Advanced Lyapunov stability theory

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Autonomous systems

- Lyapunov stability
- La Salle's Lemma
- Signal chasing
- Important Examples

Consider a dynamical system with equilibrium point $\bar{x}$:

$$\dot{x} = f(x), \qquad x(0) = x_0, \qquad f(\bar{x}) = 0. \qquad (1)$$

Define a change of variables: $\tilde{x} = x - \bar{x}$, so $x = \tilde{x} + \bar{x}$. Then we have

$$\dot{\tilde{x}} = \tilde{f}(\tilde{x}) = f(\tilde{x} + \bar{x}), \qquad \tilde{x}(0) = \tilde{x}_0 = x_0 - \bar{x}, \qquad \tilde{f}(0) = 0.$$

In the remainer we assume w.l.o.g. that $\bar{x} = 0$.

The equilibrium point $x = \bar{x} = 0$ of the system (1) is

- **stable** If $\forall \epsilon > 0$, $\exists \delta(\epsilon) > 0$ such that $\|x(0)\| < \delta \Rightarrow \|x(t)\| \leq \epsilon$ for all $t \geq 0$.
- **unstable** If it is not stable
- **asymptotically stable** If it is stable and $\exists \delta > 0$ such that $\|x(0)\| < \delta \Rightarrow \lim_{t\to\infty} \|x(t)\| = 0$.

Khalil, Nonlinear Systems, Theorem 4.2 (3rd ed.)

Consider (1). Let $V : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable such that

$$V(0) = 0 \qquad V(x) > 0 \qquad \forall x \neq 0 \qquad (2a)$$
$$\|x\| \to \infty \Rightarrow V(x) \to \infty \qquad (2b)$$
$$\dot{V} < 0 \qquad \forall x \neq 0 \qquad (2c)$$

then $x = 0$ is globally asymptotically stable

Important Example (Hahn, Stability of Motion)

Consider the system

$$\dot{x}_1 = \frac{-6x_1}{(1+x_1^2)^2} + 2x_2 \qquad \dot{x}_2 = \frac{-2(x_1+x_2)}{(1+x_1^2)^2}$$

Differentiating the Lyapunov function candidate $V = \frac{x_1^2}{1+x_1^2} + x_2^2 > 0$

along solutions results in $\dot{V} = -\frac{4(x_2^2+x_1^4x_2^2+x_1^2(3+2x_2^2))}{(1+x_1^2)^4} < 0$.

On hyperbola $x_2 = \frac{2}{x_1-\sqrt{2}}$ we have $\frac{\dot{x}_2}{\dot{x}_1} = -\frac{1}{(x_1\sqrt{2}+1)^2}$, but slope of tangent: $\frac{dx_2}{dx_1} = -\frac{1}{(x_1\sqrt{2}-2)^2}$. So for $x_1 > \sqrt{2}$ and $x_2 > \frac{2}{x_1-\sqrt{2}}$ we can never cross the hyperbola $x_2 = \frac{2}{x_1-\sqrt{2}}$. Therefore we do not have global asymptotic stability of $x = 0$.

Converse Lyapunov Theorem (Khalil, Th. 4.17)

Let $x = 0$ be an asymptotically stable equilibrium point of $\dot{x} = f(x)$.

Let R_A be the region of attraction of $x = 0$.

There exist smooth $V(x)$ and continuous positive definite $W(x)$ (both defined for $x \in R_a$) such that:

$$\begin{aligned} V(x) &\to \infty && \text{as } x \to \partial R_A \\ \frac{\partial V}{\partial x} f(x) &\leq -W(x) && \forall x \in R_A \end{aligned}$$

and for any $c > 0$: $\{x \in R_A \mid V(x) \leq c\}$ is a compact subset of R_A.

For $R_A = \mathbb{R}^n$, $V(x)$ is radially unbounded.

We can use Lyapunov functions for showing asymptotic stability.

When the origin is asymptotically stable, a Lyapunov function does exist.

Problem

How to find a Lyapunov function?

Typical (first) candidates for V:

- Position error (squared)
- Energy

Often encountered problem

$\dot{V}$ is only negative *semi*definite.

Example: mobile robot (circle, constant velocity)

Consider the following dynamics

$$\begin{aligned} \dot{x} &= v\cos\theta & \dot{x}_r &= v_r\cos\theta_r \\ \dot{y} &= v\sin\theta & \dot{y}_r &= v_r\sin\theta_r \\ \dot{\theta} &= \omega & \dot{\theta}_r &= \omega_r \end{aligned}$$

for constant reference inputs $v_r > 0$ and ω_r.

How to define error?

Often seen: $x_e = x - x_r$, $y_e = y - y_r$, $\theta_e = \theta - \theta_r$.

What happens if we change the inertial frame? Errors become different. . .

Example: mobile robot (circle, constant velocity)

Kanayama et al. (1990) defined errors in body-frame of mobile robot:

$$\begin{bmatrix} x_e \\ y_e \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_r - x \\ y_r - y \end{bmatrix}$$
$$\theta_e = \theta_r - \theta$$

resulting in the error dynamics

$$\begin{aligned} \dot{x}_e &= \omega y_e - v + v_r \cos\theta_e \\ \dot{y}_e &= -\omega x_e + v_r \sin\theta_e \\ \dot{\theta}_e &= \omega_r - \omega \end{aligned}$$

Example: mobile robot (circle, constant velocity)

Following Jiang, Nijmeijer (1997), differentiating the Lyapunov function candidate

$$V = \tfrac{1}{2}x_e^2 + \tfrac{1}{2}y_e^2 + \tfrac{1}{2c_3}\theta_e^2$$

along solutions yields

$$\begin{aligned} \dot{V} &= x_e(-v + v_r\cos\theta_e) + v_r y_e \sin\theta_e + \tfrac{1}{c_3}\theta_e(\omega_r - \omega) \\ &= x_e(-v + v_r\cos\theta_e) + \tfrac{1}{c_3}\theta_e(c_3 v_r y_e \tfrac{\sin\theta_e}{\theta_e} + \omega_r - \omega) \\ &= -c_1 x_e^2 - \tfrac{c_2}{c_3}\theta_e^2 \leq 0 \end{aligned}$$

in case we take as input

$$v = v_r\cos\theta_e + c_1 x_e \qquad \omega = \omega_r + c_2\theta_e + c_3 v_r y_e \tfrac{\sin\theta_e}{\theta_e}$$

Example: mobile robot (circle, constant velocity)

Problem: $\dot{V} = -c_1 x_e^2 - \frac{c_2}{c_3}\theta_e^2$ is negative *semi*definite.

We need something for “repairing” our proof:

LaSalle's invariance principle (1959)

Let Ω be a compact set that is positively invariant with respect to $\dot{x} = f(x)$.
Let V be a continuously differentiable function such that $\dot{V}(x) \leq 0$ in Ω.
Let E be the set of points in Ω where $\dot{V} = 0$.
Let M be the largest invariant set in E.
Then every solution starting in Ω approaches M as $t \to \infty$.

Example: mobile robot (circle, constant velocity)

Dynamics:

$$\begin{aligned} \dot{x}_e &= (\omega_r + c_2\theta_e + c_3 v_r y_e \tfrac{\sin\theta_e}{\theta_e})y_e - c_1 x_e \\ \dot{y}_e &= -(\omega_r + c_2\theta_e + c_3 v_r y_e \tfrac{\sin\theta_e}{\theta_e})x_e + v_r \sin\theta_e \\ \dot{\theta}_e &= -c_2\theta_e - c_3 v_r y_e \tfrac{\sin\theta_e}{\theta_e} \end{aligned}$$

Furthermore: $\dot{V} = -c_1 x_e^2 - \frac{c_2}{c_3}\theta_e^2 \leq 0$.

We have $E = \{(x_e, y_e, \theta_e) \mid x_e = \theta_e = 0\}$. From $x_e(t) \equiv 0$ and $\theta_e \equiv 0$ we obtain

$$\begin{aligned} 0 &= (\omega_r + c_2 \cdot 0 + c_3 v_r y_e \cdot 1)y_e - c_1 \cdot 0 \\ 0 &= -c_2 \cdot 0 - c_3 v_r y_e \cdot 1 \end{aligned}$$

and therefore $M = \{(x_e, y_e, \theta_e) \mid x_e = y_e = \theta_e = 0\}$ and global asymptotic stability.

Signal chasing: another example

In Lefeber, Robertsson (1998) we analysed the following dynamics:

$$\dot{w} = \begin{bmatrix} -b_1 & -b_2 & 0 & \cdots & 0 \\ 1 & 0 & -b_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & 0 & -b_n \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} w$$

where $b_i > 0$. Differentiating

$$V = b_1 w_1^2 + b_1 b_2 w_2^2 + \cdots + b_1 b_2 \cdots b_{n-1} w_{n-1}^2 + b_1 b_2 \cdots b_n w_n^2$$

along solutions results in

$$\dot{V} = -b_1^2 w_1^2$$

Signal chasing: another example

We have $\dot{V} = -b_1^2 w_1^2 = 0$, as well as

$$\dot{w}_1 = -b_1 w_1 - b_2 w_2, \qquad \dot{w}_2 = w_1 - b_3 w_3, \qquad \cdots \qquad \dot{w}_{n-1} = w_{n-2} - b_n w_n.$$

Then, from $0 = -b_1 \cdot 0 - b_2 w_2$ we obtain $w_2 = 0$.

Then, from $0 = 0 - b_3 w_3$ we obtain $w_3 = 0$.

$\vdots$

Finally, from $0 = 0 - b_n w_n$ we obtain $w_n = 0$.

And therefore: global asymptotic stability.

Important Example

Consider the dynamics

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1^2 x_2 \\ \dot{x}_2 &= u \end{aligned}$$

in closed-loop with the input $u = -x_2$.

We want to investigate asymptotic stability of the origin of the closed-loop system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1^2 x_2 && \text{outer loop} \\ \dot{x}_2 &= -x_2 && \text{inner loop} \end{aligned}$$

Important Example

(Erroneous) reasoning sometimes found in papers:

"Assume that x_1 is bounded, i.e. $\exists M > 0$ such that $\|x_1(t)\| \leq M$ (e.g., physical system).

Differentiating the Lyapunov function $V = \frac{1}{2}x_1^2 + \frac{M^4}{2}x_2^2$ along the dynamics

$$\dot{x}_1 = -x_1 + x_1^2 x_2 \qquad \dot{x}_2 = -x_2$$

results in

$$\begin{aligned} \dot{V} &= -x_1^2 + x_1^3 x_2 - M^4 x_2^2 \leq -x_1^2 + M^2 |x_1 x_2| - M^4 x_2^2 \\ &\leq -\tfrac{1}{2}x_1^2 \underbrace{-\tfrac{1}{2}x_1^2 + \tfrac{1}{2} \cdot 2 \cdot |x_1| \cdot M^2 |x_2| - \tfrac{1}{2} M^4 x_2^2}_{-\frac{1}{2}(|x_1| - M^2|x_2|)^2} - \tfrac{1}{2} M^4 x_2^2 < 0 \end{aligned}$$

So therefore x_1 does indeed remain bounded and we have global asymptotic stability."

Important Example

Reasoning on previous slide is wrong! Solving the ODE

$$\dot{x}_1 = -x_1 + x_1^2 x_2 \qquad x_1(0) = x_{10}$$
$$\dot{x}_2 = -x_2 \qquad x_2(0) = x_{20}$$

results in

$$x_1(t) = \frac{2x_{10}}{x_{10}x_{20}e^{-t} + [2 - x_{10}x_{20}]e^t} \qquad x_2(t) = x_{20}e^{-t}$$

For $x_{10}x_{20} > 2$ the denominator becomes zero at $t_{\text{esc}} = \frac{1}{2}\log\left(\frac{x_{10}x_{20}}{x_{10}x_{20}-2}\right)$.

So instead of having asymptotic stability, we have a finite escape time!

Non-autonomous systems

- Even more important examples
- Signal chasing using Barbălat's Lemma and Lemma of Micaelli and Samson.
- Signal chasing using (generalisation of) Matrosov's Theorem

Important example (Khalil, 3rd ed, Example 4.22)

Consider the following dynamics

$$\dot{x} = A(t)x \qquad A(t) = \begin{bmatrix} -1 + \frac{3}{2}\cos^2 t & 1 - \frac{3}{2}\sin t \cos t \\ -1 - \frac{3}{2}\sin t \cos t & -1 + \frac{3}{2}\sin^2 t \end{bmatrix}$$

Characteristic polynomial of matrix $A(t)$: $\det[\lambda I - A(t)] = \lambda^2 + \frac{1}{2}\lambda + \frac{1}{2}$

Eigenvalues: $\lambda_i = -\frac{1}{4} \pm \frac{1}{4}\sqrt{7}i$. However

$$x(t) = \begin{bmatrix} e^{\frac{1}{2}t}\cos t & e^{-t}\sin t \\ -e^{\frac{1}{2}t}\sin t & e^{-t}\cos t \end{bmatrix} x(0),$$

so therefore the system is unstable.

Mobile robot: revisited

Assume $v_r(t)$, $\omega_r(t)$ satisfying $0 < v^{\max} \leq v_r(t) \leq v^{\max}$, $|\dot{v}_r| \leq a^{\max}$ and $|\omega_r(t)| \leq \omega^{\max}$.

Consider the dynamics

$$\dot{x}_e = \omega y_e - v + v_r\cos\theta_e \qquad \dot{y}_e = -\omega x_e + v_r \sin\theta_e \qquad \dot{\theta}_e = \omega_r - \omega$$

in closed-loop with the input

$$v = v_r\cos\theta_e + c_1 x_e \qquad \omega = \omega_r + c_2\theta_e + c_3 v_r y_e \frac{\sin\theta_e}{\theta_e}$$

Differentiating $V = \frac{1}{2}x_e^2 + \frac{1}{2}y_e^2 + \frac{1}{2c_3}\theta_e^2$ along solutions results in $\dot{V} = -c_1 x_e^2 - \frac{c_2}{c_3}\theta_e^2 \leq 0$.

LaSalle (1959) is for autonomous systems, but our closed-loop system is non-autonomous...

Questions

1. We have that $V(t)$ is monotone and bounded, so therefore $V(t)$ converges to a constant. Can we deduce that $\dot{V}(t)$ converges to zero (and therefore that x_e and θ_e converge to zero)?
2. If we have that $x_e(t)$ converges to zero, can we conclude that $\dot{x}_e$ converges to zero and use signal chasing for concluding that y_e converges to zero?

Both boil down to: Assume that $\lim_{t\to\infty} x(t) = 0$. Do we have $\lim_{t\to\infty} \dot{x}(t) = 0$?

No: Consider $x(t) = e^{-t}\sin e^{2t}$ for which $\dot{x}(t) = -e^{-t}\sin e^{2t} + 2e^{t}\cos e^{2t}$.

Reverse question: Assume that $x(t)$ is bounded and $\lim_{t\to\infty} \dot{x}(t) = 0$. Do we have $\lim_{t\to\infty} x(t) = C$ for some constant C?

No: Consider $\dot{x}(t) = \frac{\cos(\ln(t+1))}{t+1}$ for which $x(t) = \sin(\ln(1+t))$.

We need some results to complete the proof...

Commonly used tools for completing the proof

Lemma (Barbălat, 1959)

Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be a uniformly continuous function (e.g., $\dot{\phi}$ bounded). Suppose that $\lim_{t\to\infty} \int_0^t \phi(\tau)\,\mathrm{d}\tau$ exists and is finite. Then $\lim\limits_{t\to\infty} \phi(t) = 0$.

Idea: For $\phi(t)$ use $\dot{V}(t)$.

Lemma (Micaelli, Samson, 1993)

Let $f : \mathbb{R}_+ \to \mathbb{R}$ be any differentiable function. If $\lim_{t\to\infty} f(t) = 0$ and

$$\dot{f}(t) = f_0(t) + \eta(t) \qquad t \geq 0$$

where f_0 is a uniformly continuous function (e.g., $\dot{f}_0$ is bounded) and $\lim_{t\to\infty} \eta(t) = 0$, then $\lim_{t\to\infty} \dot{f}(t) = \lim_{t\to\infty} f_0(t) = 0$.

Idea: Signal chasing by (repeatedly) applying to signals that converge to zero

Mobile robot revisited

Since $\dot{V} \leq 0$ we have: x_e, y_e, θ_e bounded.

Step 1: Apply Barbălat to $\phi(t) = \dot{V}(t)$

We have:

$$\begin{aligned}\dot{\phi} = \ddot{V} &= -2c_1 x_e \dot{x}_e - \tfrac{2c_2}{c_3}\theta_e\dot{\theta}_e = \\ &= -2c_1 x_e[(\omega_r + c_2\theta_e + c_3 v_r y_e \tfrac{\sin\theta_e}{\theta_e})y_e - c_1 x_e] - \tfrac{2c_2}{c_3}\theta_e[-c_2\theta_e - c_3 v_r y_e \tfrac{\sin\theta_e}{\theta_e}]\end{aligned}$$

which is bounded. Therefore, $\dot{V}$ is uniformly continuous.
Furthermore, $\lim_{t\to\infty} \int_0^t \dot{V}\,\mathrm{d}\,t = \lim_{t\to\infty} V(t) - V(0)$ exists and is finite.
Therefore, using Barbălat, $\lim_{t\to\infty} \dot{V}(t) = 0$, and therefore $\lim_{t\to\infty} x_e(t) = \lim_{t\to\infty} \theta_e(t) = 0$.

Mobile robot revisited

Step 2: Signal chasing using Lemma of Micaelli and Samson

We have $\theta_e \to 0$, so we consider $\dot{\theta}_e$:

$$\dot{\theta}_e = -c_2\theta_e - c_3 v_r y_e \frac{\sin\theta_e}{\theta_e} = \underbrace{-c_3 v_r y_e}_{f_0(t)} \underbrace{-c_2\theta_e - c_3 v_r y_e\left(\frac{\sin\theta_e}{\theta_e} - 1\right)}_{\eta(t)}$$

Since $-c_3\dot{v}_r y_e - c_3 v_r \dot{y}_e = -c_3\dot{v}_r y_e - c_3 v_r[-(\omega_r + c_2\theta_e + c_3 v_r y_e \frac{\sin\theta_e}{\theta_e})x_e + v_r\sin\theta_e]$ is bounded, we have that $f_0(t)$ is uniformly continuous.
Furthermore, we have $\lim_{t\to\infty} \eta(t) = 0$.
Therefore, using Micaelli and Samson, $\lim_{t\to\infty} f_0(t) = 0$, and therefore $\lim_{t\to\infty} y_e(t) = 0$.

We have asymptotic stability, provided $0 < v^{\min} \leq v_r(t) \leq v^{\max}$, $|\dot{v}_r| \leq a^{\max}$ and $|\omega_r(t)| \leq \omega^{\max}$.

Signal chasing: another example

In Lefeber, Robertsson (1998) we analysed the following dynamics:

$$\dot{w} = \begin{bmatrix} -b_1 & -b_2 u_{1,r} & 0 & \cdots & 0 \\ u_{1,r} & 0 & -b_3 u_{1,r} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & u_{1,r} & 0 & -b_n u_{1,r} \\ 0 & \cdots & 0 & u_{1,r} & 0 \end{bmatrix} w$$

where $b_i > 0$, as well as $0 < u_{1,r}^{\min} \leq u_{1,r}(t) \leq u_{1,r}^{\max}$ and $|\dot{u}_{1,r}| \leq M$. Differentiating

$$V = b_1 w_1^2 + b_1 b_2 w_2^2 + \cdots + b_1 b_2 \cdots b_{n-1} w_{n-1}^2 + b_1 b_2 \cdots b_n w_n^2$$

along solutions results in

$$\dot{V} = -b_1^2 w_1^2$$

Signal chasing: another example

We have $\dot{V} = -b_1^2 w_1^2 = 0$, as well as

$$\dot{w}_1 = -b_1 w_1 - b_2 u_{1,r} w_2, \quad \dot{w}_2 = u_{1,r} w_1 - b_3 u_{1,r} w_3, \quad \cdots \quad \dot{w}_{n-1} = u_{1,r} w_{n-2} - b_n u_{1,r} w_n.$$

From $\dot{V} \leq 0$ we obtain that w remains bounded.

Using Barbălat, we obtain $w_1 \to 0$.

Applying Micaelli-Samson on equation for $\dot{w}_1$ we obtain $b_2 u_{1,r} w_2 \to 0$ and therefore $w_2 \to 0$.

Applying Micaelli-Samson on equation for $\dot{w}_2$ we obtain $b_3 u_{1,r} w_3 \to 0$ and therefore $w_3 \to 0$.

$\vdots$

Applying Micaelli-Samson on equation for $\dot{w}_{n-1}$ we obtain $b_n u_{1,r} w_n \to 0$.

And therefore: global asymptotic stability.

Standard form

Previous example illustrates general approach: starting from signals that go to zero, determine other signals that go to zero.

More general: $\dot{x}_1 = f_1(t, x_1, x_2, x_3)$, $\dot{x}_2 = f_2(t, x_1, x_2, x_3)$, $\dot{x}_3 = f_3(t, x_1, x_2, x_3)$

- Lyapunov function: $V(t, x_1, x_2, x_3)$ positive definite.
- Derivative along dynamics: $\dot{V}(t, x_1)$ negative semi-definite.
- Using Barbălat: $\dot{V}(t, x_1) \to 0$, which implies $x_1 \to 0$.
- Using Micaelli, Samson: $f_1(t, 0, x_2, x_3) \to 0$, which implies $x_2 \to 0$.
- Using Micaelli, Samson: $f_2(t, 0, 0, x_3) \to 0$, which implies $x_3 \to 0$.

Or even more general...

Using this approach we can show global asymptotic stability. However, is that what we want?

Example (Panteley, Loría, Teel, 1999)

Consider the system

$$\dot{x} = \begin{cases} \frac{1}{1+t} & \text{if } x \leq -\frac{1}{1+t} \\ -x & \text{if } |x| \leq \frac{1}{1+t} \\ -\frac{1}{1+t} & \text{if } x \geq \frac{1}{1+t} \end{cases}$$

For each $r > 0$ and $t_0 \geq 0$ there exist $k > 0$ and $\gamma > 0$ such that for all $t \geq t_0$ and $|x(t_0)| \leq r$:

$$|x(t)| \leq k|x(t_0)|e^{-\gamma(t-t_0)} \qquad \forall t \geq t_0 \geq 0$$

However, always a bounded (arbitrarily small) additive perturbation $\delta(t, x)$ and a constant $t_0 \geq 0$ exist such that the trajectories of the perturbed system $\dot{x} = f(t, x) + \delta(t, x)$ are unbounded.

Main reason for this negative result: the constants k and γ are allowed to depend on t_0, i.e., for each value of t_0 different constants k and γ may be chosen.

Some definitions

Continuous function $\alpha : [0, a) \to [0, \infty)$ class $\mathcal{K}$-function ($\alpha \in \mathcal{K}$): $\alpha(0) = 0$, α strictly increasing.

Continuous function $\alpha : [0, \infty) \to [0, \infty)$ class $\mathcal{K}_\infty$-function ($\alpha \in \mathcal{K}$): $\alpha(s) \to \infty$ as $s \to \infty$.

Continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ class $\mathcal{KL}$-function ($\beta \in \mathcal{KL}$): $\beta(r, s) \in \mathcal{K}$ w.r.t. r, for each fixed r: decreasing w.r.t. s and $\beta(r, s) \to 0$ as $s \to \infty$.

Globally asymptotically stable (GAS):
$\forall t_0$: $\exists \beta \in \mathcal{KL}$ such that $\forall x(t_0) : \|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0)$.

Uniformly globally asymptotically stable (UGAS):
$\exists \beta \in \mathcal{KL}$ such that $\forall (t_0, x(t_0)) : \|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0)$.

Lyapunov theorem (Khalil, Theorem 4.9)

Let $x(t)$ be a solution of $\dot{x} = f(t, x)$. Let V be a continuously differentiable function satisfying

$$W_1(x) \leq V(t, x) \leq W_2(x) \qquad \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x)$$

where W_1, W_2, W_3, positive definite functions, then $x = 0$ is UGAS.

Converse Lyapunov theorem (Khalil, Theorem 4.16)

If $x = 0$ is a UGAS equilibrium point of $\dot{x} = f(t, x)$, then there exists V such that

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|) \qquad \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -\alpha_3(\|x\|) \qquad \left\|\frac{\partial V}{\partial x}\right\| \leq \alpha_4(\|x\|)$$

where α_1, α_2, α_3, α_4 are class $\mathcal{K}_\infty$ functions.

Robustness to perturbations for UGAS

Lemma (Khalil 1996 (2nd ed), Lemma 5.3; Khalil 2002 (3rd ed), Lemma 9.3)

Let $x = 0$ be a uniformly asymptotically stable equilibrium point of the nominal system $\dot{x} = f(t, x)$ where $f : \mathbb{R}_+ \times B_r \to \mathbb{R}^n$ is continuously differentiable, and the Jacobian $\left[\frac{\partial f}{\partial x}\right]$ is bounded on B_r, uniformly in t. Then one can determine constants $\Delta > 0$ and $R > 0$ such that for all perturbations $\delta(t, x)$ that satisfy the uniform bound $\|\delta(t, x)\| \leq \delta < \Delta$ and all initial conditions $\|x(t_0)\| \leq R$, the solution $x(t)$ of the perturbed system $\dot{x} = f(t, x) + \delta(t, x)$ satisfies

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) \quad \forall t_0 \leq t \leq t_1 \quad \text{and} \quad \|x(t)\| \leq \rho(\delta) \quad \forall t \geq t_1$$

for some $\beta \in \mathcal{KL}$ and some finite time t_1, where $\rho(\delta)$ is a class $\mathcal{K}$ function of δ. Furthermore, if $x = 0$ is a uniformly globally exponentially stable equilibrium point, we can allow for arbitrarily large δ by choosing $R > 0$ large enough.

Problem

Lesson learned from example

For robustness we need uniform global asymptotic stability.

Main take away from remainder of these lectures

How to show UGAS when we do not have a proper Lyapunov function, i.e, when $\dot{V}$ is negative semi-definite.

Matrosov like theorem (Loría et.al., 2005)

Consider the dynamical system

$$\dot{x} = f(t, x) \qquad x(t_0) = x_0 \qquad f(t, 0) = 0 \tag{3}$$

$f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ loc. bounded, continuous a.e., loc. unif. continuous in t. If there exist

- ○ j differentiable functions $V_j : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$, bounded in t, and
- ○ continuous functions $Y_i : \mathbb{R}^n \to \mathbb{R}$ for $i \in \{1, 2, \dots j\}$ such that
- V_1 is positive definite and radially unbounded,
- $\dot{V}_i(t, x) \le Y_i(x)$, for all $i \in \{1, 2, \dots, j\}$,
- $Y_i(x) = 0$ for $i \in \{1, 2, \dots, k-1\}$ implies $Y_k(x) \le 0$, for all $k \in \{1, 2, \dots, j\}$,
- $Y_i(x) = 0$ for all $i \in \{1, 2, \dots, j\}$ implies $x = 0$,

then the origin $x = 0$ of (3) is uniformly globally asymptotically stable.

Question: how to determine suitable functions V_i and Y_i (for $i > 1$)?

Mobile robot: revisited again

Assume $v_r(t)$, $\omega_r(t)$ satisfying $0 < v^{\min} \le v_r(t) \le v^{\max}$, $|\dot{v}_r| \le a^{\max}$ and $|\omega_r(t)| \le \omega^{\max}$.

Consider the dynamics $\dot{x}_e = \omega y_e - c_1 x_e$, $\dot{y}_e = -\omega x_e + v_r \sin\theta_e$, $\dot{\theta}_e = -c_2\theta_e - c_3 v_r y_e \frac{\sin\theta_e}{\theta_e}$.

Differentiating $V_1 = \frac{1}{2}x_e^2 + \frac{1}{2}y_e^2 + \frac{1}{2c_3}\theta_e^2$ results in $\dot{V}_1 = -c_1 x_e^2 - \frac{c_2}{c_3}\theta_e^2 = Y_1(x_e, y_e, \theta_e)$.

Consider $V_2 = -\theta_e\dot{\theta}_e$. Then

$$\begin{aligned}\dot{V}_2 &= -\dot{\theta}_e^2 - \theta_e\ddot{\theta}_e = -[-c_3 v_r y_e + \eta(t)]^2 - \theta_e\ddot{\theta}_e = -(c_3 v_r y_e)^2 + 2c_3 v_r y_e \eta(t) - \eta(t)^2 - \theta_e\ddot{\theta}_e \\ &\le -c_3^2 (v_r^{\min})^2 y_e^2 + M_1\|\bar{\eta}(x_e, y_e, \theta_e)\| + \|\bar{\eta}(x_e, y_e, \theta_e)\|^2 + M_2\|\theta_e\| = Y_2(x_e, y_e, \theta_e).\end{aligned}$$

Note that $Y_1 = 0$ implies $Y_2 \le 0$. Furthermore, $Y_1 = Y_2 = 0$ implies $x_e = y_e = \theta_e = 0$.

Therefore: uniform global asymptotic stability (applying Matrosov-like theorem).

NB: Instead of taking $V_2 = -\theta_e \cdot \dot{\theta}_e$ we can also taking the "simpler" $V_2 = -\theta_e \cdot f_0$.

Signal chasing: another example revisited

For $b_i > 0$, as well as $0 < u_{1,r}^{\min} \le u_{1,r}(t) \le u_{1,r}^{\max}$ and $|\dot{u}_{1,r}| \le M$, differentiating $V_1 = b_1 w_1^2 + b_1 b_2 w_2^2 + \cdots + b_1 b_2 \cdots b_{n-1} w_{n-1}^2 + b_1 b_2 \cdots b_n w_n^2$ along solutions of

$$\dot{w}_1 = -b_1 w_1 - b_2 u_{1,r} w_2, \quad \dot{w}_2 = u_{1,r} w_1 - b_3 u_{1,r} w_3, \quad \cdots \quad \dot{w}_{n-1} = u_{1,r} w_{n-2} - b_n u_{1,r} w_n.$$

results in $\dot{V}_1 = -b_1^2 w_1^2 = Y_1(w)$.

Differentiating $V_2 = b_2 u_{1,r} w_2 \cdot w_1$ along solutions results in

$$\dot{V}_2 = b_2(\dot{u}_{1,r} w_2 + u_{1,r}\dot{w}_2)w_1 + b_2 u_{1,r} w_2[-b_1 w_1 - b_2 u_{1,r} w_2] \le -b_2^2 (u_{1,r}^{\min})^2 w_2^2 + \bar{M}|w_1| = Y_2(w).$$

Differentiating $V_i = b_i u_{1,r} w_i \cdot w_{i-1}$ $(i = 3, 4, \dots, n)$ along solutions results in

$$\dot{V}_i \le -b_i^2 (u_{1,r}^{\min})^2 w_i^2 + \bar{M}_{i-2}|w_{i-2}| + \bar{M}_{i-1}|w_{i-1}| = Y_i(w).$$

Therefore: uniform global asymptotic stability of $w = 0$ (applying Matrosov-like theorem).

My standard approach for arriving at uniform results

More general case: $\dot{x}_1 = f_1(t, x_1, x_2, x_3)$, $\dot{x}_2 = f_2(t, x_1, x_2, x_3)$, $\dot{x}_3 = f_3(t, x_1, x_2, x_3)$

- Lyapunov function: $V_1(t, x_1, x_2, x_3)$ positive definite.
- Derivative along dynamics: $\dot{V}_1(t, x_1) = \cdots \le Y_1(x_1)$ negative semi-definite.
- Use $V_2 = -x_1^T\dot{x}_1$. Then $\dot{V}_2 \le -f_1(t, 0, x_2, x_3)^T f_1(t, 0, x_2, x_3) + F_2(\|x_1\|) \le Y_2(x)$.
- $Y_1 = 0$ implies $Y_2 \le 0$. Furthermore $Y_1 = Y_2 = 0$ implies $x_1 = x_2 = 0$.
- Use $V_3 = -x_2^T\dot{x}_2$. Then $\dot{V}_3 \le -f_2(t, 0, 0, x_3)^T f_2(t, 0, 0, x_3) + F_3(\|x_1\|, \|x_2\|) \le Y_3(x)$.
- $Y_1 = Y_2 = 0$ implies $Y_3 \le 0$. Also, $Y_1 = Y_2 = Y_3 = 0$ implies $x_1 = x_2 = x_3 = 0$.
- Conclusion: uniform global asymptotic stability.

NB: Often simpler functions can be found for V_i, e.g., $V_2 = -f_1(t, 0, x_2, x_3)^T\dot{x}_1$, etc.

Suggestions for exercises

- Consider a dynamic extension of a mobile robot:

$$\dot{x} = v\cos\theta \qquad \dot{y} = v\sin\theta \qquad \dot{\theta} = \omega \qquad \dot{v} = u_1 \qquad \dot{\omega} = u_2$$

  and consider the problem of tracking a (time-varying) feasible reference trajectory

$$\dot{x}_r = v_r\cos\theta_r \qquad \dot{y}_r = v_r\sin\theta_r \qquad \dot{\theta}_r = \omega_r \qquad \dot{v}_r = u_{1,r} \qquad \dot{\omega}_r = u_{2,r}$$

  Use one of the controllers for the mobile robot from this presentation as a starting point for backstepping to arrive at a tracking controller. Show uniform global asymptotic stability by means of the Matrosov-like theorem and make explicit what assumptions you need to make on signals of the reference trajectory.
- Search for "Barbalat" on the USB-stick with papers of a recent (pre-Covid) CDC or IFAC World Congress. Most likely the authors only show (global) asymptotic stability. Update the proof of the authors so that you can conclude *uniform* (global) asymptotic stability.

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