}\| \le 2$, we are guaranteed from (41) to meet the input constraints. The resulting performance is depicted in Figure 6.
We see that the first phase controller reduces the tracking errors in its attempt to steer the system towards the origin. Due to the saturation we are guaranteed to meet the input constraints (48). At $t = 14.0$ the regulating first phase controller reduced the tracking errors enough to switch to the adaptive tracking controller.

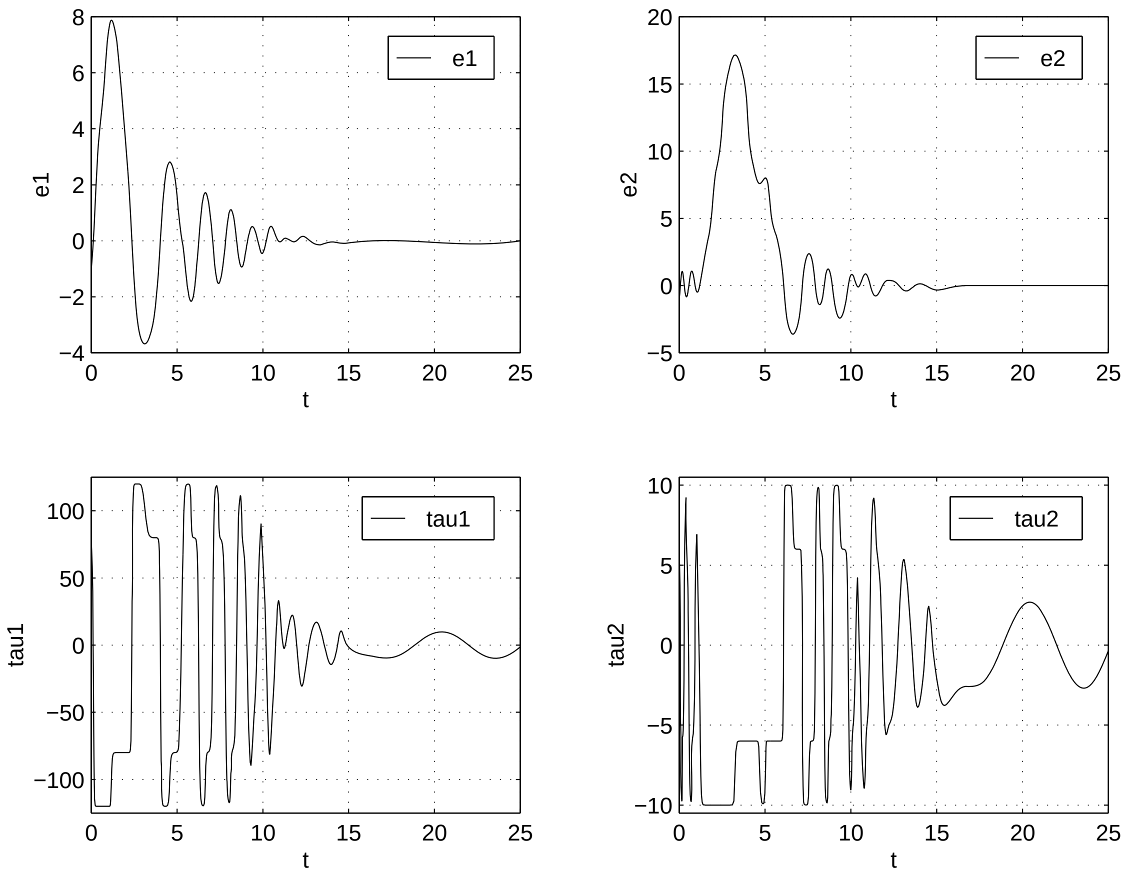

Figure 6: The adaptive composite controller

7 Concluding remarks

In this paper we designed bounded global tracking controllers for rigid robot manipulators using composite controllers. We first use a bounded controller to steer our system closer to the desired trajectory. Then we smoothly change to a local tracking controller. We showed our idea to be succesful for deriving state feedback controllers, output (position) feedback controllers and adaptive controllers. While deriving the composite controllers we extended results from [1, 4, 3, 5, 6, 7, 12, 13, 15, 17] to a much broader class of controllers.

The idea of composite controllers is useful in a far more general context than only for rigid robot manipulators, as we showed in [10]. In this paper we restricted ourselves to first controlling the system towards the origin to reduce the tracking error, before switching to a local tracking controller. More sophisticated strategies can be followed to improve performance.

In this paper we showed that given a rigid robot system, we can derive bounds within which we guarantee our controller to remain. In practice however, we have to deal with prescribed bounds on our control input and wonder if global tracking is possible. Our simulations showed how to deal with that question.

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