TU/e EINDHOVEN UNIVERSITY OF TECHNOLOGY

Lyapunov stability: common mistakes and useful machinery

Lectures for PhD students at LTH, Automatic Control

Erjen Lefeber

Slides available via `https://dc.wtb.tue.nl/lefeber/research/talks/`

Department of Mechanical Engineering

TU/e EINDHOVEN UNIVERSITY OF TECHNOLOGY

Autonomous systems

- Lyapunov stability
- La Salle's Lemma
- Signal chasing
- Backstepping
- Important Examples

Consider a dynamical system with **equilibrium point** $\bar{x}$:

$$\dot{x} = f(x), \qquad f(\bar{x}) = 0, \qquad x(0) = x_0. \tag{1}$$

Define a change of variables: $\tilde{x} = x - \bar{x}$, so $x = \tilde{x} + \bar{x}$. Then we have

$$\dot{\tilde{x}} = \tilde{f}(\tilde{x}) = f(\tilde{x} + \bar{x}), \qquad \tilde{f}(0) = 0, \qquad \tilde{x}(0) = \tilde{x}_0,$$

In the remainer we **assume w.l.o.g. that** $\bar{x} = 0$.

The equilibrium point $x = \bar{x} = 0$ of the system (1) is

stable If $\forall \epsilon > 0, \exists \delta(\epsilon) > 0$ such that $\|x(0)\| < \delta \Rightarrow \|x(t)\| \leq \epsilon$ for all $t \geq 0$.

unstable If it is not stable

asymptotically stable If it is stable and $\exists \delta > 0$ such that $\|x(0)\| < \delta \Rightarrow \lim_{t\to\infty} \|x(t)\| = 0$.

Khalil, Nonlinear Systems, Theorem 4.2 (3rd ed.)

Consider (1). Let $V : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable such that

$$V(0) = 0 \qquad V(x) > 0 \qquad \forall x \neq 0 \tag{2}$$

$$\|x\| \to \infty \Rightarrow V(x) \to \infty \tag{3}$$

$$\dot{V} < 0 \qquad \forall x \neq 0 \tag{4}$$

then $x = 0$ is globally asymptotically stable

Important Example (Hahn, Stability of Motion)

Consider the system

$$\dot{x}_1 = \frac{-6x_1}{(1+x_1^2)^2} + 2x_2 \qquad \dot{x}_2 = \frac{-2(x_1+x_2)}{(1+x_1^2)^2}$$

Differentiating the Lyapunov function candidate $V = \frac{x_1^2}{1+x_1^2} + x_2^2 > 0$

along solutions results in $\dot{V} = -\frac{4(x_2^2+x_1^4x_2^2+x_1^2(3+2x_2^2))}{(1+x_1^2)^4} < 0$.

On hyperbola $x_2 = \frac{2}{x_1-\sqrt{2}}$ we have $\frac{\dot{x}_2}{\dot{x}_1} = -\frac{1}{(x_1\sqrt{2}+1)^2}$, but slope of tangent: $\frac{dx_2}{dx_1} = -\frac{1}{(x_1\sqrt{2}-2)^2}$.
So for $x_1 > \sqrt{2}$ and $x_2 > \frac{2}{x_1-\sqrt{2}}$ we can never cross the hyperbola $x_2 = \frac{2}{x_1-\sqrt{2}}$. Therefore we do **not** have global asymptotic stability of $x = 0$.

Converse Lyapunov Theorem (Khalil, Th. 4.17)

Let $x = 0$ be an asymptotically stable equilibrium point of $\dot{x} = f(x)$.

Let R_A be the region of attraction of $x = 0$.

There exist smooth $V(x)$ and continuous positive definite $W(x)$ (both defined for $x \in R_a$) such that:

$$\begin{aligned} V(x) &\to \infty && \text{as } x \to \partial R_A \\ \frac{\partial V}{\partial x} f(x) &\leq -W(x) && \forall x \in R_A \end{aligned}$$

and for any $c > 0$: $\{x \in R_A \mid V(x) \leq c\}$ is a compact subset of R_A.

For $R_A = \mathbb{R}^n$, $V(x)$ is radially unbounded.

We can use Lyapunov functions for showing asymptotic stability.

When the origin is asymptotically stable, a Lyapunov function does exist.

Problem

How to find a Lyapunov function?

Typical (first) candidates for V:

- Position error (squared)
- Energy

Often encountered problem

$\dot{V}$ is only negative *semi*definite.

Example: mobile robot (circle, constant velocity)

Consider the following dynamics

$$\begin{aligned} \dot{x} &= v\cos\theta & \dot{x}_r &= v_r \cos\theta_r \\ \dot{y} &= v\sin\theta & \dot{y}_r &= v_r \sin\theta_r \\ \dot{\theta} &= \omega & \dot{\theta}_r &= \omega_r \end{aligned}$$

for constant reference inputs $v_r > 0$ and ω_r.

How to define error?

Often seen: $x_e = x - x_r$, $y_e = y - y_r$, $\theta_e = \theta - \theta_r$.

What happens if we change the inertial frame? Errors become different...

Example: mobile robot (circle, constant velocity)

Kanayama et al. (1990) defined errors in body-frame of mobile robot:

$$\begin{bmatrix} x_e \\ y_e \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_r - x \\ y_r - y \end{bmatrix}$$
$$\theta_e = \theta_r - \theta$$

resulting in the error dynamics

$$\begin{aligned} \dot{x}_e &= \omega y_e - v + v_r \cos\theta_e \\ \dot{y}_e &= -\omega x_e + v_r \sin\theta_e \\ \dot{\theta}_e &= \omega_r - \omega \end{aligned}$$

Example: mobile robot (circle, constant velocity)

Following Jiang, Nijmeijer (1997), differentiating the Lyapunov function candidate

$$V = \tfrac{1}{2}x_e^2 + \tfrac{1}{2}y_e^2 + \tfrac{1}{2c_3}\theta_e^2$$

along solutions yields

$$\begin{aligned} \dot{V} &= x_e(-v + v_r\cos\theta_e) + v_r y_e \sin\theta_e + \tfrac{1}{c_3}\theta_e(\omega_r - \omega) \\ &= x_e(-v + v_r\cos\theta_e) + \tfrac{1}{c_3}\theta_e(c_3 v_r y_e \tfrac{\sin\theta_e}{\theta_e} + \omega_r - \omega) \\ &= -c_1 x_e^2 - \tfrac{c_2}{c_3}\theta_e^2 \leq 0 \end{aligned}$$

in case we take as input

$$v = v_r \cos\theta_e + c_1 x_e \qquad \omega = \omega_r + c_2\theta_e + c_3 v_r y_e \tfrac{\sin\theta_e}{\theta_e}$$

Example: mobile robot (circle, constant velocity)

Problem: $\dot{V} = -c_1 x_e^2 - \frac{c_2}{c_3}\theta_e^2$ is negative *semi*definite.

We need something for “repairing” our proof:

LaSalle’s invariance principle (1959)

Let Ω be a compact set that is positively invariant with respect to $\dot{x} = f(x)$.
Let V be a continuously differentiable function such that $\dot{V}(x) \leq 0$ in Ω.
Let E be the set of points in Ω where $\dot{V} = 0$.
Let M be the largest invariant set in E.
Then every solution starting in Ω approaches M as $t \to \infty$.

Example: mobile robot (circle, constant velocity)

Dynamics:

$$\begin{aligned} \dot{x}_e &= (\omega_r + c_2\theta_e + c_3 v_r y_e \tfrac{\sin\theta_e}{\theta_e})y_e - c_1 x_e \\ \dot{y}_e &= -(\omega_r + c_2\theta_e + c_3 v_r y_e \tfrac{\sin\theta_e}{\theta_e})x_e + v_r \sin\theta_e \\ \dot{\theta}_e &= -c_2\theta_e - c_3 v_r y_e \tfrac{\sin\theta_e}{\theta_e} \end{aligned}$$

Furthermore: $\dot{V} = -c_1 x_e^2 - \frac{c_2}{c_3}\theta_e^2 \leq 0$.

We have $E = \{(x_e, y_e, \theta_e) \mid x_e = \theta_e = 0\}$. From $x_e(t) \equiv 0$ and $\theta_e \equiv 0$ we obtain

$$\begin{aligned} 0 &= (\omega_r + c_2 \cdot 0 + c_3 v_r y_e \cdot 1)y_e - c_1 \cdot 0 \\ 0 &= -c_2 \cdot 0 - c_3 v_r y_e \cdot 1 \end{aligned}$$

and therefore $M = \{(x_e, y_e, \theta_e) \mid x_e = y_e = \theta_e = 0\}$ and global asymptotic stability.

Defining error coordinates for angles

We picked displacement errors independent of the inertial frame. But how about angles?

More natural error: associate angle with point on unit disc and use distance between such points.

Differentiating $V = \frac{1}{2}x_e^2 + \frac{1}{2}y_e^2 + \frac{1}{c_3}(1 - \cos\theta_e)$ along

$$\dot{x}_e = \omega y_e - v + v_r \cos\theta_e \qquad \dot{y}_e = -\omega x_e + v_r \sin\theta_e \qquad \dot{\theta}_e = \omega_r - \omega$$

results in

$$\dot{V} = x_e[-v + v_r \cos\theta_e] + \tfrac{1}{c_3}\sin\theta_e[c_3 v_r y_e + \omega_r - \omega]$$

so we get $\dot{V} \le 0$ for the input

$$v = v_r \cos\theta_e + c_1 x_e \qquad \omega = \omega_r + c_2 \sin\theta_e + c_3 v_r y_e$$

Completing the proof using LaSalle

We have $\dot{V} = -c_1 x_e^2 - \frac{c_2}{c_3}\sin^2\theta_e \le 0$.

From $\sin\theta_e \equiv 0$ we obtain

$$0 = \cos\theta_e \dot{\theta}_e = -c_2 \sin\theta_e \cos\theta_e - c_3 v_r y_e \cos\theta_e = -c_3 v_r y_e \cos\theta_e$$

and as $\cos\theta_e = \pm 1$ for $\sin\theta_e = 0$, we obtain $y_e = 0$ and therefore $M = \{(x_e, y_e, \theta_e) \mid x_e = y_e = 0, \sin\theta_e = 0\}$.

Remaining question: **Do we converge to the desired equilibrium point?**

Stability of equilibrium points

Linearisation of closed-loop dynamics around $x_e = 0$, $y_e = 0$, $\sin\theta_e = 0$:

$$\begin{bmatrix}\dot{\bar{x}}_e \\ \dot{\bar{y}}_e \\ \dot{\bar{\theta}}_e\end{bmatrix} = \underbrace{\begin{bmatrix}-c_1 & \omega_r & 0 \\ -\omega_r & 0 & 1 \\ 0 & -c_3 v_r & -c_2\end{bmatrix}}_{\text{in case } \cos\theta_e = 1}\begin{bmatrix}\bar{x}_e \\ \bar{y}_e \\ \bar{\theta}_e\end{bmatrix} \qquad \begin{bmatrix}\dot{\bar{x}}_e \\ \dot{\bar{y}}_e \\ \dot{\bar{\theta}}_e\end{bmatrix} = \underbrace{\begin{bmatrix}-c_1 & \omega_r & 0 \\ -\omega_r & 0 & -1 \\ 0 & -c_3 v_r & c_2\end{bmatrix}}_{\text{in case } \cos\theta_e = -1}\begin{bmatrix}\bar{x}_e \\ \bar{y}_e \\ \bar{\theta}_e\end{bmatrix}$$

Characteristic polynomial for $\cos\theta_e = -1$:

$$\lambda^3 + (c_1 - c_2)\lambda^2 + (\omega_r^2 - c_2 c_1 - c_3 v_r)\lambda \underbrace{-(c_1 c_3 v_r + c_2 \omega_r^2)}_{<0:\ \text{unstable}}$$

Characteristic polynomial for $\cos\theta_e = 1$:

$$\lambda^3 + (c_1 + c_2)\lambda^2 + (\omega_r^2 + c_2 c_1 + c_3 v_r)\lambda + (c_1 c_3 v_r + c_2 \omega_r^2)$$

Stable, as also $(c_1 + c_2)(\omega_r^2 + c_2 c_1 + c_3 v_r) > (c_1 c_3 v_r + c_2 \omega_r^2)$.

Signal chasing: another example

In Lefeber, Robertsson (1998) we analysed the following dynamics:

$$\dot{w} = \begin{bmatrix}-b_1 & -b_2 & 0 & \cdots & 0 \\ 1 & 0 & -b_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & 0 & -b_n \\ 0 & \cdots & 0 & 1 & 0\end{bmatrix} w$$

where $b_i > 0$. Differentiating

$$V = b_1 w_1^2 + b_1 b_2 w_2^2 + \cdots + b_1 b_2 \cdots b_{n-1} w_{n-1}^2 + b_1 b_2 \cdots b_n w_n^2$$

along solutions results in

$$\dot{V} = -b_1^2 w_1^2$$

Signal chasing: another example

We have $\dot{V} = -b_1^2 w_1^2 = 0$, as well as

$$\dot{w}_1 = -b_1 w_1 - b_2 w_2, \qquad \dot{w}_2 = w_1 - b_3 w_3, \qquad \cdots \qquad \dot{w}_{n-1} = w_{n-2} - b_n w_n.$$

Then, from $0 = -b_1 \cdot 0 - b_2 w_2$ we obtain $w_2 = 0$.

Then, from $0 = 0 - b_3 w_3$ we obtain $w_3 = 0$.

$\vdots$

Finally, from $0 = 0 - b_n w_n$ we obtain $w_n = 0$.

And therefore: global asymptotic stability.

Important Example

Consider the dynamics

$$\begin{aligned} \dot{x}_1 &= -x_1 \ + \ x_1^2 x_2 \\ \dot{x}_2 &= u \end{aligned}$$

in closed-loop with the input $u = -x_2$.

We want to investigate asymptotic stability of the origin of the closed-loop system

$$\begin{aligned} \dot{x}_1 &= -x_1 \ + \ x_1^2 x_2 && \text{outer loop} \\ \dot{x}_2 &= -x_2 && \text{inner loop} \end{aligned}$$

Important Example

(Erroneous) reasoning sometimes found in papers:

“Assume that x_1 is bounded, i.e. $\exists M > 0$ such that $\|x_1(t)\| \leq M$ (e.g., physical system).

Differentiating the Lyapunov function $V = \frac{1}{2}x_1^2 + \frac{M^4}{2}x_2^2$ along the dynamics

$$\dot{x}_1 = -x_1 \ + \ x_1^2 x_2 \qquad\qquad \dot{x}_2 = -x_2$$

results in

$$\begin{aligned} \dot{V} &= -x_1^2 + x_1^3 x_2 - M^4 x_2^2 \leq -x_1^2 + M^2 |x_1 x_2| - M^4 x_2^2 \\ &\leq -\tfrac{1}{2}x_1^2 \underbrace{-\tfrac{1}{2}x_1^2 + \tfrac{1}{2} \cdot 2 \cdot |x_1| \cdot M^2 |x_2| - \tfrac{1}{2} M^4 x_2^2}_{-\frac{1}{2}(|x_1| - M^2|x_2|)^2} - \tfrac{1}{2} M^4 x_2^2 < 0 \end{aligned}$$

So therefore x_1 does indeed remain bounded and we have global asymptotic stability.”

Important Example

Reasoning on previous slide is wrong! Solving the ODE

$$\begin{aligned} \dot{x}_1 &= -x_1 \ + \ x_1^2 x_2 && x_1(0) = x_{10} \\ \dot{x}_2 &= -x_2 && x_2(0) = x_{20} \end{aligned}$$

results in

$$x_1(t) = \frac{2x_{10}}{x_{10}x_{20}e^{-t} + [2 - x_{10}x_{20}]e^{t}} \qquad\qquad x_2(t) = x_{20}e^{-t}$$

For $x_{10}x_{20} > 2$ the denominator becomes zero at $t_{\text{esc}} = \frac{1}{2} \log \left(\frac{x_{10}x_{20}}{x_{10}x_{20} - 2} \right)$.

So instead of having asymptotic stability, we have a finite escape time!

Non-autonomous systems

- Even more important examples
- Signal chasing using Barbălat's Lemma and Lemma of Micaelli and Samson.
- Signal chasing using (generalisation of) Matrosov's Theorem

Important example (Khalil (3rd ed), Example 4.22)

Consider the following dynamics

$$\dot{x} = A(t)x \qquad A(t) = \begin{bmatrix} -1+\frac{3}{2}\cos^2 t & 1-\frac{3}{2}\sin t\cos t \\ -1-\frac{3}{2}\sin t\cos t & -1+\frac{3}{2}\sin^2 t \end{bmatrix}$$

Characteristic polynomial of matrix $A(t)$: $\det[\lambda I - A(t)] = \lambda^2 + \frac{1}{2}\lambda + \frac{1}{2}$

Eigenvalues: $\lambda_i = -\frac{1}{4} \pm \frac{1}{4}\sqrt{7}i$. However

$$x(t) = \begin{bmatrix} e^{\frac{1}{2}t}\cos t & e^{-t}\sin t \\ -e^{\frac{1}{2}t}\sin t & e^{-t}\cos t \end{bmatrix} x(0),$$

so therefore the system is **unstable**.

Mobile robot: revisited

Assume $v_r(t), \omega_r(t)$ satisfying $0 < v^{\min} \leq v_r(t) \leq v^{\max}$, $|\dot{v}_r| \leq a^{\max}$ and $|\omega_r(t)| \leq \omega^{\max}$.

Consider the dynamics

$$\dot{x}_e = \omega y_e - v + v_r\cos\theta_e \qquad \dot{y}_e = -\omega x_e + v_r\sin\theta_e \qquad \dot{\theta}_e = \omega_r - \omega$$

in closed-loop with the input

$$v = v_r\cos\theta_e + c_1 x_e \qquad \omega = \omega_r + c_2\theta_e + c_3 v_r y_e \frac{\sin\theta_e}{\theta_e}$$

Differentiating $V = \frac{1}{2}x_e^2 + \frac{1}{2}y_e^2 + \frac{1}{2c_3}\theta_e^2$ along solutions results in $\dot{V} = -c_1 x_e^2 - \frac{c_2}{c_3}\theta_e^2 \leq 0$.

LaSalle (1959) is for autonomous systems, but our closed-loop system is non-autonomous...

Questions

1. We have that $V(t)$ is monotone and bounded, so therefore $V(t)$ converges to a constant. Can we deduce that $\dot{V}(t)$ converges to zero (and therefore that x_e and θ_e converge to zero)?
2. If we have that $x_e(t)$ converges to zero, can we conclude that $\dot{x}_e$ converges to zero and use signal chasing for concluding that y_e converges to zero?

Both boil down to: Assume that $\lim_{t\to\infty} x(t) = 0$. Do we have $\lim_{t\to\infty} \dot{x}(t) = 0$?

No: Consider $x(t) = e^{-t}\sin e^{2t}$ for which $\dot{x}(t) = -e^{-t}\sin e^{2t} + 2e^{t}\cos e^{2t}$.

Reverse question: Assume that $x(t)$ is bounded and $\lim_{t\to\infty} \dot{x}(t) = 0$. Do we have $\lim_{t\to\infty} x(t) = C$ for some constant C?

No: Consider $\dot{x}(t) = \frac{\cos(\ln(t+1))}{t+1}$ for which $x(t) = \sin(\ln(1+t))$.

We need some results to complete the proof...

Commonly used tools for completing the proof

Lemma (Barbălat, 1959)

Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be a uniformly continuous function (e.g., $\dot{\phi}$ bounded). Suppose that $\lim_{t\to\infty} \int_0^t \phi(\tau)\,\mathrm{d}\,\tau$ exists and is finite. Then $\lim\limits_{t\to\infty} \phi(t) = 0$.

Idea: For $\phi(t)$ use $\dot{V}(t)$.

Lemma (Micaelli, Samson, 1993)

Let $f : \mathbb{R}_+ \to \mathbb{R}$ be any differentiable function. If $\lim_{t\to\infty} f(t) = 0$ and

$$\dot{f}(t) = f_0(t) + \eta(t) \qquad\qquad t \geq 0$$

where f_0 is a uniformly continuous function (e.g., $\dot{f}_0$ is bounded) and $\lim_{t\to\infty} \eta(t) = 0$, then $\lim_{t\to\infty} \dot{f}(t) = \lim_{t\to\infty} f_0(t) = 0$.

Idea: Signal chasing by (repeatedly) applying to signals that converge to zero

Mobile robot revisited

Since $\dot{V} \leq 0$ we have: x_e, y_e, θ_e bounded.

Step 1: Apply Barbălat to $\phi(t) = \dot{V}(t)$

We have:

$$\begin{aligned}\dot{\phi} &= \ddot{V} = -2c_1 x_e \dot{x}_e - \tfrac{2c_2}{c_3}\theta_e\dot{\theta}_e = \\ &= -2c_1 x_e[(\omega_r + c_2\theta_e + c_3 v_r y_e \tfrac{\sin\theta_e}{\theta_e})y_e - c_1 x_e] - \tfrac{2c_2}{c_3}\theta_e[-c_2\theta_e - c_3 v_r y_e \tfrac{\sin\theta_e}{\theta_e}]\end{aligned}$$

which is bounded. Therefore, $\dot{V}$ is uniformly continuous.
Furthermore, $\lim_{t\to\infty} \int_0^t \dot{V}\,\mathrm{d}\,t = \lim_{t\to\infty} V(t) - V(0)$ exists and is finite.
Therefore, using Barbălat, $\lim_{t\to\infty} \dot{V}(t) = 0$, and therefore $\lim_{t\to\infty} x_e(t) = \lim_{t\to\infty} \theta_e(t) = 0$.

Mobile robot revisited

Step 2: Signal chasing using Lemma of Micaelli and Samson

We have $\theta_e \to 0$, so we consider $\dot{\theta}_e$:

$$\dot{\theta}_e = -c_2\theta_e - c_3 v_r y_e \frac{\sin\theta_e}{\theta_e} = \underbrace{-c_3 v_r y_e}_{f_0(t)} \underbrace{-c_2\theta_e - c_3 v_r y_e \left(\frac{\sin\theta_e}{\theta_e} - 1\right)}_{\eta(t)}$$

Since $-c_3\dot{v}_r y_e - c_3 v_r \dot{y}_e = -c_3 \dot{v}_r y_e - c_3 v_r[-(\omega_r + c_2\theta_e + c_3 v_r y_e \frac{\sin\theta_e}{\theta_e})x_e + v_r \sin\theta_e]$ is bounded, we have that $f_0(t)$ is uniformly continuous.
Furthermore, we have $\lim_{t\to\infty} \eta(t) = 0$.
Therefore, using Micaelli and Samson, $\lim_{t\to\infty} f_0(t) = 0$, and therefore $\lim_{t\to\infty} y_e(t) = 0$.

We have asymptotic stability, provided $0 < v^{\min} \leq v_r(t) \leq v^{\max}$, $|\dot{v}_r| \leq a^{\max}$ and $|\omega_r(t)| \leq \omega^{\max}$.

Mobile robot revisited: definition angular error

Differentiating $V = \frac{1}{2}x_e^2 + \frac{1}{2}y_e^2 + \frac{1}{c_3}(1 - \cos\theta_e)$ along

$$\dot{x}_e = \omega y_e - v + v_r\cos\theta_e \qquad \dot{y}_e = -\omega x_e + v_r \sin\theta_e \qquad \dot{\theta}_e = \omega_r - \omega$$

results for $v = v_r\cos\theta_e + c_1 x_e$, $\omega = \omega_r + c_2\sin\theta_e + c_3 v_r y_e$ in $\dot{V} = -c_1 x_e^2 + \frac{c_2}{c_3}\sin^2\theta_e$.

Using Barbălat we obtain $\lim_{t\to\infty} x_e(t) = \lim_{t\to\infty} \sin\theta_e(t) = 0$.

Applying Micaelli-Samson to $f(t) = \sin\theta_e(t)$ gives

$$\dot{f} = \underbrace{-c_3 v_r y_e \cos\theta_e}_{f_0(t)} \underbrace{-c_2\cos\theta_e\sin\theta_e}_{\eta(t)}$$

And we can conclude $\lim_{t\to\infty} y_e(t) = 0$.

Signal chasing: another example

In Lefeber, Robertsson (1998) we analysed the following dynamics:

$$\dot{w} = \begin{bmatrix} -b_1 & -b_2 u_{1,r} & 0 & \cdots & 0 \\ u_{1,r} & 0 & -b_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & u_{1,r} & 0 & -b_n u_{1,r} \\ 0 & \cdots & 0 & u_{1,r} & 0 \end{bmatrix} w$$

where $b_i > 0$, as well as $0 < u_{1,r}^{\text{min}} \leq u_{1,r}(t) \leq u_{1,r}^{\text{max}}$ and $|\dot{u}_{1,r}| \leq M$. Differentiating

$$V = b_1 w_1^2 + b_1 b_2 w_2^2 + \cdots + b_1 b_2 \cdots b_{n-1} w_{n-1}^2 + b_1 b_2 \cdots b_n w_n^2$$

along solutions results in

$$\dot{V} = -b_1^2 w_1^2$$

Signal chasing: another example

We have $\dot{V} = -b_1^2 w_1^2 = 0$, as well as

$$\dot{w}_1 = -b_1 w_1 - b_2 u_{1,r} w_2, \quad \dot{w}_2 = u_{1,r} w_1 - b_3 u_{1,r} w_3, \quad \cdots \quad \dot{w}_{n-1} = u_{1,r} w_{n-2} - b_n u_{1,r} w_n.$$

From $\dot{V} \leq 0$ we obtain that w remains bounded.

Using Barbălat, we obtain $w_1 \to 0$.

Applying Micaelli-Samson on equation for $\dot{w}_1$ we obtain $b_2 u_{1,r} w_2 \to 0$ and therefore $w_2 \to 0$.

Applying Micaelli-Samson on equation for $\dot{w}_2$ we obtain $b_3 u_{1,r} w_3 \to 0$ and therefore $w_3 \to 0$.

$\vdots$

Applying Micaelli-Samson on equation for $\dot{w}_{n-1}$ we obtain $b_n u_{1,r} w_n \to 0$.

And therefore: **global asymptotic stability**.

Standard form

Previous example illustrates general approach: starting from signals that go to zero, determine other signals that go to zero.

More general: $\dot{x}_1 = f_1(t, x_1, x_2, x_3)$, $\dot{x}_2 = f_2(t, x_1, x_2, x_3)$, $\dot{x}_3 = f_3(t, x_1, x_2, x_3)$

- Lyapunov function: $V(t, x_1, x_2, x_3)$ positive definite.
- Derivative along dynamics: $\dot{V}(t, x_1)$ negative semi-definite.
- Using Barbălat: $\dot{V}(t, x_1) \to 0$, which implies $x_1 \to 0$.
- Using Micaelli, Samson: $f_1(t, 0, x_2, x_3) \to 0$, which implies $x_2 \to 0$.
- Using Micaelli, Samson: $f_2(t, 0, 0, x_3) \to 0$, which implies $x_3 \to 0$.

Or even more general...

Using this approach we can show **global asymptotic stability**. However, is that what we want?

Example (Panteley, Loría, Teel, 1999)

Consider the system

$$\dot{x} = \begin{cases} \frac{1}{1+t} & \text{if } x \leq -\frac{1}{1+t} \\ -x & \text{if } |x| \leq \frac{1}{1+t} \\ -\frac{1}{1+t} & \text{if } x \geq \frac{1}{1+t} \end{cases}$$

For each $r > 0$ and $t_0 \geq 0$ there exist $k > 0$ and $\gamma > 0$ such that for all $t \geq t_0$ and $|x(t_0)| \leq r$:

$$|x(t)| \leq k|x(t_0)|e^{-\gamma(t-t_0)} \qquad \forall t \geq t_0 \geq 0$$

However, always a bounded (arbitrarily small) additive perturbation $\delta(t, x)$ and a constant $t_0 \geq 0$ exist such that the trajectories of the perturbed system $\dot{x} = f(t, x) + \delta(t, x)$ are unbounded.

Main reason for this negative result: the constants k and γ are allowed to depend on t_0, i.e., for each value of t_0 different constants k and γ may be chosen.

Some definitions

Continuous function $\alpha : [0, a) \to [0, \infty)$ **class $\mathcal{K}$-function** ($\alpha \in \mathcal{K}$): $\alpha(0) = 0$, α strictly increasing.

Continuous function $\alpha : [0, \infty) \to [0, \infty)$ **class $\mathcal{K}_\infty$-function** ($\alpha \in \mathcal{K}$): $\alpha(s) \to \infty$ as $s \to \infty$.

Continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ **class $\mathcal{KL}$-function** ($\beta \in \mathcal{KL}$): $\beta(r, s) \in \mathcal{K}$ w.r.t. r, for each fixed r: decreasing w.r.t. s and $\beta(r, s) \to 0$ as $s \to \infty$.

Globally asymptotically stable (**GAS**):
$\forall t_0$: $\exists \beta \in \mathcal{KL}$ such that $\forall x(t_0) : \|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0)$.

Uniformly globally asymptotically stable (**UGAS**):
$\exists \beta \in \mathcal{KL}$ such that $\forall (t_0, x(t_0)) : \|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0)$.

Lyapunov theorem (Khalil, Theorem 4.9)

Let $x(t)$ be a solution of $\dot{x} = f(t, x)$. Let V be a continuously differentiable function satisfying

$$W_1(x) \leq V(t, x) \leq W_2(x) \qquad \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x)$$

where W_1, W_2, W_3, positive definite functions, then $x = 0$ is UGAS.

Converse Lyapunov theorem (Khalil, Theorem 4.16)

If $x = 0$ is a UGAS equilibrium point of $\dot{x} = f(t, x)$, then there exists V such that

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|) \qquad \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -\alpha_3(\|x\|) \qquad \left\|\frac{\partial V}{\partial x}\right\| \leq \alpha_4(\|x\|)$$

where α_1, α_2, α_3, α_4 are class $\mathcal{K}_\infty$ functions.

Robustness to perturbations for UGAS

Lemma (Khalil 1996 (2nd ed), Lemma 5.3; Khalil 2002 (3rd ed), Lemma 9.3)

*Let $x = 0$ be a **uniformly asymptotically stable** equilibrium point of the nominal system $\dot{x} = f(t, x)$ where $f : \mathbb{R}_+ \times B_r \to \mathbb{R}^n$ is continuously differentiable, and the Jacobian $\left[\frac{\partial f}{\partial x}\right]$ is bounded on B_r, uniformly in t. Then one can determine constants $\Delta > 0$ and $R > 0$ such that for all perturbations $\delta(t, x)$ that satisfy the uniform bound $\|\delta(t, x)\| \leq \delta < \Delta$ and all initial conditions $\|x(t_0)\| \leq R$, the solution $x(t)$ of **the perturbed system** $\dot{x} = f(t, x) + \delta(t, x)$ **satisfies***

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) \quad \forall t_0 \leq t \leq t_1 \quad \text{and} \quad \|x(t)\| \leq \rho(\delta) \quad \forall t \geq t_1$$

*for some $\beta \in \mathcal{KL}$ and some finite time t_1, where $\rho(\delta)$ is a class $\mathcal{K}$ function of δ.
Furthermore, if $x = 0$ is a uniformly globally exponentially stable equilibrium point, we can allow for arbitrarily large δ by choosing $R > 0$ large enough.*

Problem

Lesson learned from example

For robustness we need **uniform** global asymptotic stability.

Main take away from remainder of this lecture series

How to show UGAS when we do **not** have a proper Lyapunov function, i.e, when $\dot{V}$ is negative **semi**-definite.

Matrosov like theorem (Loría et.al., 2005)

Consider the dynamical system

$$\dot{x} = f(t, x) \qquad x(t_0) = x_0 \qquad f(t, 0) = 0 \qquad (5)$$

$f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ loc. bounded, continuous a.e., loc. unif. continuous in t. If there exist

- j differentiable functions $V_i : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$, bounded in t, and
- continuous functions $Y_i : \mathbb{R}^n \to \mathbb{R}$ for $i \in \{1, 2, \ldots j\}$ such that
- V_1 is positive definite and radially unbounded,
- $\dot{V}_i(t, x) \leq Y_i(x)$, for all $i \in \{1, 2, \ldots, j\}$,
- $Y_i(x) = 0$ for $i \in \{1, 2, \ldots, k-1\}$ implies $Y_k(x) \leq 0$, for all $k \in \{1, 2, \ldots, j\}$,
- $Y_i(x) = 0$ for all $i \in \{1, 2, \ldots, j\}$ implies $x = 0$,

then the origin $x = 0$ of (5) is **uniformly** globally asymptotically stable.

Question: how to determine suitable functions V_i and Y_i (for $i > 1$)?

Mobile robot: revisited again

Assume $v_r(t), \omega_r(t)$ satisfying $0 < v^{\min} \leq v_r(t) \leq v^{\max}$, $|\dot{v}_r| \leq a^{\max}$ and $|\omega_r(t)| \leq \omega^{\max}$.

Consider the dynamics $\dot{x}_e = \omega y_e - c_1 x_e$, $\dot{y}_e = -\omega x_e + v_r \sin\theta_e$, $\dot{\theta}_e = -c_2\theta_e - c_3 v_r y_e \frac{\sin\theta_e}{\theta_e}$.

Differentiating $V_1 = \frac{1}{2}x_e^2 + \frac{1}{2}y_e^2 + \frac{1}{2c_3}\theta_e^2$ results in $\dot{V}_1 = -c_1 x_e^2 - \frac{c_2}{c_3}\theta_e^2 = Y_1(x_e, y_e, \theta_e)$.

Consider $V_2 = -\theta_e\dot{\theta}_e$. Then

$$\begin{aligned}\dot{V}_2 &= -\dot{\theta}_e^2 - \theta_e\ddot{\theta}_e = -[-c_3 v_r y_e + \eta(t)]^2 - \theta_e\ddot{\theta}_e = -(c_3 v_r y_e)^2 + 2c_3 v_r y_e \eta(t) - \eta(t)^2 - \theta_e\ddot{\theta}_e \\ &\leq -c_3^2(v_r^{\min})^2 y_e^2 + M_1\|\bar{\eta}(x_e, y_e, \theta_e)\| + \|\bar{\eta}(x_e, y_e, \theta_e)\|^2 + M_2\|\theta_e\| = Y_2(x_e, y_e, \theta_e).\end{aligned}$$

Note that $Y_1 = 0$ implies $Y_2 \leq 0$. Furthermore, $Y_1 = Y_2 = 0$ implies $x_e = y_e = \theta_e = 0$.

Therefore: **uniform** global asymptotic stability (applying Matrosov-like theorem).

NB: Instead of taking $V_2 = -\theta_e \cdot \dot{\theta}_e$ we can also taking the "simpler" $V_2 = -\theta_e \cdot f_0$.

Mobile robot revisited again: definition angular error

Differentiating $V_1 = \frac{1}{2}x_e^2 + \frac{1}{2}y_e^2 + \frac{1}{c_3}(1 - \cos\theta_e)$ along $\dot{x}_e = \omega y_e - c_1 x_e$, $\dot{y}_e = -\omega x_e + v_r \sin\theta_e$, $\dot{\theta}_e = -c_2\sin\theta_e - c_3 v_r y_e$, results in $\dot{V}_1 = -c_1 x_e^2 + \frac{c_2}{c_3}\sin^2\theta_e = Y_1(x_e, y_e, \sin\theta_e)$[1].

Differentiating $V_2 = c_3 v_r y_e \cos\theta_e \cdot \sin\theta_e$ along solutions results in

$$\begin{aligned}\dot{V}_2 &= c_3[\dot{v}_r y_e + v_r\dot{y}_e - v_r y_e \sin\theta_e\dot{\theta}_e]\sin\theta_e + c_3 v_r y_e \cos^2\theta_e[-c_2\sin\theta_e - c_3 v_r y_e] \\ &\leq -c_3^2(v_r^{\min})^2 y_e^2 + M\|\sin\theta_e\| = Y_2(x_e, y_e, \sin\theta_e).\end{aligned}$$

Therefore: **uniform** global asymptotic stability of $(x_e, y_e, \sin\theta_e)$ (applying Matrosov-like theorem).

[1]Formally: we lift the path of $\sin\theta_e$ to a path in S^1

Signal chasing: another example revisited

For $b_i > 0$, as well as $0 < u_{1,r}^{\min} \leq u_{1,r}(t) \leq u_{1,r}^{\max}$ and $|\dot{u}_{1,r}| \leq M$, differentiating $V_1 = b_1 w_1^2 + b_1 b_2 w_2^2 + \cdots + b_1 b_2 \cdots b_{n-1} w_{n-1}^2 + b_1 b_2 \cdots b_n w_n^2$ along solutions of

$$\dot{w}_1 = -b_1 w_1 - b_2 u_{1,r} w_2, \quad \dot{w}_2 = u_{1,r} w_1 - b_3 u_{1,r} w_3, \quad \cdots \quad \dot{w}_{n-1} = u_{1,r} w_{n-2} - b_n u_{1,r} w_n.$$

results in $\dot{V}_1 = -b_1^2 w_1^2 = Y_1(w)$.

Differentiating $V_2 = b_2 u_{1,r} w_2 \cdot w_1$ along solutions results in

$$\dot{V}_2 = b_2(\dot{u}_{1,r} w_2 + u_{1,r}\dot{w}_2)w_1 + b_2 u_{1,r} w_2[-b_1 w_1 - b_2 u_{1,r} w_2] \leq -b_2^2(u_{1,r}^{\min})^2 w_2^2 + \bar{M}|w_1| = Y_2(w).$$

Differentiating $V_i = b_i u_{1,r} w_i \cdot w_{i-1}$ $(i = 3, 4, \ldots, n)$ along solutions results in

$$\dot{V}_i \leq -b_i^2(u_{1,r}^{\min})^2 w_i^2 + \bar{M}_{i-2}|w_{i-2}| + \bar{M}_{i-1}|w_{i-1}| = Y_i(w).$$

Therefore: **uniform** global asymptotic stability of $w = 0$ (applying Matrosov-like theorem).

My standard approach for arriving at uniform results

More general case: $\dot{x}_1 = f_1(t, x_1, x_2, x_3)$, $\dot{x}_2 = f_2(t, x_1, x_2, x_3)$, $\dot{x}_3 = f_3(t, x_1, x_2, x_3)$

- Lyapunov function: $V_1(t, x_1, x_2, x_3)$ positive definite.
- Derivative along dynamics: $\dot{V}_1(t, x_1) = \cdots \leq Y_1(x_1)$ negative semi-definite.
- **Use** $V_2 = -x_1^T \dot{x}_1$. Then $\dot{V}_2 \leq -f_1(t, 0, x_2, x_3)^T f_1(t, 0, x_2, x_3) + F_2(\|x_1\|) \leq Y_2(x)$.
- $Y_1 = 0$ implies $Y_2 \leq 0$. Furthermore $Y_1 = Y_2 = 0$ implies $x_1 = x_2 = 0$.
- **Use** $V_3 = -x_2^T \dot{x}_2$. Then $\dot{V}_3 \leq -f_2(t, 0, 0, x_3)^T f_2(t, 0, 0, x_3) + F_3(\|x_1\|, \|x_2\|) \leq Y_3(x)$.
- $Y_1 = Y_2 = 0$ implies $Y_3 \leq 0$. Also, $Y_1 = Y_2 = Y_3 = 0$ implies $x_1 = x_2 = x_3 = 0$.
- Conclusion: **uniform** global asymptotic stability.

NB: Often simpler functions can be found for V_i, e.g., $V_2 = -f_1(t, 0, x_2, x_3)^T \dot{x}_1$, etc.

Uncovered subjects/Extra material

- Backstepping
- Cascaded systems
- References/Recommended reading material
- Suggestions for exercises

Integrator Backstepping

Consider the dynamics

$$\dot{x}_1 = -x_1 + x_1^2 x_2 \qquad \dot{x}_2 = u$$

Take x_2 as a **virtual input**.

Possible candidates for stabilizing the x_1 dynamics: $x_2 = 0$, or $x_2 = -c_1 x_1$.

Differentiating the Lyapunov function candidate $V_1 = \frac{1}{2}x_1^2$ along solutions results in $\dot{V}_1 = -x_1^2$ respectively $\dot{V}_1 = -x_1^2 - c_1 x_1^4$.

Three steps:

Step 1 Define new coordinate: difference between state and desired state

Step 2 Define (inverse) change of coordinates and write dynamics in new coordinates

Step 3 Extend Lyapunov function and make its time-derivative negative definite.

Integrator backstepping: case 1: $x_2 = 0$

Define $z_2 = x_2 - 0 = x_2$. Inverse change of coordinates: $x_2 = z_2$. Dynamics in new coordinates:

$$\dot{x}_1 = -x_1 + z_2 \cdot x_1^2 \qquad \dot{z}_2 = u$$

Differentiating $V_2 = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2$ along solutions yields

$$\dot{V}_2 = -x_1^2 + z_2 x_1^3 + z_2 u = -x_1^2 + z_2 \cdot (x_1^3 + u)$$

which can be rendered negative definite by taking

$$u = -x_1^3 - kz_2 = -x_1^3 - kx_2 \qquad k > 0$$

Integrator backstepping: case 2: $x_2 = -c_1 x_1$

Define $z_2 = x_2 - (-c_1 x_1) = x_2 + c_1 x_1$. Inverse change of coordinates: $x_2 = z_2 - c_1 x_1$. Dynamics in new coordinates:

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_1^2(z_2 - c_1 x_1) = -x_1 - c_1 x_1^3 \;+\; z_2 \cdot x_1^2\\ \dot{z}_2 &= u + c_1 \dot{x}_1 = u - c_1 x_1 - c_1^2 x_1^3 + c_1 x_1^2 z_2\end{aligned}$$

Differentiating $V_2 = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2$ along solutions yields

$$\dot{V}_2 = -x_1^2 - c_1 x_1^4 \;+\; z_2(x_1^3 + u - c_1 x_1 - c_1^2 x_1^3 + c_1 x_1^2 z_2)$$

which can be rendered negative definite by taking

$$u = -x_1^3 + c_1 x_1 + c_1^2 x_1^3 - c_1 x_1^2 z_2 - c_2 z_2 = -x_1^3 + c_1(1 - c_2 - x_1 x_2)x_1 - c_2 x_2$$

General backstepping

Consider for $w \in \mathbb{R}^m$, and scalar x_i ($i = 1, 2, \dots, n$) the dynamics

$$\begin{aligned}\dot{w} &= f_0(w) + g_0(w)x_1\\ \dot{x}_1 &= f_1(w, x_1) + g_1(w, x_1)x_2\\ \dot{x}_2 &= f_2(w, x_1, x_2) + g_2(w, x_1, x_2)x_3\\ &\vdots\\ \dot{x}_{n-1} &= f_{n-1}(w, x_1, x_2, \dots, x_{n-1}) + g_{n-1}(w, x_1, x_2, \dots x_{n-1})x_n\\ \dot{x}_n &= f_n(w, x_1, x_2, \dots, x_{n-1}, x_n) + g_n(w, x_1, x_2, \dots, x_{n-1}, x_n)u\end{aligned}$$

where $f_i(0, 0, \dots, 0) = 0$ and $g_i(w, x_1, x_2, \dots, x_i) \neq 0$ on the domain of interest.

Furthermore, assume that the dynamics $\dot{w} = f_0(w) + g_0(w)u$ can be stabilized to $w = 0$ by a known feedback $u = u_0(w)$, for which a Lyapunov function $V_0(w)$ is also known.

General backstepping

We therefore have that $\frac{\partial V_0}{\partial w}[f_0(w) + g_0(w)u_0(w)] < 0$. Define the change of coordinates $z_1 = x_1 - u_0(w)$ with inverse change of coordinates $x_1 = z_1 + u_0(w)$. Then we get

$$\dot{w} = f_0(w) + g_0(w)u_0(w) + z_1 \cdot g_0(w) \qquad \dot{z}_1 = f_1(w, x_1) + g_1(w, x_1)x_2 + \frac{\partial u_0}{\partial w}\dot{w}$$

Differentiating $V_1(w, z_1) = V_0(w) + \frac{1}{2}z_1^2$ along solutions, results in

$$\begin{aligned}\dot{V}_1 &= \tfrac{\partial V_0}{\partial w}[f_0(w) + g_0(w)u_0(w)] + \tfrac{\partial V_0}{\partial w} z_1 g_0(w) + z_1\left[f_1(w, z_1 + u_0(w)) + g_1(w, z_1 + u_0(w))x_2 + \tfrac{\partial u_0}{\partial w}\dot{w}\right]\\ &= \tfrac{\partial V_0}{\partial w}[f_0(w) + g_0(w)u_0(w)] + z_1\left[\tfrac{\partial V_0}{\partial w} g_0(w) + f_1(w, z_1 + u_0(w)) + g_1(w, z_1 + u_0(w))x_2 + \tfrac{\partial u_0}{\partial w}\dot{w}\right]\end{aligned}$$

which is negative definite if we take

$$x_2 = -\left[c_1 z_1 + \tfrac{\partial V_0}{\partial w} g_0(w) + f_1(w, z_1 + u_0(w)) + \tfrac{\partial u_0}{\partial w}\dot{w}\right] \Big/ g_1(w, z_1 + u_0(w)).$$

General backstepping

If we now define $\bar{w} = \begin{bmatrix} w \\ x_1 \end{bmatrix}$, $\bar{f}_0(\bar{w}) = \begin{bmatrix} f_0(w) + g_0(w)x_1 \\ f_1(w, x_1) \end{bmatrix}$, $\bar{g}_0(\bar{w}) = \begin{bmatrix} 0 \\ g_1(w, x_1) \end{bmatrix}$, as well as $\bar{x}_i = x_{i+1}$, $\bar{f}_i = f_{i+1}$, $\bar{g}_i = g_{i+1}$ for $i = 1, 2, \dots, n-1$, we obtain

$$\begin{aligned}\dot{\bar{w}} &= \bar{f}_0(\bar{w}) + \bar{g}_0(\bar{w})\bar{x}_1\\ \dot{\bar{x}}_1 &= \bar{f}_1(\bar{w}, \bar{x}_1) + \bar{g}_1(\bar{w}, \bar{x}_1)\bar{x}_2\\ &\vdots\\ \dot{\bar{x}}_{n-2} &= \bar{f}_{n-2}(\bar{w}, \bar{x}_1, \bar{x}_2, \dots, \bar{x}_{n-2}) + \bar{g}_{n-2}(\bar{w}, \bar{x}_1, \bar{x}_2, \dots \bar{x}_{n-2})\bar{x}_{n-1}\\ \dot{\bar{x}}_{n-1} &= \bar{f}_{n-1}(\bar{w}, \bar{x}_1, \bar{x}_2, \dots, \bar{x}_{n-1}) + \bar{g}_{n-1}(\bar{w}, \bar{x}_1, \bar{x}_2, \dots \bar{x}_{n-1})u\end{aligned}$$

So continuing this procedure $n-1$ times more, we obtain a stabilizing controller for the system (as well as a Lyapunov function proving this).

Important remark about backstepping

Though backstepping provides a means to arrive at stabilizing controller, including a Lyapunov proof, the resulting controllers usually are quite **difficult expressions** (in particular if expressed in the original coordinates).

Cascaded systems

Recall the example studied earlier

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1^2 x_2 && \text{outer loop/} \\ \dot{x}_2 &= -x_2 && \text{inner loop} \end{aligned}$$

Even though the subsystems

$$\dot{x}_1 = -x_1 \qquad \dot{x}_2 = -x_2$$

were exponentially stable, the cascaded system can have a finite escape time.

Assume you have been able to show asymptotic stability of the unperturbed x_1 system using a Lyapunov function for which $\dot{V}$ is only negative semi-definite.

Then a useful result to analyze stability of the cascade is given on the next slide

Panteley, Loría (corollary of more general result)

Consider a system $\dot{z} = f(t,z)$ that can be written as

$$\begin{aligned} \dot{z}_1 &= f_1(t,z_1) + g(t,z_1,z_2)z_2 \\ \dot{z}_2 &= f_2(t,z_2) \end{aligned}$$

where the systems $\dot{z}_1 = f_1(t,z_1)$ and $\dot{z}_2 = f_2(t,z_2)$ are UGAS. Then we have UGAS of the cascaded system if the following conditions are satisfied:

1. We have a positive definite V with negative *semi*-definite $\dot{V}$ along solutions of $\dot{z}_1 = f_1(t,z_1)$, satisfying $c_1\|z_1\|^2 \leq V$ and $\left\|\frac{\partial V}{\partial z}\right\| \leq c_4\|z_1\|$,
2. $\|g(t,z_1,z_2)\| \leq k_1(\|z_2\|) + k_2(\|z_2\|)\|z_1\|$,
3. $\int_0^\infty \|z_2(t)\|\,\mathrm{d}\,t \leq \phi(\|z_2(t_0)\|)$ (e.g. when $\dot{z}_2 = f_2(t,z_2)$ is ULES).

Alternative way out

Panteley and Loría proved that showing boundness of z_1 suffices to conclude UGAS.

The conditions 1–3 on previous slide guarantee boundedness of z_1.

What if condition 1 and/or condition 2 are not satisfied?

Option 1: See if one of the other conditions in their paper works for you
Option 2: Show boundedness of z_1 by evaluating V for $\dot{z}_1 = f(t,z_1)$ along the cascade. If you can find a function ϕ such that $\|\frac{\partial V}{\partial z_1} g(t,z_1,z_2)\| \leq \phi(V)\|z_2(t)\|$ (e.g., $\phi(V) = \sqrt{V}$ or $\phi(V) = V$), then:

$$\dot{V} = \underbrace{\tfrac{\partial V}{\partial z_1} f_1(t,z_1)}_{\leq 0} + \tfrac{\partial V}{\partial z_1} g(t,z_1,z_2) \leq \phi(V)\|z_2(t)\| \qquad \text{so} \qquad \int_0^t \dot{V}/\phi(V) \leq \int_0^t \|z_2(\tau)\|\,\mathrm{d}\,\tau$$

If the primitive of $1/\phi$ is bounded on bounded intervals, you have boundedness of V and therefore of z_1.

Suggestions for exercises

- Consider a dynamic extension of a mobile robot:

$$\dot{x} = v\cos\theta \qquad \dot{y} = v\sin\theta \qquad \dot{\theta} = \omega \qquad \dot{v} = u_1 \qquad \dot{\omega} = u_2$$

  and consider the problem of tracking a (time-varying) feasible reference trajectory

$$\dot{x}_r = v_r\cos\theta_r \qquad \dot{y}_r = v_r\sin\theta_r \qquad \dot{\theta}_r = \omega_r \qquad \dot{v}_r = u_{1,r} \qquad \dot{\omega}_r = u_{2,r}$$

  Use one of the controllers for the mobile robot from this presentation as a starting point for backstepping to arrive at a tracking controller. Show uniform global asymptotic stability by means of the Matrosov-like theorem and make explicit what assumptions you need to make on signals of the reference trajectory.
- Search for "Barbalat" on the USB-stick with papers of a recent (pre-Covid) CDC or IFAC World Congress. Most likely the authors only show (global) asymptotic stability. Update the proof of the authors so that you can conclude *uniform* (global) asymptotic stability.

References/Recommended reading material

- Greiff, M. (2021). Nonlinear control of unmanned aerial vehicles. PhD thesis. ISBN: 978-91-8039-047-7. https://portal.research.lu.se/files/109517053/MG_thesis_final.pdf
- Hahn, W. (1967). Stability of motion. Vol. 138. Springer Verlag, Berlin, Germany. ISBN: 978-3-642-50085-5.
- Khalil, H. (1996). Nonlinear systems. 2nd ed. Prentice hall Upper Saddle River, New Jersey, USA. ISBN: 978-0-132-28024-2.
- Khalil, H. (2002). Nonlinear systems. 3rd ed. Prentice hall Upper Saddle River, New Jersey, USA. ISBN: 978-0-130-67389-3.
- Lefeber, A.A.J. (2000). Tracking control of nonlinear mechanical systems. PhD thesis. ISBN: 90-365-1426-6. url: https://dc.wtb.tue.nl/lefeber/do_download_pdf.php?id=48

References/Recommended reading material

- Loria, A., E. Panteley, D. Popovic, and A. R. Teel (2005). "A nested Matrosov theorem and persistency of excitation for uniform convergence in stable nonautonomous systems". IEEE Transactions on Automatic Control 50:2, pp. 183–198. doi: 10.1109/TAC.2004.841939.
- Micaelli, A. (1993). Trajectory tracking for unicycle-type and two-steering wheels mobile robots. Tech. rep. RR-2097, INRIA. inria-00074575. url: hal.inria.fr/inria-00074575.
- Panteley, E. and A. Loria (1998). "On global uniform asymptotic stability of nonlinear time-varying systems in cascade". Systems & Control Letters 33:2, pp. 131–138. doi: 10.1016/S0167-6911(97)00119-9.
- Panteley, E. and A. Loria (2001). "Growth rate conditions for uniform asymptotic stability of cascaded". Automatica 37:3, pp. 453–460. doi: 10.1016/S0005-1098(00)00169-2.
- Parks, P. C. (1962). "A new proof of the Routh-Hurwitz stability criterion using the second method of Liapunov". In: Mathematical Proceedings of the Cambridge Philosophical Society. Vol. 58. 4. Cambridge University Press, pp. 694–702. doi: 10.1017/S030500410004072X.