Rigid body attitude observers

for multirotor aircraft

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Abstract

The attitude dynamics of a rigid body can be used as a model for systems such as multirotor aircraft, rockets and artificial satellites. Those systems are often actively controlled. However, many control policies require to know the states of the system. Estimates of those states can be obtained using an appropriately designed observer.

This thesis considers the problem of estimating the states related to the attitude dynamics of a rigid body with an observer. This dynamics is formulated in three different ways, whose derivations are based on the available sensors. Two different approaches of constructing observers is discussed for each formulation of the dynamics. The first approach transforms the model of the dynamics into a linear time varying model, for which known observers from the literature can be used. The second approach starts with a Lyapunov function after which the correction terms are chosen such the time derivative of this Lyapunov function becomes negative semi-definite. Furthermore, it is also discussed how each observer can be implemented in numerical simulations and how one could go about assessing their performance in an objective way.

Contents

Nomenclature

Abbreviations

CQLF	Common quadratic Lyapunov function
LTV	Linear time varying
UCO	Uniformly completely observable

Operators

$\det(A)$	Function that returns the determinant of matrix A.
$\dot{x}$	Time derivative of the variable x.
$\|A\|$	Norm of a matrix defined as the largest value for $\|A\,x\|$ given that $\|x\| = 1$.
$\mathrm{diag}(x)$	Function that returns a square matrix, whose diagonal elements are equal to the elements of x.
$A \otimes B$	Returns the Kronecker product between matrices A and B.
$A \succ B$	Inequality between square matrices in $\mathbb{R}^{n\times n}$ stating that $x^\top(A - B)\,x > 0,\ \forall\ \|x\| \neq 0$.
$A \succeq B$	Inequality between square matrices in $\mathbb{R}^{n\times n}$ stating that $x^\top(A - B)\,x \geq 0,\ \forall\, x \in \mathbb{R}^n$.
$S(x)$	Function that returns the skew symmetric matrix of size 3×3 corresponding to the thee-dimensional vector x.
$x^\top$	Returns the transpose of the vector or matrix x.

Number Sets

$\mathbb{Z}$	Integers
$\mathbb{R}$	Real numbers
$\mathbb{C}$	Complex Numbers
$\mathbb{H}$	Quaternions

$\mathbb{R}^n$	The n-dimensional Euclidean space
$\mathbb{R}^{n\times m}$	Matrix of n by m populated with real numbers
$SO(3)$	Special orthogonal group for three-dimensional space

Constants and variables

$0_{n\times m}$	Zero matrix of n by m
I_n	Identity matrix of n by n
v^a	Coordinates of the vector v in Euclidean three-dimensional space expressed in Cartesian frame a.

Chapter 1

Introduction

Multirotor aircraft are a flying machines that uses multiple rotors to generate lift. In 1907 one of the first flying machines that could be classified as such was already constructed, called the Breguet-Richet Gyroplane No.I and only hovered 0.6 meters above the ground [37]. One of the first multirotor aircraft that used variable thrust of the main rotors to control its attitude is the Model A Quadrotor build in 1956 [32]. Controlling the attitude with variable thrust allows for less moving parts in multirotor aircraft compared to a helicopter, which uses a swash plate to alter the pitch and thus lift of its main rotor blades while they rotate. Multirotor aircraft gained a lot of popularity in roughly the last decade as unmanned aerial vehicles (UAV). This could be explained by the relatively small amount of required moving parts combined with the rise in popularity of smart phones. The small amount of moving parts reduces manufacturing cost and the development of smart phones stimulated battery technology and the production of inertial measurement unit (IMU) chips. The rise in computation capabilities also allowed those UAV's to become more and more autonomous. Initially mainly linear control is used to save on computation time. However, to make those autonomous UAV's more robust against disturbances and initial conditions one could use nonlinear control, since linear control might become unstable if those disturbances or initial conditions cause large deviations away from the linearization point. Most nonlinear control theory uses state feedback [11], [21], [4], [12], [27], but the available sensors usually do not allow to recover full state information from measurements at a single instance and are also perturbed by disturbances.

In order to recover the full state information of multirotor aircraft from sensor measurements, one would need to use an algorithm that can combine the measurements over time. The dynamics of a multirotor aircraft can be modeled as a rigid body, which can be split into the dynamics related to its position and its attitude. It can be noted that, when ignoring any kind of nonlinear friction, the dynamics related to the attitude contains the majority of the nonlinear terms of the combined dynamics of a rigid body model. To limit the scope of this thesis, it is therefore chosen to only focus on finding such algorithms for the attitude part of the state. It can also be noted that such rigid body attitude model can not only be used for multirotor aircraft, but also for things like rockets and artificial satellites. Furthermore, in this thesis it is assumed that the rigid body model does not contain any stochastic processes, in which case such algorithms can also be called observers. For linear systems one can use certainty equivalence [1] to decouple the observer and state feedback dynamics. However, for nonlinear systems this can not be used in general. Therefore, both the observer and the state feedback, whose combination is also called output feedback, have to be taken into consideration simultaneously in order to show stability. This implies that the proposed observers in this thesis do not guarantee that when combined with state feedback the resulting output feedback is stable.

Before attempting to formulate observers for the considered dynamics, the literature is consulted about already existing observers for (parts) of the attitude dynamics of a rigid body. From this it can be shown that the following observers have been proposed for the angular velocity [22], [28] and for the attitude kinematics [5], [9], [24], [23], [25]. However, not many observers have been constructed that consider the combined dynamics [8], [14].

In this thesis the following structure is used. In Chapter 2 three different models are discussed which can be used the describe the attitude dynamics of a rigid body. This is followed by Chapter 3 and 4 in which two different observer structures are derived. The structure from Chapter 3 assumes that each model can be formulated as linear time varying for which a known observer from the literature can be used. The structure from Chapter 4 uses a Lyapunov function at its basis to shape the observer dynamics. Both these observer structures are applied to each of the three proposed models from Chapter 2 and analyzed in more detail in Chapter 5, 6 and 7 respectively. In Chapter 8 it is discussed how the proposed observers can be implemented numerically and how those numerical results can be used to help choose an observer and its parameters.

Chapter 2

Rigid body models

The goal of this thesis is to construct observers which can estimate the states of a rigid body related to its attitude dynamics using the applied torque, attitude measurements and angular velocity measurements, if available. In order to do this Section 2.1 first introduces the definitions used for the state of a rigid body. This is followed by Section 2.2 in which the considered possible model outputs are discussed. All this is combined in Section 2.3 into three different models for describing the attitude dynamics of a rigid body.

2.1 Rigid body state description

In classical mechanics the motion of bodies takes place in three-dimensional Euclidean space. To describe geometrical and mechanical relations a reference frame is required. Commonly a Cartesian frame $\{O, \vec{e}\}$ is used, which is characterized by the location of its origin O and by the orientation of a set $\vec{e} \coloneqq \{e_1, e_2, e_3\}$ of three orthogonal unit vectors as shown in Figure 2.1. Furthermore, in this thesis only right-handed frames are used, so $e_1 \times e_2 = e_3$, $e_2 \times e_3 = e_1$ and $e_3 \times e_1 = e_2$ [29]. In

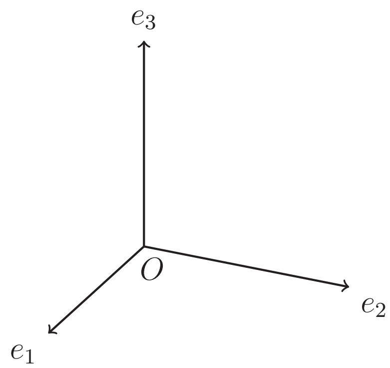

Figure 2.1: *Right handed Cartesian frame $\{O, \vec{e}\}$.*

classical mechanics it is convenient to work with multiple Cartesian frames. Each of these frames are denoted using a different symbol, for example $\{O^a, \vec{a}\}$ and

$\{O^b, \vec{b}\}$ represent frame a and b respectively. A point and a vector in Euclidean three-dimensional space can be defined relative to a given frame using three coordinates. For a given frame, denoted with a, such coordinates for a vector are defined as $v^a := \begin{bmatrix} v_1^a & v_2^a & v_3^a \end{bmatrix}^\top \in \mathbb{R}^3$, which represents the vector $v_1^a\, a_1 + v_2^a\, a_2 + v_1^a\, a_2$. Similar coordinates could also be defined for a point, but this thesis only focuses on the mechanics related to the attitude of a rigid body, which only requires vectors and no positions.

One sufficient set of states that can be used to describe the attitude dynamics of a rigid body is an attitude representation and a measure of the rate of rotation. In order to represent those states with coordinates an inertial and body fixed Cartesian reference frame are used, denoted with e and b respectively.

For the majority of this thesis rotation matrices are used as the attitude representation. For local stability analysis and numerical implementation also other representations are used, but those are introduced when they are used. Any rotation matrix can formally be defined as a member of the third special orthogonal group, or $SO(3)$ for short, which can be defined as [16]

$$SO(3) := \left\{ X \in \mathbb{R}^{3\times3} \,|\, X^\top X = I_3, \det X = 1 \right\}. \tag{2.1}$$

Rotation matrices function as linear transformations for vector coordinates between two frames. In the remainder of this thesis the notation R^{eb} is used to represent a rotation matrix which can transform coordinates expressed using the body fixed reference frame b into coordinates expressed using the inertial reference frame e according to $v^e = R^{eb} v^b$.

Common ways of describing the rate of rotation of a rigid body is with the angular velocity or angular momentum. Such an angular velocity is a vector which defines the axis and magnitude of the instantaneous rate of rotation in radians per second between two reference frames according to the right hand rule. In the remainder of this thesis the notation ${}^{be}\omega$ is used to represent the rate of rotation of the body frame relative to the inertial frame. The coordinates of such an angular velocity in the body frame would thus be denoted with ${}^{be}\omega^b$. The coordinates of the angular momentum of a rigid body in the inertial frame, denoted with L^e, can be related to the angular velocity coordinates ${}^{be}\omega^b$ using [15, p. 134]

$$L^e = R^{eb} J\, {}^{be}\omega^b, \tag{2.2}$$

with $J = J^\top \succ 0 \in \mathbb{R}^{3\times3}$ the body fixed mass moment of inertia matrix. The assumption that $J \succ 0$ is only violated when the mass of the rigid body is concentrated onto a line. However, that would either require zero mass or infinite density, both of which are in most cases not physically meaningful. Therefore, when the angular momentum L^e, rotation matrix R^{eb} and moment of inertia matrix J are known it is also possible to solve (2.2) for the angular velocity ${}^{be}\omega^b$ using

$${}^{be}\omega^b = J^{-1} {R^{eb}}^\top L^e. \tag{2.3}$$

This transformation from angular velocity to angular momentum is for example also used for an angular velocity observer [28].

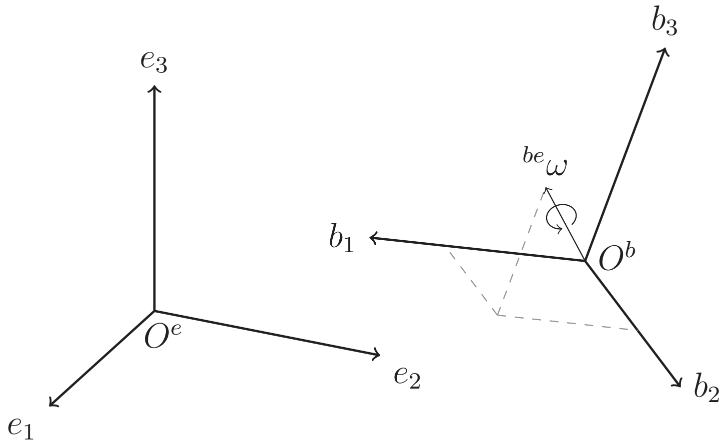

Figure 2.2: *Illustration of inertial and body fixed frames e and b, and angular velocity vector ${}^{be}\omega$ decomposed into body fixed coordinates.*

An example of an inertial frame e, body fixed frame b and angular velocity is given in Figure 2.2.

The time derivative of the attitude of a rigid body, expressed as a rotation matrix, is also known as the Poisson equation and can be described with [29, p. 81]

$$\dot{R}^{eb} = R^{eb}\, S({}^{be}\omega^b), \tag{2.4}$$

where $S(x)$ is a function which transforms any $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top \in \mathbb{R}^3$ into a corresponding skew symmetric matrix

$$S(x) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}. \tag{2.5}$$

The time derivative of the angular velocity coordinates ${}^{be}\omega^b$ is given by [15, p. 143]

$$J\, {}^{be}\dot{\omega}^b = S\big(J\, {}^{be}\omega^b\big)\, {}^{be}\omega^b + \tau^b, \tag{2.6}$$

with τ^b the sum of all torques acting on the body expressed using body fixed coordinates. The time derivative of the angular momentum coordinates L^e is given by [15, p. 137]

$$\dot{L}^e = \tau^e, \tag{2.7}$$

with τ^e the sum of all torques acting on the body expressed as coordinates in the inertial frame e, which is equivalent to $\tau^e = R^{eb}\tau^b$ with τ^b the same as in (2.6).

2.2 Measurements

For all models it is assumed that the entire attitude can be measured as a rotation matrix R^{eb}. This can for example indirectly be obtained from multiple direction measurements, where the coordinates of those direction are measured in one frame and are known in another frame. Examples of such directions are the directions of gravity or Earth's magnetic field. The attitude can be reconstructed from these measured direction by solving Wahba's problem, which can be formulated as follows [36]

$$R^{eb} = \arg \min_{R \in SO(3)} \sum_{n=1}^{N} \lambda_n \left\| v_n^e - R\, v_n^b \right\|^2, \tag{2.8}$$

were $N \geq 2$ and $\lambda_n > 0$ are weights depending on the relative confidence of each vector pair. In order to always be able to have one unique solution to Wahba's problem it is required that there are at least two linearly independent directions from which its coordinates are known and measured in both frames. So the directions of gravity or Earth's magnetic field would not be sufficient at the magnetic poles of the Earth.

It has been shown that Wahba's problem can be solved using [26]

$$B = \sum_{n=1}^{N} \lambda_n\, v_n^e\, {v_n^b}^\top = U\, S\, V^\top, \tag{2.9}$$

where $U\, S\, V^\top$ is the singular value decomposition of B. The solution for R^{eb} can be found with

$$R^{eb} = U \operatorname{diag}\left(\begin{bmatrix} 1 & 1 & \det(U) \det(V) \end{bmatrix}\right) V^\top. \tag{2.10}$$

If the appropriate sensor would be available it is assumed that the angular velocity can be measured in the body fixed frame. An example of such sensor would be a gyroscope attached to the body. However, the measurements from those gyroscopes are often subjected to a bias [23]. That bias is denoted with $b^b \in \mathbb{R}^3$ and is assumed to be unknown but constant. Therefore, the biased angular velocity measurement can be defined as

$$z^b := {}^{be}\omega^b + b^b. \tag{2.11}$$

By substituting (2.3) in (2.11), it is also possible to express the biased angular velocity measurement using the angular momentum

$$z^b := J^{-1} {R^{eb}}^\top L^e + b^b. \tag{2.12}$$

Besides the bias of the angular velocity measurements it is assumed that all measurements are not subjected to other sources of noise or disturbance. This is not a realistic assumption, but helps to limit the scope of this thesis.

2.3 System models

The combined system of differential equations that needs to be considered by an observer for the attitude dynamics, as described in Section 2.1, differs depending on which measurements are used. Namely, if the biased angular velocity measurement is used the constant but unknown bias b^b should also be taken into consideration. This can be included into the model of the system by adding b^b as part of the state and setting the time derivative of b^b to zero. When the biased angular velocity measurement is used it is also possible to only consider the attitude kinematics [23]. Together, this allows for three different ways of modeling the system, each of which can be used to construct observers. The first model of the system is discussed in Subsection 2.3.1 and considers the minimal dynamics and uses the applied torque and attitude measurements. The second model of the system is discussed in Subsection 2.3.2 and expands on the first model by additionally considering the bias and using the biased angular velocity measurement. The third model of the system is discussed in Section 2.3.3 and considers the attitude kinematics and uses the attitude measurements and biased angular velocity measurements.

2.3.1 Model with minimal dynamics

For the model with minimal dynamics it is assumed that only the rotation matrix R^{eb} is measured and the input τ^b is known. In order to fully define the state of the attitude dynamics one could use R^{eb} and either the angular velocity ${}^{be}\omega^b$ or angular momentum L^e. However, when comparing their time derivatives from (2.6) and (2.7) respectively it can be concluded that the resulting dynamics is simpler when using L^e. Therefore, the combined dynamics of this minimal model can be obtained using (2.4), (2.3) and (2.7) yielding

$$\dot{R}^{eb} = R^{eb}\, S(J^{-1} {R^{eb}}^\top L^e), \tag{2.13a}$$

$$\dot{L}^e = R^{eb} u, \tag{2.13b}$$

$$u = \tau^b, \tag{2.13c}$$

$$y = R^{eb}, \tag{2.13d}$$

with u the known input of the system and y the output of the system. Furthermore, it is also assumed that J is known. Therefore, if an observer is able to estimate R^{eb} and L^e then ${}^{be}\omega^b$ could also be estimated using (2.3).

2.3.2 Model with biased dynamics

For the model with biased dynamics it is assumed that the rotation matrix R^{eb} and the biased angular velocity z^b are measured and the input τ^b is known. As stated before, this model expands on the model from Section 2.3.1 by additionally considering b^b and using the measurement z^b. The combined dynamics of this model can be obtained using (2.13), (2.11), (2.3) and the assumption that b^b is constant, yielding

$$\dot{R}^{eb} = R^{eb}\, S(J^{-1} {R^{eb}}^{\top} L^e), \tag{2.14a}$$
$$\dot{L}^e = R^{eb} u, \tag{2.14b}$$
$$\dot{b}^b = 0, \tag{2.14c}$$
$$u = \tau^b, \tag{2.14d}$$
$$y_1 = R^{eb}, \tag{2.14e}$$
$$y_2 = J^{-1} {R^{eb}}^{\top} L^e + b^b, \tag{2.14f}$$

with u the known input of the system and, y_1 and y_2 the outputs of the system.

2.3.3 Model with kinematics

For the model with kinematics it is assumed that the rotation matrix R^{eb} and the biased angular velocity z^b are measured. The input τ^b and matrix J do not need to be known. In order to transform the dynamics into kinematics z^b can instead be seen as the input to the system and the dynamics associated with the rate of rotation can be omitted. The combined model can therefore be described by solving (2.11) for ${}^{be}\omega^b$ and using (2.4) and (2.14c) yielding

$$\dot{R}^{eb} = R^{eb}\, S(u - b^b), \tag{2.15a}$$
$$\dot{b}^b = 0, \tag{2.15b}$$
$$u = z^b, \tag{2.15c}$$
$$y = R^{eb}, \tag{2.15d}$$

with u the effective known input and y the output of the system. An observer could obtain an estimate for the angular velocity by subtracting the estimate for the bias from the biased angular velocity measurement z^b.

2.4 Outline of proposed observers

In each of the next two chapters a class of observers is proposed which are applied to each of the models from Section 2.3 in Chapter 5, 6 and 7. In Chapter 3 it is assumed that each model can be formulated as linear time varying for which a known observer structure is used. In Chapter 4 a nonlinear observer structure is proposed, with a Lyapunov function at its basis.

Chapter 3

Linear time varying observer outline

This chapter discusses the derivation of the first structure of an observer outline used in this thesis. This structure is applied to each of the three models from Section 2.3 in Section 5.1, 6.1 and 7.1. This observer structure is based on describing the considered model as linear time varying (LTV). If such LTV model can be shown to be uniformly completely observable (UCO) one can use a known observer structure from the literature. The dynamical description of this observer structure is discussed in Section 3.1. This is followed by a discussion of how each model from Section 2.3 could be formulated as LTV. Lastly, in Section 3.3 it is discussed how the UCO condition could be checked for each model.

3.1 Observer dynamics

The system matrices of each LTV model are assumed to be an implicit function of time. Namely, it is assumed that each system matrix can be written as a function of the rotation matrix R^{eb}, which is measured as a function of time, yielding the following LTV-like model

$$\begin{cases} \dot{x}(t) = A(R^{eb})\,x(t) + B(R^{eb})\,u(t), \\ y(t) = C(R^{eb})\,x(t), \end{cases} \tag{3.1}$$

with $x(t) \in \mathbb{R}^n$ the state vector, $A(R^{eb}) : SO(3) \to \mathbb{R}^{n\times n}$, $u(t) \in \mathbb{R}^m$ the input vector, $B(R^{eb}) : SO(3) \to \mathbb{R}^{n\times m}$, $y(t) \in \mathbb{R}^p$ the output vector and $C(R^{eb}) : SO(3) \to \mathbb{R}^{p\times n}$.

If the considered LTV model has bounded system matrices and is shown to be UCO one can use an observer of the form [6, pp. 16-17]

$$\dot{\hat{x}}(t) = A(R^{eb})\,\hat{x}(t) + B(R^{eb})\,u(t) - K(t)\left[C(R^{eb})\,\hat{x}(t) - y(t)\right], \tag{3.2}$$

with $\hat{x}(t)$ the estimate of $x(t)$ from (3.1) and $K(t)$ is given by

$$\begin{cases} \dot{M}(t) = A(R^{eb})M(t)+M(t)A^\top(R^{eb})-M(t)C^\top(R^{eb})W^{-1}C(R^{eb})M(t)+V+\delta M(t), \\ M(t_0) = M_0 = M_0^\top \succ 0, \quad W = W^\top \succ 0, \\ K(t) = M(t)\, C^\top(R^{eb})\, W^{-1}, \end{cases} \tag{3.3}$$

with $K(t) \in \mathbb{R}^{n\times p}$, $M(t) = M^\top(t) \in \mathbb{R}^{n\times n}$ $\forall\, t \geq t_0$, $W \in \mathbb{R}^{p\times p}$, $V \in \mathbb{R}^{n\times n}$ and $\delta \in \mathbb{R}$. Furthermore, for general LTV models it is required that either $\delta > 2\,\|A(R^{eb})\|$ for all $R^{eb} \in SO(3)$, or $V = V^\top \succ 0$.

The observer, defined by (3.2) and (3.3), can recover the full state information, so that the estimated state $\hat{x}(t)$ has global exponential convergence to the true state $x(t)$. The rate of this convergence can be tuned by δ or the minimal eigenvalue of V [6, pp. 16-17]. It can be noted that scaling W, V and M_0 from (3.3) by the same positive scalar yields the same scaled solution for $M(t)$. However, in (3.2) only the observer gain $K(t)$ is used, in which this scalar would cancels with itself. Therefore, the performance of the observer is not affected by such scaling and could be used to normalize W or V with respect to some measure, such as the largest eigenvalue of W.

For $\delta = 0$ the observer becomes equivalent to the continuous time Kalman filter without the stochastic terms [6]. The continuous time Kalman filter is also known as the Kalman-Bucy filter, which can be shown to be exponentially stable if $\big(A(R^{eb}), V\big)$ is uniformly completely controllable (dual of UCO) [10]. Therefore, when $\delta = 0$ does not require that $V = V^\top \succ 0$ as stated, only that $\big(A(R^{eb}), V\big)$ is uniformly completely controllable. However, it is guaranteed that this condition is satisfied when $V = V^\top \succ 0$. If stochastic terms where acting on the inputs and outputs of the system then the covariance of the estimation error is minimized when the values for W and V are chosen to be equal to the covariance of output and input noises respectively. However, adding noise to the measured rotation matrix also affects the matrices used for the LTV model in the observer, which could affect what the optimal W and V would be.

The bound on δ when $V = 0$ is stated as $\delta > 2\,\|A(R^{eb})\|$. That bound is based on the inequality $\|\Phi(t, t-T)\| \leq e^{\sigma T}$ with $\sigma = \sup \|A(R^{eb})\|$ and $\Phi(t, t-T)$ the state transition matrix from time $t - T$ to time t [7]. Such state transition matrix is defined as the map from the state at time $t - T$ to the state at time t while the system is subjected to an input of zero. However, the homogeneous attitude dynamics is Lyapunov stable, since in that case the angular momentum should remain constant and the rotation matrix is a member of $SO(3)$ thus remains bounded. Therefore, the norm of the state transition matrix can always be upper bounded by a constant. This relaxes the lower bound on the observer parameter to $\delta > 0$.

3.2 Attitude representation in state vector

The LTV model considered in (3.1) uses one vector $x(t)$ to represent the entire state. Therefore, for each model from Section 2.3 all variables used to represent the state should be combined into one such vector in order to be able to transform each model into the form of (3.1). All state variables of each model from Section 2.3, besides the one used for the attitude representation, are vectors which can easily be transformed into one larger vector by stacking them on top of each other. For the attitude representation all three model formulations use a rotation matrix, which cannot be incorporated into that larger vector by stacking, since it is a three by three matrix and not a vector. This subsection discusses how those rotation matrices can be turned into a vector, such that the entire state can be captured by one state vector. Additionally, it is also discussed how the dynamics of that vectorized attitude representation can be described and how the state vector can be reduced in size, which can help reduce the computational cost of using the proposed observer structure from (3.2) and (3.3).

By using the vectorization operator one can turn any rotation matrix $R^{eb} \in SO(3) \subset \mathbb{R}^{3\times 3}$ into a vector

$$\rho := \mathrm{vec}(R^{eb}) \in \mathbb{R}^9, \tag{3.4}$$

where $\mathrm{vec}(M)$ denotes the vectorization operator with as input a matrix M and outputs the columns of matrix M stacked on top of each other into a single vector. This vectorization operation is constant and linear. Therefore, the time derivative of ρ can be obtained by also vectorizing the time derivative of R^{eb}. The time derivative of R^{eb} for each considered model from Section 2.3 are all of the form

$$\dot{R}^{eb} = R^{eb} S(\omega), \tag{3.5}$$

with $\omega = J^{-1} {R^{eb}}^\top L^e$ for the minimal and biased model and $\omega = z^b - b^b$ for the kinematic model. When vectorizing the product of matrices one can use the Kronecker product, which for the product between matrices $A = [a_{ij}] \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{o\times p}$ is defined as [19, p. 303]

$$A \otimes B := [a_{ij}\, B] = \begin{bmatrix} a_{11}\, B & \cdots & a_{1n}\, B \\ \vdots & & \vdots \\ a_{m1}\, B & \cdots & a_{mn}\, B \end{bmatrix} \in \mathbb{R}^{m\,o \times n\,p}. \tag{3.6}$$

The vectorization of the product of three matrices can also be written using the Kronecker product as [19, p. 306]

$$\mathrm{vec}(A\, C\, B) = \big(B^\top \otimes A\big)\, \mathrm{vec}(C). \tag{3.7}$$

The general expression for the time derivative of R^{eb} from (3.5) contains only a product between two matrices. Thus, one of the matrices can be chosen to be the identity matrix. This gives three ways of writing the vectorization, each of which can be seen as part of a LTV state space model. For this model $\mathrm{vec}(C)$ and $B^\top \otimes A$ can be interpreted as part of the state and a time varying matrix respectively. However, it is assumed that each time varying matrix is only a function of R^{eb} and not ω. Therefore, the only the option with $C = S(x)$ remains. Therefore, the dynamics of ρ can be written as

$$\dot{\rho} = (I_3 \otimes R^{eb})\, \mathrm{vec}(S(\omega)). \tag{3.8}$$

It can be noted that the ith column of $S(\omega)$ can also be written as $\omega \times e_i = S(\omega)\, e_i$, with e_i the ith column of I_3. By using that the cross product is anti-commutative, this gives $\omega \times e_i = -e_i \times \omega$. Substituting this back in for the columns of $S(\omega)$ gives $\omega \times e_i = -S(e_i)\, \omega$. Therefore, the vectorization of $S(\omega)$ can also be written as

$$\operatorname{vec}(S(\omega)) = \begin{bmatrix} -S(e_1)\, \omega \\ -S(e_2)\, \omega \\ -S(e_3)\, \omega \end{bmatrix},$$

which can be simplified to

$$\operatorname{vec}(S(x)) = \Gamma\, \omega, \tag{3.9}$$

with

$$\Gamma = -\begin{bmatrix} S(e_1) \\ S(e_2) \\ S(e_3) \end{bmatrix} = \begin{bmatrix} S(e_1) & S(e_2) & S(e_3) \end{bmatrix}^\top. \tag{3.10}$$

Substituting (3.9) in (3.8) therefore gives

$$\dot{\rho} = (I_3 \otimes R)\, \Gamma\, \omega. \tag{3.11}$$

The vectorized rotation matrix ρ could in its current form be used as part of the state vector $x(t) \in \mathbb{R}^n$. However, the vector ρ contains redundant information about the attitude [33] and the proposed observer contains a differential equation in $M(t) \in \mathbb{R}^{n\times n}$, thus the computational resources required to use the observer are roughly proportional to n^2. Therefore, it would be desirable if ρ could be condensed into an attitude representation which uses less coordinates. One way to reduce the amount of coordinates used for the attitude would be to remove one of the columns or rows of the rotation matrix, since that column or row can be reconstructed by taking the cross product between the remaining columns or rows respectively.

Instead of just omitting one of the columns or rows from the rotation matrix R^{eb}, the six parameters can more generally defined as a pair of linearly independent vectors, which are rotated by the rotation matrix or the transpose of that rotation matrix. The rotation matrix can be reconstructed from those initial and rotated vectors by solving Wahba's problem as defined by (2.8). Those two pairs of possibly time-varying vectors are denoted in the body fixed frame and inertial frame with $V^b(t) = \begin{bmatrix} v_1^b(t) & v_2^b(t) \end{bmatrix} \in \mathbb{R}^{3\times 2}$ and $V^e(t) = \begin{bmatrix} v_1^e(t) & v_2^e(t) \end{bmatrix} \in \mathbb{R}^{3\times 2}$ respectively, where $v_i^j(t) = \begin{bmatrix} v_{i,1}^j(t) & v_{i,2}^j(t) & v_{i,3}^j(t) \end{bmatrix}^\top \in \mathbb{R}^3, j \in \{b, e\}, i \in \{1, 2\}$. These pairs of vectors are related to each other using the rotation matrix $R^{eb}(t)$ according to

$$V^e(t) = R^{eb}(t)\, V^b(t). \tag{3.12}$$

When $V^b(t)$ is given and known $V^e(t)$ can be used for the six parameters to represent the attitude and vice versa. This reduced attitude representation can be also be vectorized. Vectorizing both sides of (3.12) using (3.4) and (3.7) gives

$$\operatorname{vec}(V^e(t)) = \left(V^{b^\top}(t) \otimes I_3\right) \rho(t), \tag{3.13}$$

where $\text{vec}(V^e(t)) = \begin{bmatrix} v^e_{1,1}(t) & v^e_{1,2}(t) & v^e_{1,3}(t) & v^e_{2,1}(t) & v^e_{2,2}(t) & v^e_{2,3}(t) \end{bmatrix}^\top$ are the reduced coordinates for the attitude state vector. If $V^b(t)$ is constant in time the matrix

$$H := {V^b}^\top \otimes I_3, \tag{3.14}$$

is also constant in time and would define a constant linear transformation from the vectorized rotation matrix from (3.4) to the reduced coordinates from (3.13).

Solving (3.12) for $V^b(t)$ and taking the transpose gives

$$ {V^b}^\top(t) = {V^e}^\top(t)\, R^{eb}(t). \tag{3.15}$$

If $V^e(t)$ instead of $V^b(t)$ is constant in time and vectorizing both sides of (3.15) would, similar to (3.13) and (3.14), yield

$$H := I_3 \otimes {V^e}^\top, \tag{3.16}$$

where the reduced coordinates for the attitude state vector would now be given by $\text{vec}({V^b}^\top(t)) = \begin{bmatrix} v^b_{1,1}(t) & v^b_{2,1}(t) & v^b_{1,2}(t) & v^b_{2,2}(t) & v^b_{1,3}(t) & v^b_{2,3}(t) \end{bmatrix}^\top$.

For both constant weighting matrices H defined in (3.14) and (3.16) the six-dimensional attitude parameterization is written as

$$\rho_r := H\, \text{vec}(R^{eb}). \tag{3.17}$$

The dynamics of (3.17) can therefore be expressed by using (3.11) as

$$\dot{\rho}_r = H\ (I_3 \otimes R)\, \Gamma\, \omega. \tag{3.18}$$

3.3 Sufficient condition verification

The proposed observer structure from (3.2) and (3.3) can only guarantee convergence of the estimation error to zero if the considered LTV model satisfies two conditions. The first condition requires that every matrix of the LTV model is bounded for all times. The second condition requires that the LTV model is UCO.

In order to check the boundedness of each system matrix any norm could be used, since for matrices of finite size all norms are equivalent when showing boundedness. Therefore, boundedness would be shown if every entry of the matrices is bounded, which is equivalent to saying that each sub-matrix is bounded. It can be noted that a finite product of bounded matrices should also yield a bounded matrix. Furthermore, the rotation matrix R^{eb} belongs to the bounded set $SO(3)$. Therefore, if every sub-matrix of each system matrix of the LTV model can be written as a finite product of matrices and each of those matrices is bounded then the boundedness condition would be satisfied.

The definition of UCO for a LTV model can be defined as that there exists positive constants $\alpha, \delta > 0$ such that for all $t \geq t_0$ [2]

$$\int_t^{t+\delta} \Phi^\top(\tau, t)\, C^\top(\tau)\, C(\tau)\, \Phi(\tau, t)\, d\tau \succeq \alpha\, I_n,$$

where $\Phi(t_1, t_2)$ is the state transition matrix associated with $A(t)$ from t_1 to t_2. However, this condition is in general is not easy to check for LTV models whose system matrices are explicit functions of time. This condition becomes even harder to evaluate for implicit system matrices, as is assumed to be the case for (3.1). Instead, a relaxed condition for UCO can be used. That condition uses the following generalized definition of an observability matrix, which for (3.1) can be defined as

$$Q(t) = \begin{bmatrix} L_0(t) \\ L_1(t) \\ \vdots \\ L_q(t) \end{bmatrix}, \tag{3.19a}$$

$$\begin{cases} L_0(t) = C(R^{eb}), \\ L_k(t) = L_{k-1}(t)\,A(R^{eb}) + \dot{L}_{k-1}(t), \quad k = 1, \ldots, q. \end{cases} \tag{3.19b}$$

The relaxed condition for UCO requires that bounded constants $q \geq 0$, $\delta > 0$ and $\alpha > 0$ exists such that [2]

$$\|L_q(t_1) - L_q(t_2)\| \leq c_q\,|t_1 - t_2|,\; c_q > 0, \tag{3.20}$$

for all $t_1, t_2 \geq t_0$ and it is possible to choose $\tau \in [t, t+\delta]\ \forall\, t \geq t_0$ such that

$$Q^\top(\tau)\,Q(\tau) \succeq \alpha\, I_n, \tag{3.21}$$

with $Q(t) \in \mathbb{R}^{w \times n}$ and $w \geq n$. However, the implicit time dependency of the system matrices would require that (3.21) has to be satisfied at all times.

The inequality from (3.20) is violated either when $L_q(t_1) - L_q(t_2)$ is unbounded for bounded $t_1 - t_2$ or when the time derivative of $L_q(t)$ is unbounded. For the first condition it can be noted that if $L_q(t)$ is bounded $\forall\, t \geq t_0$ then $L_q(t_1) - L_q(t_2)$ should also remain bounded. The second condition can be obtained by dividing both sides of (3.20) by $|t_1 - t_2|$ and taking the limit of $t_1 - t_2$ to zero, which effectively results in taking the derivative of $L_q(t)$ with respect to time.

It is assumed that for each model $q = 1$ is sufficient to show UCO. This would yield the following expression for generalized observability matrix from (3.19)

$$Q(t) = \begin{bmatrix} C(R^{eb}) \\ C(R^{eb})\,A(R^{eb}) + \dot{C}(R^{eb}) \end{bmatrix}, \tag{3.22}$$

with the expression used in (3.20) given by $L_q(t) = C(R^{eb})\,A(R^{eb}) + \dot{C}(R^{eb})$. It is proposed that for each model the left hand side of (3.21) contains

$$\Psi(R^{eb}) = \Gamma^\top \left(I_3 \otimes R^{eb\top}\right) H^\top H \left(I_3 \otimes R^{eb}\right) \Gamma \tag{3.23}$$

and that it is key in verifying the inequality of (3.21) when using (3.22).

It can be shown that when using either (3.14) or (3.16) for H that (3.23) is positive definite matrix. By using two Kronecker product properties, given by

$(A \otimes B)\,(C \otimes D) = (A\,C \otimes B\,D)$ and $(A \otimes B)^\top = A^\top \otimes B^\top$, one can combine all Kronecker products of (3.23) into one [19, p. 306].

Substituting (3.16) for H in (3.23) and combining Kronecker products gives

$$\Psi(R^{eb}) = \Gamma^\top \left(I_3 \otimes U(R^{eb})\,U^\top(R^{eb})\right)\Gamma, \tag{3.24}$$

with $U(R^{eb}) = \begin{bmatrix}u_1 & u_2\end{bmatrix} = {R^{eb}}^\top V^e \in \mathbb{R}^{3\times 2}$. From the definition of the Kronecker product, given in (3.6), it follows that $I_3 \otimes U(R^{eb})\,U^\top(R^{eb})$ is block diagonal, due to I_3. Using this block diagonal structure and (3.10) for Γ allows for (3.24) to also be written as

$$\Psi(R^{eb}) = \sum_{i=1}^{3} S(e_i)\,U(R^{eb})\,\left(S(e_i)\,U(R^{eb})\right)^\top. \tag{3.25}$$

Using the anti-commutative property of the skew symmetric matrix in $S(e_i)\,U(R^{eb})$ gives

$$S(e_i)\,U(R^{eb}) = -\begin{bmatrix}S(u_1)\,e_i & S(u_2)\,e_i\end{bmatrix}. \tag{3.26}$$

Substituting (3.26) into the ith summation term of (3.25) gives

$$S(e_i)\,U(R^{eb})\,\left(S(e_i)\,U(R^{eb})\right)^\top = -S(u_1)\,e_i\,e_i^\top S(u_1) - S(u_2)\,e_i\,e_i^\top S(u_2). \tag{3.27}$$

By using (3.27) and $\sum_{i=1}^{3} e_i\,e_i^\top = I_3$ the summation in (3.25) can be simplified to

$$\Psi(R^{eb}) = -S^2(u_1) - S^2(u_2). \tag{3.28}$$

Each term $-S^2(u_i)$ from (3.28) is positive semi-definite. Namely,

$$-x^\top S^2(u_i)\,x = \|u_i \times x\|^2,$$

which is only zero if x and u_i are linearly dependent. However, it is assumed that the columns of V^e are linearly independent. The property of linear independence is preserved under rotation, so the columns of $U(R^{eb})$ are also linearly independent. From this property of $U(R^{eb})$ it follows that for $\|x\| \neq 0$ whenever $\|u_1 \times x\|^2 = 0$ it has to be true that $\|u_2 \times x\|^2 > 0$ and vice versa. Therefore, it can be concluded that (3.23) is a positive definite matrix for all $R^{eb} \in SO(3)$ when using (3.16) for H.

Substituting (3.14) instead of (3.16) in (3.23) and combining Kronecker products gives

$$\Psi(R^{eb}) = \Gamma^\top \left(V^b {V^b}^\top \otimes I_3\right)\Gamma, \tag{3.29}$$

which is independent of R^{eb}. The term $V^b {V^b}^\top \otimes I_3$ in (3.29) can be expanded into

$$V^b {V^b}^\top \otimes I_3 = \left({V^b}^\top \otimes I_3\right)^\top \left({V^b}^\top \otimes I_3\right),$$

such that (3.29) can also be written as $\Psi(R^{eb}) = \Omega^\top\Omega$ with $\Omega = ({V^b}^\top \otimes I_3)\,\Gamma$. It can be noted that the ith column of Γ can be written as $\Gamma\,e_i = \mathrm{vec}(S(e_i))$.

Therefore, the ith column of Ω can be written as $\Omega\, e_i = (V^{b\top} \otimes I_3)\,\mathrm{vec}(S(e_i))$ which according to (3.7) is equivalent to

$$\Omega\, e_i = \mathrm{vec}\big(S(e_i)\, V^b\big)\,. \tag{3.30}$$

The element of (3.29) at the ith row and jth column can be obtained by using $e_i^\top \Psi(R^{eb})\, e_j$. This element can also be obtained by multiplying the ith row of $\Omega^\top$ with the jth column of Ω. Using (3.30) for those rows and columns gives

$$e_i^\top \Psi(R^{eb})\, e_j = \mathrm{vec}\big(S(e_i)\, V^b\big)^\top \mathrm{vec}\big(S(e_j)\, V^b\big)\,. \tag{3.31}$$

By using that $V^b = \begin{bmatrix} v_1^b & v_2^b \end{bmatrix}$ and the anti-commutative property of the skew symmetric matrix it is possible to write (3.31) also as

$$e_i^\top \Psi(R^{eb})\, e_j = -e_i^\top S^2(v_1^b)\, e_j - e_i^\top S^2(v_2^b)\, e_j. \tag{3.32}$$

By factoring out $e_i^\top$ and e_j on the left and right side respectively of (3.32) gives

$$\Psi(R^{eb}) = -S^2(v_1^b) - S^2(v_2^b), \tag{3.33}$$

which, similar as for (3.28), can be shown to be positive definite given that the columns of V^b are linearly independent.

Instead of only showing that $\Psi(R^{eb})$ is positive definite it is also possible to calculate the lower bound of $\Psi(R^{eb})$. This lower bound can be obtained from rewriting (3.28) and (3.33) using $S(x)\, S(y) = y\, x^\top - y^\top x\, I_3$ [16, p. 13], which yields

$$\Psi(R^{eb}) = \big(w_1^\top w_1 + w_2^\top w_2\big)\, I_3 - \big(w_1\, w_1^\top + w_2\, w_2^\top\big)\,, \tag{3.34}$$

with $\{w_1, w_2\} = \{u_1, u_2\}$ for (3.28) and $\{w_1, w_2\} = \{v_1^b, v_2^b\}$ for (3.33). It can be noted that (3.34) is symmetric, so the lower bound is equal to the smallest eigenvalue. The largest eigenvalue of (3.34) can be shown to be equal to $w_1^\top w_1 + w_2^\top w_2$, with corresponding eigenvector that is perpendicular to both w_1 and w_2. A symmetric matrix also has orthogonal eigenvectors. Therefore, given the eigenvector of the largest eigenvalue yields that the remaining eigenvectors should be of the form $\alpha\, w_1 + \beta\, w_2$ with $\alpha, \beta \in \mathbb{R}$. This reduces the eigenvalue equation for the remaining two eigenvalues to

$$\big(\alpha\, w_2^\top w_2 - \beta\, w_1^\top w_2\big)\, w_1 + \big(\beta\, w_1^\top w_1 - \alpha\, w_2^\top w_1\big)\, w_2 = \lambda\, \alpha\, w_1 + \lambda\, \beta\, w_2,$$

for which it can be shown that it has the following characteristic polynomial

$$\lambda^2 - (w_1^\top w_1 + w_2^\top w_2)\, \lambda + w_1^\top w_1\, w_2^\top w_2 - (w_1^\top w_2)^2 = 0. \tag{3.35}$$

Solving (3.35) yields

$$\lambda = \frac{w_1^\top w_1 + w_2^\top w_2 \pm \sqrt{(w_1^\top w_1 - w_2^\top w_2)^2 + 4\, (w_1^\top w_2)^2}}{2}. \tag{3.36}$$

When w_1 and w_2 are linearly independent they should satisfy the following inequality [19, p. 171]

$$(w_1^\top w_2)^2 < w_1^\top w_1\, w_2^\top w_2. \tag{3.37}$$

Substituting (3.37) into the expression inside the square root of (3.36) gives the following inequality

$$0 \leq (w_1^\top w_1 - w_2^\top w_2)^2 + 4\,(w_1^\top w_2)^2 < (w_1^\top w_1 + w_2^\top w_2)^2. \tag{3.38}$$

Therefore, under the assumption that w_1 and w_2 are linearly independent it can be shown that the smallest eigenvalue of (3.34) should be positive. Since V^e or V^b can be chosen one could also choose them to be orthogonal, such that $w_1^\top w_2 = 0$. Substituting this in (3.36) gives that the two smaller eigenvalues are equal to $w_1^\top w_1$ and $w_2^\top w_2$.

All eigenvalues of (3.34) are only a function of $w_1^\top w_1$, $w_1^\top w_2$ and $w_2^\top w_2$. For (3.28) this is equivalent to the eigenvalues only being a function of $u_1^\top u_1$, $u_1^\top u_2$ and $u_2^\top u_2$, all of which are invariant under rotation. Therefore, the eigenvalues of (3.28) can instead also be calculated using $v_1^{e\top} v_1^e$, $v_1^{e\top} v_2^e$ and $v_2^{e\top} v_2^e$. Thus the eigenvalues of both (3.28) and (3.33) are all positive and independent of R^{eb}.

3.4 Remarks

This section discusses some remarks that can be made on the observer structure from this chapter. It is discussed which alternatives to the proposed attitude representation there are and why most of them are not suitable, and what a downside might be to not constraining the estimated attitude representation.

Since a smaller parameterization of the attitude would reduce computational cost, one might think that even smaller attitude representations such as unit quaternion and exponential coordinates [33] might be better options. However, those representations have singularities or do not have a unique mapping to and from $SO(3)$. Namely, a singularity would lead to unbounded matrices for the LTV model and thus violate one of the conditions from Section 3.3. And if there is no unique mapping either it is not possible to represent all attitudes, such as for the Cayley parametrization [33]. Or there are multiple representations that map to the same from attitude, such as for the unit quaternions where q and $-q$ represent the same attitude. When it is not possible to represent all attitudes the observer cannot give a global result. When there are multiple mappings the state $x(t)$ and output $y(t)$ are not uniquely defined, and the observer might not converge to any of the valid state representations. There are unique and non-singular five-dimensional parameterizations [33]. However, the dynamics of such parameterizations are more complicated so might outweigh the benefit of a slightly smaller state vector. The mapping of $\mathbb{R}^5$ to the "nearest" five-dimensional parameterization is also not trivial. Therefore, it is chosen to use the six-dimensional parameterization ρ_r, though the five-dimensional parameterization might be worth investigating in future research.

An even better option might be to use a reduced order observer, such that the attitude representation can be completely excluded from the observer state [20] [30]. However, this approach cannot be used because the time dependency of the matrices of the LTV model is only implicitly known, such that their time derivatives are not necessarily known. The implication of not knowing those time derivatives is demonstrated in Appendix A.

Lastly, it can be noted that even though the proposed observer structure in this chapter is exponentially stable, the obtained estimation for ρ_r has to be projected to the nearest rotation matrix. This projected rotation matrix might not always be continuous in time, even though the estimation for ρ_r should vary continuously in time. Such discontinuity can be demonstrated by using

$$v_1^b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2^b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_1^e(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2^e(t) = \begin{bmatrix} 1 \\ t \\ 0 \end{bmatrix},$$

with $\lambda_1 = \lambda_2 = 1$ in (2.8), which yields the following solution

$$B(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & t & 0 \\ 0 & 0 & 0 \end{bmatrix} = U(t)\, S(t)\, V^\top(t), \tag{3.39}$$

with

$$U(t) = \begin{bmatrix} \frac{2-t^2+\sqrt{4+t^4}}{\sqrt{(2-t^2+\sqrt{4+t^4})^2+4\,t^2}} & \frac{2-t^2-\sqrt{4+t^4}}{\sqrt{(2-t^2-\sqrt{4+t^4})^2+4\,t^2}} & 0 \\ \frac{2\,t}{\sqrt{(2-t^2+\sqrt{4+t^4})^2+4\,t^2}} & \frac{2\,t}{\sqrt{(2-t^2-\sqrt{4+t^4})^2+4\,t^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.40a}$$

$$S(t) = \begin{bmatrix} \sqrt{\frac{2+t^2+\sqrt{4+t^4}}{2}} & 0 & 0 \\ 0 & \sqrt{\frac{2+t^2-\sqrt{4+t^4}}{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{3.40b}$$

$$V(t) = \begin{bmatrix} \frac{-t^2+\sqrt{4+t^4}}{\sqrt{(t^2-\sqrt{4+t^4})^2+4}} & \frac{-t^2-\sqrt{4+t^4}}{\sqrt{(t^2+\sqrt{4+t^4})^2+4}} & 0 \\ \frac{2}{\sqrt{(t^2-\sqrt{4+t^4})^2+4}} & \frac{2}{\sqrt{(t^2+\sqrt{4+t^4})^2+4}} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.40c}$$

The determinant of $V(t)$ from (3.40c) can be shown to be always equal to one. It can be noted that the limit of $t \to 0$ of the element at the second row and second column of $U(t)$ from (3.40a), denoted with $U_{2,2}(t)$, is not well defined. Namely, it can be shown that $\lim_{t\to 0^-} U_{2,2}(t) = -1$ and $\lim_{t\to 0^+} U_{2,2}(t) = 1$. This not well defined limit also causes the limit of $t \to 0$ of the determinant of $U(t)$ to be not well defined. Namely, it can be shown that $\lim_{t\to 0^-} \det(U(t)) = -1$ and $\lim_{t\to 0^+} \det(U(t)) = 1$. Substituting these values of the determinants together with (3.40c) and (3.40a) in (2.10) also yields that the limit of $t \to 0$ of the solution to Wahba's problem is not well defined. Namely, it can be shown that limits of that rotation matrix are given by

$$\lim_{t \to 0^-} R_y(t) = \frac{1}{2} \begin{bmatrix} \sqrt{2} & \sqrt{2} & 0 \\ \sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & -2 \end{bmatrix},$$

$$\lim_{t \to 0^+} R_y(t) = \frac{1}{2} \begin{bmatrix} \sqrt{2} & \sqrt{2} & 0 \\ -\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Therefore, it does not immediately follow that the projected rotation matrix exponentially converges to the true rotation matrix. It is hypothesized that those discontinuities only happen for large estimation errors in ρ_r and that after some possible transient the projected rotation matrix eventually does continuously exponentially converge to the true rotation matrix. This hypothesis would have to be investigated in future research.

3.5 Summary

In this chapter an observer structure is proposed for each of the three models from Section 2.3. This observer structure relies on transforming each model to LTV, for which the observer dynamics from (3.2) and (3.3) can be used. This transformation requires that the state of the initial models is combined into a single vector. For this state vector construction the attitude representation is reduced in size, which also aid in lowering computation cost of the observer. Furthermore, it is discussed under what assumptions each LTV model guarantees that the estimated states of those observers would converge to the true states.

Chapter 4

Lyapunov based observer outline

This chapter discusses the derivation of the second structure of an observer outline used in this thesis. In Section 5.2, 6.2 and 7.2 this structure is applied to each of the three models from Section 2.3. This structure uses a Lyapunov function as starting point. For the dynamics of the estimated states a copy of the original dynamics is used to which correction terms are added. The value of those correction terms are chosen such that negative semi-definite time derivatives of the Lyapunov function is obtained. In Section 4.1 it is first discussed how the observer dynamics can be defined. This is followed by Section 4.2 in which the proposed Lyapunov function is introduced.

4.1 Observer dynamics

In contrast to the observers proposed in Chapter 3, the correction term used in the dynamics of the attitude estimation for the observer structure from this chapter are chosen such that no projection is needed. The dynamics of the attitude representation for each model from Section 2.3 are all of the form

$$\dot{R}^{eb} = R^{eb} S(\omega), \tag{4.1}$$

with $\omega = J^{-1} {R^{eb}}^{\top} L^e$ for the minimal and biased model and $\omega = z^b - b^b$ for the kinematic model. No projection of the estimated attitude representation $\hat{R}^{eb}$ is required if the correction term added to the dynamics of $\hat{R}^{eb}$ ensures that $\hat{R}^{eb}$ remains a rotation matrix. The most general way one can be written the dynamics of $\hat{R}^{eb}$ would be using

$$\dot{\hat{R}}^{eb} = \hat{R}^{eb} S(\hat{\omega}) + \Delta, \tag{4.2}$$

with $\hat{\omega}$ the estimate of ω and $\Delta \in \mathbb{R}^{3\times 3}$ is a function of the measured outputs and estimated states.

This form of the correction term does not in general assure that $\hat{R}^{eb}$ also remains a rotation matrix. The set of all rotation matrices is called $SO(3)$ and is in (2.1) defined as:

$$SO(3) \coloneqq \left\{ X \in \mathbb{R}^{3\times 3} \,|\, X^{\top} X = I_3, \det X = 1 \right\}$$

A sufficient condition, which assures that $\hat{R}^{eb}$ remains a rotation matrix, is that the time derivative of $\hat{R}^{eb\top}\hat{R}^{eb}$ is equal to zero. Namely, if the initial condition of $\hat{R}^{eb}$ is a rotation matrix then the sufficient condition assures that $\hat{R}^{eb\top}\hat{R}^{eb} = I_3$ is always satisfied. That relation also guarantees that the determinant of $\hat{R}^{eb}$ is equal to plus or minus one. In order for that determinant to switch from plus one to minus one requires that $\hat{R}^{eb}$ is discontinuous. However, if (4.2) is bounded would assure that $\hat{R}^{eb}$ is continuous. Therefore, the stated sufficient condition guarantees that $\hat{R}^{eb}$ remains a member of $SO(3)$. Substituting (4.2) into that condition, while using that $S(\hat{\omega})$ is a skew symmetric matrix, yields

$$\frac{d}{dt}\hat{R}^{eb\top}\hat{R}^{eb} = \dot{\hat{R}}^{eb\top}\hat{R}^{eb} + \hat{R}^{eb\top}\dot{\hat{R}}^{eb}, \tag{4.3a}$$

$$= \left(\Delta^\top - S(\hat{\omega})\hat{R}^{eb\top}\right)\hat{R}^{eb} + \hat{R}^{eb\top}\left(\hat{R}^{eb}S(\hat{\omega}) + \Delta\right), \tag{4.3b}$$

$$= \Delta^\top \hat{R}^{eb} + \hat{R}^{eb\top}\Delta. \tag{4.3c}$$

In order for (4.3) to be equal to zero for all possible $\hat{R}^{eb}$ requires $\Delta = \hat{R}^{eb}\Lambda$ with $\Lambda^\top = -\Lambda \in \mathbb{R}^{3\times3}$. However, this constraint on Λ is equivalent to using $\Lambda = S(\delta_R)$ with $\delta_R \in \mathbb{R}^3$ in which case (4.2) can be simplified to

$$\dot{\hat{R}}^{eb} = \hat{R}^{eb}S(\hat{\omega} + \delta_R). \tag{4.4}$$

The representations used for the rotation rate in Section 2.3 are the angular momentum and subtracting the bias from the biased angular velocity measurement. Therefore, in general the estimated angular velocity can be written as

$$\hat{\omega} = W\,J^{-1}R^{eb\top}\hat{L}^e + (I_3 - W)\,(z^b - \hat{b}^b) \tag{4.5}$$

with $W \in \mathbb{R}^{3\times3}$ a weighting matrix, $\hat{L}^e$ the estimate of the angular momentum L^e, $z^b = J^{-1}R^{eb\top}L^e + b^b$ the measured biased angular velocity and $\hat{b}^b$ the estimate of the bias b^b. It can be noted that for the minimal model z^b is not measured so $W = I_3$ would be the only option for that model. Similarly, for the kinematic model L^e is not considered to be part of the state, such that $W = 0_{3\times3}$ would be the only option for that model. This same weighing matrix can also be used to define the true angular velocity, which can thus also be written as

$$\omega = W\,J^{-1}R^{eb\top}L^e + (I_3 - W)\,(z^b - b^b). \tag{4.6}$$

The angular momentum and angular velocity measurement bias are not subjected to any constraints. Therefore, the correction terms added to the dynamics of their associated estimated states can be of the same size as those states themselves. The dynamics of the angular momentum is given by (2.7) and $\tau^e = R^{eb}\tau^b$ and the bias is assumed constant. Therefore, the general dynamics of those two estimated states can be written as

$$\dot{\hat{L}}^e = R^{eb}\tau^b + \delta_L, \tag{4.7}$$

$$\dot{\hat{b}}^b = \delta_b, \tag{4.8}$$

with $\delta_L, \delta_b \in \mathbb{R}^3$ their correction terms.

The errors of the estimated states are defined as $\tilde{R}^{eb} := \hat{R}^{eb} R^{eb\top} \in SO(3)$, $\tilde{L}^e := \hat{L}^e - L^e \in \mathbb{R}^3$ and $\tilde{b}^b := \hat{b}^b - b^b \in \mathbb{R}^3$. By using (4.1), (4.4) and the rotation invariance of the cross product the dynamics of $\tilde{R}^{eb}$ can be written as

$$\dot{\tilde{R}}^{eb} = \tilde{R}^{eb} S(R^{eb}\tilde{\omega} + R^{eb}\delta_R), \tag{4.9}$$

with $\tilde{\omega} := \hat{\omega} - \omega$ using (4.5) and (4.6). When using the definitions of $\tilde{L}^e$ and $\tilde{b}^b$ the expression for $\tilde{\omega}$ can also be written as

$$\tilde{\omega} = W\, J^{-1} R^{eb\top} \tilde{L}^e + (W - I_3)\, \tilde{b}^b. \tag{4.10}$$

Similarly, the dynamics of $\tilde{L}^e$ and $\tilde{b}^b$ are given by

$$\dot{\tilde{L}}^e = \delta_L, \tag{4.11}$$

$$\dot{\tilde{b}}^b = \delta_b. \tag{4.12}$$

The observer would have a zero estimation error between the true and estimated state if $\tilde{R}^{eb} = I_3$, $\tilde{L}^e = 0$ and $\tilde{b}^b = 0$. Not all three models from Section 2.3 use all three states. Therefore, the observer for each model should only have to drive the estimation error to zero of the states relevant for that model.

4.2 Proposed Lyapunov functions

The proposed Lyapunov functions for each model from Section 2.3 should be positive definite in the relevant error coordinates, and only equal to zero when there is zero estimation error. In order to satisfy this it is chosen to use the same structure used for the Lyapunov function for the minimal model proposed by A.A.J. (Erjen) Lefeber in internal personal correspondence, resulting in

$$V = P + \frac{1}{2}\tilde{x}^\top \Gamma^{-1} \tilde{x}, \tag{4.13}$$

with $\tilde{x}$ containing $\tilde{L}^e$ and $\tilde{b}^b$ depending on whether the considered model uses those states, $\Gamma = \Gamma^\top \succ 0$ a square matrix matching the size of $\tilde{x}$ and

$$P = \sum_{i=1}^{n} \frac{k_i}{2} \left(\tilde{R}^{eb}\, v_i - v_i\right)^\top \left(\tilde{R}^{eb}\, v_i - v_i\right), \tag{4.14}$$

with $n \geq 2$, $v_i \in \mathbb{R}^3$ unit vectors and $k_i > 0$.

The correction terms need to be chosen such that the time derivative of the Lyapunov function from (4.13) is at most negative semi-definite. When evaluating this time derivative one can evaluate the time derivative of P separate from the quadratic term in $\tilde{x}$. The time derivative of P from (4.14) can be obtained using the product rule and the general dynamics of $\tilde{R}^{eb}$ defined in (4.9) yielding

$$\dot{P} = \sum_{i=1}^{n} k_i \left(\tilde{R}^{eb} S(R^{eb}\tilde{\omega} + R^{eb}\delta_R)\, v_i\right)^{\top} \left(\tilde{R}^{eb}\, v_i - v_i\right), \tag{4.15a}$$

$$= \sum_{i=1}^{n} k_i \left(-\tilde{R}^{eb} S(v_i)\,(R^{eb}\tilde{\omega} + R^{eb}\delta_R)\right)^{\top} \left(\tilde{R}^{eb}\, v_i - v_i\right), \tag{4.15b}$$

$$= \sum_{i=1}^{n} k_i \left(R^{eb}\tilde{\omega} + R^{eb}\delta_R\right)^{\top} S(v_i) \left(v_i - \tilde{R}^{eb^{\top}} v_i\right), \tag{4.15c}$$

$$= \left(R^{eb}\tilde{\omega} + R^{eb}\delta_R\right)^{\top} \sum_{i=1}^{n} -k_i\, S(v_i)\tilde{R}^{eb^{\top}} v_i, \tag{4.15d}$$

$$= \left(\tilde{\omega} + \delta_R\right)^{\top} R^{eb^{\top}} \sum_{i=1}^{n} k_i\, S(\tilde{R}^{eb^{\top}} v_i)\, v_i. \tag{4.15e}$$

By defining an intermediate variable it is also possible to write (4.15) as

$$\dot{P} = \left(\tilde{\omega} + \delta_R\right)^{\top} \Pi, \tag{4.16}$$

with $\tilde{\omega}$ as defined in (4.10) and

$$\Pi = R^{eb^{\top}} \sum_{i=1}^{n} k_i\, S(\tilde{R}^{eb^{\top}} v_i)\, v_i, \tag{4.17a}$$

$$= \sum_{i=1}^{n} k_i \left(\hat{R}^{eb^{\top}} v_i\right) \times \left(R^{eb^{\top}} v_i\right). \tag{4.17b}$$

The time derivative of the quadratic term in $\tilde{x}$ can be obtained by using the product rule and using (4.11) and (4.12) to construct the time derivative of $\tilde{x}$. Therefore, the time derivative of the proposed Lyapunov function from (4.13) can be written as

$$\dot{V} = \left(\tilde{\omega} + \delta_R\right)^{\top} \Pi + \tilde{x}^{\top}\Gamma^{-1}\delta_x, \tag{4.18}$$

with δ_x containing the corresponding corrections terms associated with error states used to construct $\tilde{x}$.

In order to be able to use the estimated dynamics for an observer it is required that the correction terms δ_R, δ_L and δ_b are only a function of known variables. These known variables are the inputs and outputs of the model and the estimated states. Furthermore, the correction terms should vanish whenever all estimated states are equal to the true states. Therefore, it is proposed that each correction term is a linear combination of Π as defined by (4.17) and the estimation error of the biased angular velocity measurement, which is defined as

$$\tilde{z}^b := J^{-1}R^{eb^{\top}}\hat{L}^e + \hat{b}^b - z^b = J^{-1}R^{eb^{\top}}\tilde{L}^e + \tilde{b}^b. \tag{4.19}$$

The minimal model does not measure the biased angular velocity z^b and the kinematic model does not use the state $\hat{L}^e$. Therefore, the correction terms for the observers of those models would reduce to only being a linear function of Π.

4.3 Equilibria of the observer

The equilibria of the error dynamics from (4.9), (4.11) and (4.12) can be obtained by equating all time derivatives to zero. For (4.9) this is equivalent to equating the term inside the skew-symmetric matrix to zero. Therefore, the combined set of equations that have to be solved in order to find the equilibria are

$$\begin{aligned} W\, J^{-1} R^{eb\top} \tilde{L}^e + (W - I_3)\, \tilde{b}^b + \delta_R &= 0_{3\times1}, && \text{(4.20a)} \\ \delta_L &= 0_{3\times1}, && \text{(4.20b)} \\ \delta_b &= 0_{3\times1}. && \text{(4.20c)} \end{aligned}$$

It is assumed that all equilibria require $\Pi = \tilde{L}^e = \tilde{b}^b = 0_{3\times1}$. Thus for each equilibrium there would be a zero estimation error in L^e and b^b. For the attitude estimation error at each equilibrium it would only be guaranteed that $\Pi = 0_{3\times1}$.

In 1776 Leonard Euler first showed that the group of all three-dimensional rotations is a three-dimensional manifold [33]. Therefore, if one would use a three-dimensional parameterization to represent the attitude no constraints, such as $R^{eb} \in SO(3) \subset \mathbb{R}^{3\times3}$, are required when solving $\Pi = 0_{3\times1}$. However, is has been shown that it is impossible to have three variables that globally parameterize such rotation without singularities [33]. A well known example of such three-dimensional parameterization are the Euler/Tait-Bryan angles, whose singularities are often referred to as gimbal lock [16, p. 43]. Another example of such three-dimensional parameterization is the exponentiation of a skew symmetric matrix [33]. This attitude representation is from now on referred to as the exponential coordinates. The singularities of the exponential coordinates are located at less inconvenient locations then those of the Euler angles with respect to the application of a three-dimensional parameterization in this thesis and are therefore chosen to be used for solving for the equilibria. Additionally, a three-dimensional parameterization for the attitude also simplifies the local stability analysis of the obtained equilibria, since the local dynamics would also be unconstrained.

It has been shown that any rotation matrix $R^{eb} \in SO(3)$ is the matrix exponential of a three by three skew symmetric matrix [33]. Such a skew symmetric matrix can be parameterized with the exponential coordinates, which is a vector of length three denoted with $u := \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^\top$. These exponential coordinates can also be interpreted as an axis-angle representation, where the normalized vector represents the axis and the norm of the vector the angle [16, p. 40]. The exponential coordinates u are used to represent the attitude estimation error. The error of the estimated rotation matrix corresponding to the exponential coordinates u can be obtained by taking the matrix exponential of $S(u)$, which can be simplified to the following expression [33]

$$\tilde{R}^{eb} = e^{S(u)} = I_3 + \frac{\sin\|u\|}{\|u\|} S(u) + \frac{1 - \cos\|u\|}{\|u\|^2} S(u)^2. \tag{4.21}$$

The exponential coordinates do not give a unique attitude representation. Therefore, the exponential coordinates are limited to $\|u\| \leq \pi$ in order to avoid as many non-unique attitude representations as possible. For this interval the zero attitude estimation error $\tilde{R}^{eb} = I_3$ is equivalent to $u = 0_{3\times1}$. However, for $\|u\| = \pi$ both u and $-u$ represent to same attitude, which should be taken into account when considering unique solutions for the attitude equilibria. Substituting (4.21) into the expression for Π from (4.17a) gives

$$\Pi = R^{eb\top} \sum_{i=1}^{n} k_i \, S\left(\left[I_3 + \frac{\sin\|u\|}{\|u\|} S(u) + \frac{1-\cos\|u\|}{\|u\|^2} S(u)^2 \right]^\top v_i \right) v_i, \tag{4.22a}$$

$$= -R^{eb\top} \sum_{i=1}^{n} k_i \left(\frac{\sin\|u\|}{\|u\|} S(v_i)^2 + \frac{1-\cos\|u\|}{\|u\|^2} S(v_i \, v_i^\top u) \right) u. \tag{4.22b}$$

Substituting (4.22) in $\Pi = 0_{3\times1}$ and factoring out $R^{eb\top}$ is equivalent to

$$\frac{1-\cos\|u\|}{\|u\|^2} \left(\sum_{i=1}^{n} k_i \, S(v_i \, v_i^\top u) \right) u = -\frac{\sin\|u\|}{\|u\|} \left(\sum_{i=1}^{n} k_i \, S(v_i)^2 \right) u. \tag{4.23}$$

The trivial solution to (4.23) would be $u = 0_{3\times1}$, which is equivalent to the rotation matrix $\tilde{R}^{eb} = I_3$. It can be noted that for both the left and right hand side of (4.23) the limit of $\|u\|$ to zero exists and is finite. When solving for all other equilibria from (4.23) one can factor out u. From the remaining expressions it can be deduced that the left hand and right hand side of (4.23) can only be equal to each other when both sides are equal to zero. Namely, the left hand side of (4.23) with u factored out contains a summation of only skew symmetric matrices, which is always equal to a skew symmetric matrix as well, while the right hand side of (4.23) with u factored contains a summation of $k_i \, S(v_i)^2$, which is shown to be negative definite in Section 3.3 if the vectors $\sqrt{k_i}\, v_i$ span at least a two dimensional subspace. The product of a skew symmetric matrix with u is always perpendicular to u. However, the product with a negative definite matrix and u should always point partially in the opposite direction of u. Therefore, it can be concluded that both sides should be equal to zero. Since the right hand side contains a negative definite matrix it can only be zero for $\|u\| \neq 0$ when $\sin\|u\| = 0$. In the considered interval this has the solution $\|u\| = \pi$. The left hand side of (4.23) can only be equal to zero when the resulting skew symmetric matrix is equal to a vector which is parallel to u converted into a skew symmetric matrix, which can also be written as

$$\sum_{i=1}^{n} k_i \, v_i \, v_i^\top u = \alpha \, u, \tag{4.24}$$

with $\alpha \in \mathbb{R}$. It can be noted that (4.24) is an eigenvalue problem for the matrix

$$M := \sum_{i=1}^{n} k_i \, v_i \, v_i^\top, \tag{4.25}$$

with α one of the eigenvalues of M and u the corresponding eigenvector. Therefore, the remaining values of u that solve (4.23) are the eigenvectors of M normalized to a length of π. It can be noted that negating such normalized eigenvector also solves (4.23), however such solution represents the same attitude and thus can be omitted. The matrix from (4.25) is a three by three positive (semi)-definite matrix, which has three orthogonal eigenvectors. Therefore, there should be at least three solutions to (4.23) with $\|u\| = \pi$. However, if M has repeated eigenvalues then any combination of its corresponding eigenvectors would also be a solution. This can be avoided when M has three distinct eigenvalues. A similar result was also obtained [23] when using rotation matrices instead of the exponential coordinates for the attitude representation. It can be noted that one would still also have three solutions with $\|u\| = \pi$ when one of the eigenvalues of M is zero (and the other eigenvalues are positive and distinct). This could be the case when using $n = 2$ and appropriately chosen k_i and v_i.

4.4 Stability of equilibria

If the time derivative of the proposed Lyapunov function, given in (4.18), can only be made negative semi-definite by appropriately choosing expressions for the correction terms then one can initially only guarantee Lyapunov stability of the estimation error of the observer. However, negative semi-definiteness might enable one to show that the error dynamics cannot have limit cycles, in which case the error should eventually settle at one of the equilibria of the error dynamics. By analyzing the local dynamics of these equilibria one can say more about at which equilibrium the error dynamics is most likely to settle.

A common method of analyzing the local stability of equilibria of a dynamical model is by looking at the linearized dynamics of that model at each equilibrium. In Section 4.1 the error dynamics is given by (4.9), (4.11) and (4.12). However, in Section 4.3 it is proposed to use exponential coordinates instead of a rotation matrix for the attitude representation, thus a transformation of attitude dynamics is also required.

The time derivative of the attitude of a rigid body, expressed as exponential coordinates $v \in \mathbb{R}^{3\times 1}$, equivalent to (4.1) such that $R^{eb} = e^{S(v)}$ is given by [33]

$$\dot{X} = \Omega - \frac{1}{2}\left(\Omega\, X - X\, \Omega\right) + \frac{2 - \theta\cot(\theta/2)}{2\,\theta^2}\left(X^2\,\Omega + \Omega\, X^2 - 2\, X\, \Omega\, X\right),$$

with $X = S(v)$, $\Omega = S(\omega)$ and $\theta = \|v\|$, which can also be written as

$$\dot{v} = F(v)\,\omega, \tag{4.26}$$

$$F(v) = I_3 + \frac{1}{2}S(v) + \frac{2 - \theta\,\cot(\theta/2)}{2\,\theta^2}S(v)^2. \tag{4.27}$$

It can be noted that the limit of (4.27) as $\theta \to 0$ does exists. However, the limit of (4.27) does not exist for $\{\theta \in \mathbb{R} \,|\, \theta = 2\,k\,\pi,\ k \in \mathbb{Z},\ k \neq 0\}$. Those singularities can be explained by the fact that a rotation of a multiple of $2\,\pi$ is equivalent to no rotation and the axis of rotation of no rotation is not well defined. Therefore, by combining (4.9) with (4.26) the time derivative of u is given by

$$\dot{u} = F(u)\, R^{eb}(\tilde{\omega} + \delta_R). \tag{4.28}$$

The remaining dynamics is given by (4.11) and (4.12):

$$\dot{\tilde{L}}^e = \delta_L,$$
$$\dot{\tilde{b}}^b = \delta_b.$$

By using (4.10), (4.27), (4.22), (4.19) and the assumption that each correction term is expressed as a linear combination of Π and $\tilde{z}^b$, the corresponding intermediate variables used in the considered dynamics can in general be written as

$$F(u) = I_3 + \frac{1}{2}S(u) + \frac{2 - \|u\| \, \cot(\|u\|/2)}{2\,\|u\|^2}S(u)^2, \tag{4.29a}$$
$$\tilde{\omega} = W\,J^{-1}R^{eb\top}\tilde{L}^e + (W - I_3)\,\tilde{b}^b, \tag{4.29b}$$
$$\Pi = -R^{eb\top}\sum_{i=1}^{n} k_i \left(\frac{\sin\|u\|}{\|u\|}S(v_i)^2 + \frac{1-\cos\|u\|}{\|u\|^2}S(v_i\,v_i^\top u)\right)u, \tag{4.29c}$$
$$\tilde{z}^b = J^{-1}R^{eb\top}\tilde{L}^e + \tilde{b}^b, \tag{4.29d}$$
$$\delta_R = A_R\,\Pi + B_R\,\tilde{z}^b, \tag{4.29e}$$
$$\delta_L = A_L\,\Pi + B_L\,\tilde{z}^b, \tag{4.29f}$$
$$\delta_b = A_b\,\Pi + B_b\,\tilde{z}^b, \tag{4.29g}$$

with A_i and B_i for $i \in \{R, L, b\}$ matrices which are independent of u, $\tilde{L}^e$ and $\tilde{b}^b$ but can vary with time. Furthermore, in Section 4.3 it is assumed that this error dynamics has four equilibria. Each equilibrium has $\tilde{L}^e = \tilde{b}^b = 0_{3\times1}$ and either $u = 0_{3\times1}$ or u is equal to one of the three eigenvectors of the matrix from (4.25) normalized to a length of π. It can be noted that $\Pi = 0_{3\times1}$ when using the value for u at each equilibrium.

The combined error dynamics from (4.28), (4.11) and (4.12) can be approximated near the equilibria using a linearization

$$\dot{\tilde{x}} = A\,(\tilde{x} - x^*) + \text{h.o.t.}, \tag{4.30}$$

with $\tilde{x} = \begin{bmatrix} u^\top & \tilde{L}^{e\top} & \tilde{b}^{b\top} \end{bmatrix}^\top$ and x^* the equilibrium around which the model is linearized. This linearization matrix A, which might be time varying, can be obtained by evaluating the partial derivatives of the right hand sides of the combined error dynamics with respect to $\tilde{x}$ at the equilibria. The expression for A is broken down for convenience into

$$A = \begin{bmatrix} \mathcal{A}_{uu} & \mathcal{A}_{uL} & \mathcal{A}_{ub} \\ \mathcal{A}_{Lu} & \mathcal{A}_{LL} & \mathcal{A}_{Lb} \\ \mathcal{A}_{bu} & \mathcal{A}_{bL} & \mathcal{A}_{bb} \end{bmatrix}, \tag{4.31}$$

where $\mathcal{A}_{xy}$, with $x, y \in \{u, L, b\}$ where u, L and b are referring to u, $\tilde{L}^e$ and $\tilde{b}^b$ respectively, is defined as

$$\mathcal{A}_{xy} = \lim_{\tilde{x}\to x^*} \frac{\partial\,\dot{x}}{\partial y}. \tag{4.32}$$

By using (4.28) and the product rule the first sub-matrix from (4.31) is given by

$$\mathcal{A}_{uu} = \lim_{\tilde{x}\to x^*} \frac{\partial\, F(u)}{\partial u} R^{eb}(\tilde{\omega} + \delta_R) + F(u)\, R^{eb} \frac{\partial\, (\tilde{\omega} + \delta_R)}{\partial u}. \tag{4.33}$$

It can be noted that $\tilde{\omega} + \delta_R = 0_{3\times 1}$ when taking the limit of $\tilde{x}$ to any of the x^*. By combining this with (4.29b) and (4.29e) gives the following equivalent expression for (4.33)

$$\mathcal{A}_{uu} = \lim_{\tilde{x}\to x^*} F(u)\, R^{eb} A_R \frac{\partial\, \Pi}{\partial u}. \tag{4.34}$$

The expression for Π, as defined in (4.29c), can be simplified by using the intermediate variable $\theta := \|u\|$, the matrix M from (4.25) and define another matrix as

$$N := \sum_{i=1}^{n} k_i\, S(v_i)^2, \tag{4.35}$$

resulting in the following equivalent expression

$$\Pi = -R^{eb\top} \left(\frac{\sin\theta}{\theta} N + \frac{1-\cos\theta}{\theta^2} S(M\, u) \right) u. \tag{4.36}$$

The partial derivative of Π, as defined in (4.36), with respect to u can be obtained by using the product and chain rule together with the anti-commutative property $S(x)\, y = -S(y)\, x$, resulting in

$$\frac{\partial\, \Pi}{\partial u} = -R^{eb\top} \left(\frac{\sin\theta}{\theta} N + \frac{1-\cos\theta}{\theta^2} \left(S(M\, u) - S(u)\, M \right) \right) + \frac{\partial\, \Pi}{\partial \theta} \frac{\partial\, \theta}{\partial u}. \tag{4.37}$$

The partial derivative of Π with respect to θ can be shown to be equal to

$$\frac{\partial\, \Pi}{\partial \theta} = -R^{eb\top} \left(\frac{\theta\cos\theta - \sin\theta}{\theta^2} N + \frac{\theta\sin\theta - 2\,(1-\cos\theta)}{\theta^4} S(Mu) \right) u. \tag{4.38}$$

The intermediate variable θ can also be expressed as $\sqrt{u^\top u}$, for which it can be shown that it has the following partial derivative with respect to u

$$\frac{\partial\, \theta}{\partial u} = \frac{u^\top}{\theta}. \tag{4.39}$$

Evaluating the combined expression for (4.37) by substituting in (4.38) and (4.39) at the equilibria gives

$$\lim_{\tilde{x}\to x^*} \frac{\partial\, \Pi}{\partial u} = -R^{eb\top} N, \tag{4.40}$$

at the equilibrium with $\|u\| = 0$ and

$$\lim_{\tilde{x}\to x^*} \frac{\partial\, \Pi}{\partial u} = \frac{R^{eb\top}}{\pi^2} \left(2\, S(\bar{u})\, (M - \sigma\, I_3) + N\, \bar{u}\, \bar{u}^\top \right) \tag{4.41}$$

at the equilibria where u is equal to one of the eigenvectors of M normalized to a length of π, for which this normalized eigenvector is denoted with $\bar{u}$ and σ is the corresponding eigenvalue. Evaluating the expression for $F(u)$, defined in (4.29a), at the same limit values as (4.40) and (4.41) yields

$$\lim_{\tilde{x}\to x^*} F(u) = I_3, \tag{4.42}$$

$$\lim_{\tilde{x}\to x^*} F(u) = I_3 + \frac{1}{2}S(\bar{u}) + \frac{1}{\pi^2}\bar{u}\,\bar{u}^\top = \frac{1}{2}S(\bar{u}) + \frac{\bar{u}\,\bar{u}^\top}{\pi^2}, \tag{4.43}$$

respectively for $\|u\| = 0$ and $u = \bar{u}$.

The remaining sub-matrices related to the dynamics of u can be obtained by substituting (4.29b) and (4.29e) in (4.28), resulting in

$$\mathcal{A}_{uL} = \lim_{\tilde{x}\to x^*} \frac{\partial\, F(u)\, R^{eb}\Big((W+B_R)\,J^{-1}R^{eb^\top}\tilde{L}^e + (W+B_R-I_3)\,\tilde{b}^b + A_R\,\Pi\Big)}{\partial \tilde{L}^e}, \tag{4.44a}$$

$$= \lim_{\tilde{x}\to x^*} F(u)\, R^{eb}(W+B_R)\,J^{-1}R^{eb^\top}, \tag{4.44b}$$

$$\mathcal{A}_{ub} = \lim_{\tilde{x}\to x^*} \frac{\partial\, F(u)\, R^{eb}\Big((W+B_R)\,J^{-1}R^{eb^\top}\tilde{L}^e + (W+B_R-I_3)\,\tilde{b}^b + A_R\,\Pi\Big)}{\partial \tilde{b}^b}, \tag{4.45a}$$

$$= \lim_{\tilde{x}\to x^*} F(u)\, R^{eb}(W+B_R-I_3). \tag{4.45b}$$

Similarly, the sub-matrices related to the dynamics of $\tilde{L}^e$ and $\tilde{b}^b$ in (4.31) can be obtained by substituting (4.29f) and (4.29g) into (4.11) and (4.12) respectively, resulting in

$$\mathcal{A}_{Lu} = \lim_{\tilde{x}\to x^*} \frac{\partial\Big(A_L\,\Pi + B_L\,(J^{-1}R^{eb^\top}\tilde{L}^e + \tilde{b}^b)\Big)}{\partial u} = A_L \lim_{\tilde{x}\to x^*}\frac{\partial\,\Pi}{\partial u}, \tag{4.46}$$

$$\mathcal{A}_{LL} = \lim_{\tilde{x}\to x^*} \frac{\partial\Big(A_L\,\Pi + B_L\,(J^{-1}R^{eb^\top}\tilde{L}^e + \tilde{b}^b)\Big)}{\partial \tilde{L}^e} = B_L\,J^{-1}R^{eb^\top}, \tag{4.47}$$

$$\mathcal{A}_{Lb} = \lim_{\tilde{x}\to x^*} \frac{\partial\Big(A_L\,\Pi + B_L\,(J^{-1}R^{eb^\top}\tilde{L}^e + \tilde{b}^b)\Big)}{\partial \tilde{b}^b} = B_L, \tag{4.48}$$

$$\mathcal{A}_{bu} = \lim_{\tilde{x}\to x^*} \frac{\partial\Big(A_b\,\Pi + B_b\,(J^{-1}R^{eb^\top}\tilde{L}^e + \tilde{b}^b)\Big)}{\partial u} = A_b \lim_{\tilde{x}\to x^*}\frac{\partial\,\Pi}{\partial u}, \tag{4.49}$$

$$\mathcal{A}_{bL} = \lim_{\tilde{x}\to x^*} \frac{\partial\Big(A_b\,\Pi + B_b\,(J^{-1}R^{eb^\top}\tilde{L}^e + \tilde{b}^b)\Big)}{\partial \tilde{L}^e} = B_b\,J^{-1}R^{eb^\top}, \tag{4.50}$$

$$\mathcal{A}_{bb} = \lim_{\tilde{x}\to x^*} \frac{\partial\Big(A_b\,\Pi + B_b\,(J^{-1}R^{eb^\top}\tilde{L}^e + \tilde{b}^b)\Big)}{\partial \tilde{b}^b} = B_b. \tag{4.51}$$

The resulting matrix from (4.31) linearized at the equilibrium with $\|u\| = 0$, denoted with A_0, can therefore be obtained by substituting in (4.40) and (4.42) in (4.34) and (4.44) through (4.44), resulting in

$$A_0 = \begin{bmatrix} -R^{eb} A_R {R^{eb}}^\top N & R^{eb}(W + B_R)\, J^{-1} {R^{eb}}^\top & R^{eb}(W + B_R - I_3) \\ -A_L\, {R^{eb}}^\top N & B_L\, J^{-1} {R^{eb}}^\top & B_L \\ -A_b\, {R^{eb}}^\top N & B_b\, J^{-1} {R^{eb}}^\top & B_b \end{bmatrix}. \tag{4.52}$$

Similarly, the resulting matrix from (4.31) linearized at one of the equilibria with $u = \bar{u}$, denoted with A_π, can be obtained by substituting in (4.41) and (4.43) in (4.34) and (4.44) through (4.44), resulting in

$$A_\pi = \begin{bmatrix} Q\, R^{eb} A_R\, T & Q\, R^{eb}(W + B_R)\, J^{-1} {R^{eb}}^\top & Q\, R^{eb}(W + B_R - I_3) \\ A_L\, T & B_L\, J^{-1} {R^{eb}}^\top & B_L \\ A_b\, T & B_b\, J^{-1} {R^{eb}}^\top & B_b \end{bmatrix}, \tag{4.53}$$

with $Q := \frac{1}{2} S(\bar{u}) + \frac{\bar{u}\,\bar{u}^\top}{\pi^2}$ and $T := \frac{{R^{eb}}^\top}{\pi^2} \left(2\, S(\bar{u})\, (M - \sigma\, I_3) + N\, \bar{u}\, \bar{u}^\top \right)$.

The matrix M, from (4.25), is symmetric and positive (semi-)definite, which guarantees that it can also be written as $M = U\, \Omega\, U^\top$ with $U \in SO(3)$ and $\Omega = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$, where $\lambda_1 > \lambda_2 > \lambda_3 \geq 0$ are the eigenvalues of M. Using this decomposition it is also possible to define the values of the exponential coordinates at the non-zero equilibria as $\bar{u} = \pi\, U\, e_i$, with e_i the ith column of I_3 and $\alpha = \lambda_i$ the corresponding eigenvalue. By using $S(x)\, S(y) = y\, x^\top - y^\top x\, I_3$ it is also possible to write the matrix from (4.35) as $N = M - \kappa\, I_3$, with $\kappa = \sum_{i=1}^{n} k_i = \sum_{i=1}^{3} \lambda_i$. By using all this it is also possible to write Q and T as

$$Q := U \left(\frac{\pi}{2} S(e_i) + e_i\, e_i^\top \right) U^\top, \tag{4.54a}$$

$$T := {R^{eb}}^\top U \left(\frac{2}{\pi} S(e_i)(\Omega - \lambda_i\, I_3) + (\lambda_i - \kappa)\, e_i\, e_i^\top \right) U^\top. \tag{4.54b}$$

The linearizations obtained from (4.30) using (4.52) or (4.53) can be used to assess the local stability or instability of the relevant equilibria. Though, depending on which model from Section 2.3 is used, some of the sub-matrices from (4.52) and (4.53) might need to be removed if the associated state variables are not used in that model. The matrices from (4.52) and (4.53) are a function of the rotation matrix R^{eb}. Thus local stability cannot be concluded by simply calculating the their eigenvalues. Instead the local stability analysis is done by searching for a common quadratic Lyapunov function (CQLF) of the form

$$V_i(\tilde{x} - x_i^*) = (\tilde{x} - x^*)^\top P_i (\tilde{x} - x_i^*), \tag{4.55}$$

with x_i^* the considered equilibrium point, $i = 0$ for the equilibrium with $\|u\| = 0$ and $i = \pi$ for the equilibria with $\|u\| = \pi$. For the initial guess for each P_i in (4.55) the Hessian of initial Lyapunov function as defined in (4.13) evaluated at x_i^* is used. It can be noted that the Lyapunov function from (4.13) needs to be expressed using u instead of $\tilde{R}^{eb}$ by substituting in (4.22). By using the dynamics of the linearization the time derivative of (4.55) can be written as

$$\dot{V}_i(\tilde{x} - x_i^*) = (\tilde{x} - x_i^*)^\top \left(P_i A_i + A_i^\top P_i\right)(\tilde{x} - x_i^*), \tag{4.56}$$

which is thus characterized by the matrix $P_i A_i + A_i^\top P_i$.

For each equilibrium with $\|u\| = 0$ one can show exponential stability if one can find a CQLF with $P_0 \succ 0$ such that the time derivative of the CQLF is negative definite, which is the case if

$$A_0^\top P_0 + P_0\, A_0 \prec 0 \;\forall\, R^{eb} \in SO(3). \tag{4.57}$$

The starting guess for P_0 is the Hessian and can if necessary be perturbed such that (4.57) is satisfied. Evaluating this Hessian yields the following starting guess for each model

$$P_0 = \begin{bmatrix} -N & 0 \\ 0 & \Gamma^{-1} \end{bmatrix}, \tag{4.58}$$

with the zero matrices and Γ of the appropriate sizes. The perturbations to this initial guess can be aided by spotting patterns in P_0 when solving a set of linear matrix inequalities containing $P_0 \succ 0$ and (4.57) in which $\forall\, R^{eb} \in SO(3)$ is replaced with a large but finite number rotation matrices.

The remaining equilibria with $\|u\| = \pi$ can be shown to be unstable by using Chetaev's theorem [18]. This theorem considers the equilibrium point $x = 0$ of an autonomous system $\dot{x} = f(x)$ and is defined as:

Theorem 1. *Let $V : D \to \mathbb{R}$ be a continuously differentiable function on a domain $D \subset \mathbb{R}^n$ such that $V(0) = 0$ and $V(x_0) > 0$ for some x_0 with arbitrarily small $\|x_0\|$. Choose $r > 0$ such that the ball $B_r = \{x \in \mathbb{R}^n \,|\, \|x\| \le r\}$ is contained in D and let $\mathcal{U} = \{x \in B_r \,|\, V(x) > 0\}$ and suppose that $\dot{V}(x) > 0$ in U. Then, $x = 0$ is unstable.*

Applying Theorem 1 to the equilibria with $\|u\| = \pi$ using the CQLF from (4.55) for the continuously differentiable function thus requires that $\dot{V}_\pi(x) > 0$ for all $x \in \{y \in \mathbb{R}^n \,|\, \|y\| \le r \,|\, V_\pi(y) > 0\}$ with $r > 0$ but sufficiently small such that the linearization holds. The starting guess for P_π is in these cases the Hessian multiplied by minus one. Evaluating this Hessian yields the following starting guess for each model

$$P_\pi = \begin{bmatrix} U\left((\kappa - \lambda_i)\, e_i\, e_i^\top + \frac{4}{\pi^2}(\Omega - \lambda_i\, I_3)\right) U^\top & 0 \\ 0 & -\Gamma^{-1} \end{bmatrix}, \tag{4.59}$$

with the zero matrices and Γ of the appropriate sizes. By using that $\Omega = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$, $\lambda_1 > \lambda_2 > \lambda_3 \ge 0$ and $\kappa = \sum_{i=1}^{3} \lambda_i$ it follows that the upper left three by three block of the matrix in (4.59) is diagonalizable by pre-multiplying and post-multiplied it by $U^\top$ and U respectively. This diagonal matrix contains the elements $\{\lambda_j + \lambda_k, \frac{4}{\pi^2}(\lambda_j - \lambda_i), \frac{4}{\pi^2}(\lambda_k - \lambda_i)\}$ where $\{i, j, k\}$ are some permutation of $\{1, 2, 3\}$. It can be noted that the term $\lambda_j + \lambda_k$ is always positive, while zero, one or two of the other two diagonal elements are also positive when $i = 1$, $i = 2$

and $i = 3$ respectively. Therefore, the matrix from (4.59) should have at least one positive eigenvalue and thus set $\mathcal{U} = \{x \in \mathbb{R}^n \mid \|x\| \leq r \mid V(x) > 0\}$ should be nonempty.

Besides equilibria a dynamical model can also have limit cycles, which are a closed trajectories. In order for such trajectories to exist for the error dynamics of the proposed observer in this chapter it is requires that the integral of the time derivative of the Lyapunov function along such closed trajectory should be zero. Namely, a closed trajectory returns to its starting state, which means that the Lyapunov function should return to its starting value. If the time derivative of the Lyapunov function is negative semi-definite this can only be the case if this time derivative is zero along the entire trajectory.

The assumption that M from (4.25) has distinct eigenvalues is used to show that each observer has unstable equilibria and no periodic trajectories. This assumptions puts constraints on the possible values for k_i and v_i. When using $n = 2$ one of the eigenvalues is zero and the non-zero eigenvalues can be shown to satisfy

$$\lambda_i = \frac{k_1 + k_2 \pm \sqrt{(k_1 - k_2)^2 + 4\, k_1 k_2 \left(v_1^\top v_2\right)^2}}{2}.$$

Therefore, in order for M to have distinct eigenvalues would require that at least one of the inequalities $v_1^\top v_2 \neq 0$ and $k_1 \neq k_2$ is satisfied.

4.5 Summary

In this chapter an observer structure is proposed for each of the three models from Section 2.3, whose design is derived from a Lyapunov function. For this observer structure it is assumed that it has multiple equilibria. However, by linearizing the error dynamics it is possible to analyze the stability of each equilibrium, for which it is postulated that all but one can be shown to be unstable.

Chapter 5

Minimal observers

In this chapter the observer structures from Chapter 3 and 4 are applied to the minimal model for the attitude dynamics from Section 2.3.1. First, the observer structure from Chapter 3 is applied to this model in Section 5.1, in which it is shown how the model can be formulated as linear time varying (LTV) and when it satisfies the sufficient conditions. This is followed by Section 5.2, in which the observer structure from Chapter 4 is applied to the minimal model and is the local dynamics of its equilibria analyzed.

The minimal model that is considered in this chapter is defined in (2.13) as:

$$\dot{R}^{eb} = R^{eb}\, S(J^{-1} {R^{eb}}^\top L^e), \tag{5.1a}$$

$$\dot{L}^e = R^{eb} u, \tag{5.1b}$$

$$u = \tau^b, \tag{5.1c}$$

$$y = R^{eb}, \tag{5.1d}$$

with u the known input of the system, y the output of the system and J the known mass moment of inertia matrix.

5.1 Minimal linear time varying observer

In this section the observer structure from Chapter 3 is applied to the minimal model from (5.1). The transformation of the minimal model to a LTV model and the resulting observer dynamics are first discussed in Subsection 5.1.1. This is followed by Subsection 5.1.2, in which it is shown under which assumptions the sufficient conditions of the used observer structure are satisfied.

5.1.1 Observer dynamics

In order to use the observer structure from Chapter 3 it is required that the considered model can be formulated to fit the LTV model from (3.1):

$$\begin{cases} \dot{x}(t) = A(R^{eb})\, x(t) + B(R^{eb})\, u(t), \\ y(t) = C(R^{eb})\, x(t). \end{cases}$$

This can be done by defining the state vector $x(t)$ and equivalent output $y(t)$ as

$$x(t) := \begin{bmatrix} \rho_r \\ L^e \end{bmatrix} \in \mathbb{R}^9, \quad y(t) := \rho_r \in \mathbb{R}^6,$$

with ρ_r as defined in (3.17). By combining (3.18), (2.13b) and $\omega = J^{-1} {R^{eb}}^\top L^e$ it can be shown that using the following matrices for the LTV model yield a model that is equivalent to the minimal model

$$\begin{cases} A(R^{eb}) := \begin{bmatrix} 0_{6\times6} & H\,(I_3 \otimes R^{eb})\,\Gamma\, J^{-1} {R^{eb}}^\top \\ 0_{3\times6} & 0_{3\times3} \end{bmatrix}, \quad B(R^{eb}) := \begin{bmatrix} 0_{6\times3} \\ R^{eb} \end{bmatrix}, \\ C(R^{eb}) := \begin{bmatrix} I_6 & 0_{6\times3} \end{bmatrix}. \end{cases} \tag{5.2}$$

By using (5.2) the observer structure from Chapter 3 can be expressed using (3.2) and (3.3):

$$\dot{\hat{x}}(t) = A(R^{eb})\,\hat{x}(t) + B(R^{eb})\,u(t) - K(t)\,\left[C(R^{eb})\,\hat{x}(t) - y(t)\right],$$

$$\begin{cases} \dot{M}(t) = A(R^{eb})M(t) + M(t)A^\top(R^{eb}) - M(t)C^\top(R^{eb})W^{-1}C(R^{eb})M(t) + V + \delta M(t), \\ M(t_0) = M_0 = M_0^\top \succ 0, \quad W = W^\top \succ 0, \\ K(t) = M(t)\,C^\top(R^{eb})\,W^{-1}, \end{cases}$$

with $\hat{x}(t)$ the estimate of $x(t)$, $M_0, V \in \mathbb{R}^{9\times9}$ and $W \in \mathbb{R}^{6\times6}$.

5.1.2 Sufficient condition verification

The given observer dynamics in the previous subsection would yield that $\hat{x}(t)$ globally converges to $x(t)$ if the matrices, that are defined in (5.2), can be shown to be bounded and that the associated LTV model is uniformly completely observable (UCO). All sub-matrices from each matrix from (5.2) consists of a product of bounded matrices. Therefore, by using the argumentation from Section 3.3, it can be concluded that the obtained LTV model has bounded matrices. The obtained LTV model can be shown to be UCO if (3.20) and (3.21) are satisfied. Those two conditions can be evaluated for the obtained LTV model by using the generalized observability matrix obtained by substituting (5.2) in (3.22), which yields

$$Q(t) = \begin{bmatrix} I_6 & 0_{6\times3} \\ 0_{6\times6} & H\,(I_3 \otimes R^{eb})\,\Gamma\, J^{-1} {R^{eb}}^\top \end{bmatrix}, \tag{5.3}$$

with

$$L_q(t) = \begin{bmatrix} 0_{6\times6} & H\,(I_3 \otimes R^{eb})\,\Gamma\, J^{-1} {R^{eb}}^\top \end{bmatrix}. \tag{5.4}$$

It can be noted that (3.20) is equivalent to showing that $L_q(t)$ from (5.4) and its time derivative are bounded. The expression for (5.4) is essentially a sub-matrix of $A(R^{eb})$ from (5.2), which has been shown to be bounded. This implies that $L_q(t)$ is also always bounded. The only variable in (5.4) that is not necessarily constant in time is R^{eb}. Therefore, by using $\dot{R}^{eb} = R^{eb}\,S(J^{-1} {R^{eb}}^\top L^e)$ the time derivative of (5.4) can be shown to be

$$\dot{L}_q(t) = \begin{bmatrix} 0_{6\times6} & H\,\left[(I_3 \otimes R^{eb}\,S(\omega))\,\Gamma\, J^{-1} - (I_3 \otimes R^{eb})\,\Gamma\, J^{-1} S(\omega)\right] {R^{eb}}^\top \end{bmatrix}, \tag{5.5}$$

with $\omega = J^{-1}R^{eb\top}L^e$. The expression from (5.5) can only be guaranteed to be bounded if one assumes that ω or equivalently L^e is also bounded.

Substituting (5.3) in (3.21) yields

$$\begin{bmatrix} I_6 & 0_{6\times 3} \\ 0_{3\times 6} & R^{eb}J^{-1}\Psi(R^{eb})\,J^{-1}R^{eb\top} \end{bmatrix} \succeq \alpha\, I_9, \tag{5.6}$$

with $\Psi(R^{eb})$ as defined in (3.23):

$$\Psi(R^{eb}) = \Gamma^\top \left(I_3 \otimes R^{eb\top}\right) H^\top H \left(I_3 \otimes R^{eb}\right) \Gamma.$$

The matrix from (5.6) is block diagonal. Such block diagonal matrix is positive definite if and only if each diagonal block is positive definite. The upper diagonal six by six block is an identity matrix, which is positive definite. The matrix $\Psi(R^{eb})$ is symmetric, thus that matrix can be determined to be positive definite if the smallest eigenvalue is positive. In Section 3.3 it is shown that $\Psi(R^{eb}$ is positive definite with eigenvalues independent of R^{eb} both when using (3.16) or (3.14) for H. In the lower three by three block of (5.6) the positive definite matrix $\Psi(R^{eb})$ is pre- and post-multiplied by $R^{eb}J^{-1}$ and the transpose of $R^{eb}J^{-1}$ respectively. It can be noted that R^{eb} acts as a similarity transformation, which does not alter the eigenvalues. The matrix J^{-1} can change the eigenvalues and thus can affect the lower bound of the lower three by three block. A lower bound of the lower three by three block can be given as the smallest eigenvalue of J^{-2} times the smallest eigenvalue of $\Psi(R^{eb})$. This lower bound should be positive since J and thus J^{-2} are positive definite matrices. Therefore, there has to exist an $\alpha > 0$ which satisfies (5.6).

The two sufficient requirements to show UCO are thus satisfied under the assumption that L^e is always bounded. This assumption might seem reasonable from physical perspective. However, this required assumption could pose stability issues when this observer is combined with a state feedback controller into output feedback. Namely, due to this assumption the combination of the observer proposed in this section with a stabilizing state feedback, which uses the estimated state instead of the true state, does not completely decouple their dynamics as with certainty equivalence. Such interactions could then cause that $\tilde{L}^e$ diverges, which subsequently could cause L^e to diverge or vice versa.

5.2 Minimal Lyapunov based observer

In this section the observer structure from Chapter 4 is applied to the minimal model from (5.1). The proposed correction terms and the resulting observer dynamics are derived in Subsection 5.2.1. The equilibria resulting from this dynamics are analyzed in Subsection 5.2.2.

5.2.1 Observer dynamics

Adapting the the proposed observer dynamics from Section 4.1 to the minimal model from (5.1) using (4.4), (4.5), $W = I_3$ and (4.7) yields

$$\dot{\hat{R}}^{eb} = \hat{R}^{eb} S(J^{-1} R^{eb\top} \hat{L}^e + \delta_R), \tag{5.7a}$$

$$\dot{\hat{L}}^e = R^{eb} \tau^b + \delta_L, \tag{5.7b}$$

with $\delta_R, \delta_L \, \mathbb{R}^3$ the not yet defined correction terms. Expressions for these correction terms can be derived when considering the proposed Lyapunov function and its time derivative. Applying the proposed Lyapunov function from Section (4.2) to the minimal model, using $\tilde{x} = \tilde{L}^e$, gives

$$V = \sum_{i=1}^{n} \frac{k_i}{2} \left(\tilde{R}^{eb} \, v_i - v_i \right)^\top \left(\tilde{R}^{eb} \, v_i - v_i \right) + \frac{1}{2} \tilde{L}^{e\top} \Gamma^{-1} \tilde{L}^e, \tag{5.8}$$

with $\Gamma \in \mathbb{R}^{3\times 3}$. The corresponding time derivative of (5.8) can be obtained by substituting (4.10) with $W = I_3$ and $\delta_x = \delta_L$ in (4.18)

$$\dot{V} = \left(J^{-1} R^{eb\top} \tilde{L}^e + \delta_R \right)^\top \Pi + \tilde{L}^{e\top} \Gamma^{-1} \delta_L, \tag{5.9}$$

with Π as defined in (4.17):

$$\Pi = \sum_{i=1}^{n} k_i \left(\hat{R}^{eb\top} v_i \right) \times \left(R^{eb\top} v_i \right).$$

The minimal model only has the rotation matrix R^{eb} as output and therefore it is not be possible to use $\tilde{L}^e$ in the expressions for the correction terms. Under this constraint (5.9) can only be made negative semi-definite, where δ_R and δ_L can at best be chosen such that the terms linear in $\tilde{L}^e$ cancel and that (5.9) is negative definite in Π, resulting in

$$\delta_R = -\Delta \, \Pi, \tag{5.10a}$$

$$\delta_L = -\Gamma \, R^{eb} \, J^{-1} \Pi, \tag{5.10b}$$

with $\Delta \in \mathbb{R}^{3\times 3}$ and $\Delta = \Delta^\top \succ 0$. Substituting (5.10) in (5.9) gives

$$\dot{V} = -\Pi^\top \Delta \, \Pi, \tag{5.11}$$

which is negative semi-definite in the entire error state and thus shows that the error dynamics is Lyapunov stable. Substituting (5.10) into (5.7a) and (5.7b) therefore gives the following proposed observer dynamics for the minimal model from (5.1)

$$\dot{\hat{R}}^{eb} = \hat{R}^{eb} S(J^{-1} R^{eb\top} \hat{L}^e - \Delta \, \Pi), \tag{5.12a}$$

$$\dot{\hat{L}}^e = R^{eb} \tau^b - \Gamma \, R^{eb} \, J^{-1} \Pi. \tag{5.12b}$$

5.2.2 Analysis of the equilibria

The equilibria of the error dynamics of this observer can be found by solving the relevant equations from (4.20) using (5.10) and $W = I_3$, yielding

$$J^{-1} {R^{eb}}^\top \tilde{L}^e - \Delta\,\Pi = 0_{3\times1}, \tag{5.13a}$$

$$-\Gamma\, R^{eb}\, J^{-1}\Pi = 0_{3\times1}. \tag{5.13b}$$

By using the assumption that $\Gamma \succ 0$ it follows from (5.13b) that $\Pi = 0_{3\times1}$. Substituting this in (5.13a) yields $\tilde{L}^e = 0_{3\times1}$. Therefore, the assumptions on the equilibria from Section 4.3 are satisfied and the attitude at the equilibria in exponential coordinates u are the origin $\|u\| = 0$ and the eigenvectors of M defined in (4.25) normalized to the length of π.

Besides equilibria the error dynamics could also have limit cycles. As discussed in Section 4.4 this would require that the time derivative of the Lyapunov function from (5.11) is equal to zero along the entire trajectory of the limit cycle. Solving for when (5.11) is equal to zero yields $\Pi = 0_{3\times1}$, which as shown for the equilibria implies that attitude error is constant which in turn also requires $\tilde{L}^e = 0_{3\times1}$. However, this is equivalent to the equilibria and therefore no limit cycles in the error dynamics can exist.

The linearization matrices of the equilibria can be obtained by substituting (5.10) and $W = I_3$ in (4.29), such that for this observer (4.52) and (4.53), while leaving out the sub-matrices related to the unused states, can be written as

$$A_0 = \begin{bmatrix} R^{eb}\Delta {R^{eb}}^\top N & R^{eb} J^{-1} {R^{eb}}^\top \\ \Gamma\, R^{eb}\, J^{-1}\, {R^{eb}}^\top N & 0_{3\times3} \end{bmatrix}, \tag{5.14}$$

with $N = M - \kappa\, I_3$ and

$$A_\pi = \begin{bmatrix} -Q\, R^{eb}\Delta\, T & Q\, R^{eb} J^{-1} {R^{eb}}^\top \\ -\Gamma\, R^{eb}\, J^{-1}\, T & 0_{3\times3} \end{bmatrix}, \tag{5.15}$$

with Q and T as defined in (4.54) and (4.54b) respectively.

Local exponential stability of the equilibrium point with $\|u\| = 0$ can be shown by using the following matrix P_0 in the quadratic function from (4.56)

$$P_0 = \begin{bmatrix} -N & -\beta\, I_3 \\ -\beta\, I_3 & \Gamma^{-1} \end{bmatrix}, \tag{5.16}$$

where the two three by three diagonal blocks $-N$ and Γ^{-1} are from the initial guess given in (4.58). In order for (4.56) to be a common quadratic Lyapunov function (CQLF) it is required that (5.16) is positive definite. It can be noted that Γ and $-N = \kappa\, I_3 - M$ are positive definite. When using Schur's complement formula, which states that the following statements are equivalent: [31]

$$\left\{ \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix} \prec 0, \right. \quad \left\{ \begin{array}{c} \Phi_{22} \prec 0, \\ \Phi_{11} - \Phi_{12}\Phi_{22}^{-1}\Phi_{12}^\top \prec 0, \end{array} \right.$$

it can be shown that $P_0 \succ 0$ using (5.16) is equivalent to showing that $-N-\beta^2\,\Gamma \succ 0$. Since Γ is positive definite there should exist a full rank Λ such that $\Lambda\,\Gamma\,\Lambda = I_3$. By pre- and post-multiplying the equivalent inequality with Λ yields $Z-\beta^2\,I_3 \succ 0$, with $Z = -\Lambda\,N\,\Lambda$ a positive definite matrix. Therefore, the equivalent inequality is satisfied when $-\sqrt{\mu} < \beta < \sqrt{\mu}$, with μ the smallest eigenvalue of Z which should be greater than zero.

The time derivative of the CQLF from (4.56) is negative definite when (4.57) is satisfied:

$$A_0^\top P_0 + P_0\,A_0 \prec 0\;\forall\,R^{eb} \in SO(3).$$

Evaluating the left hand side of that inequality when using (5.14) and (5.16) yields

$$A_0^\top P_0 + P_0\,A_0 = -\begin{bmatrix} 2\,N\,\Delta_R N + \beta\,(\Gamma\,J_R N + N\,J_R\Gamma) & \beta\,N\,\Delta_R \\ \beta\,\Delta_R N & 2\,\beta\,J_R \end{bmatrix}, \tag{5.17}$$

with $\Delta_R = R^{eb}\Delta\,{R^{eb}}^\top$, $J_R = R^{eb}J^{-1}\,{R^{eb}}^\top$ and $\Delta_R, J_R \succ 0\;\forall\,R^{eb} \in SO(3)$. The bottom right three by three block of (5.17) is negative definite if $\beta > 0$. Therefore, by applying Schur's complement again it can be shown that the inequality from (4.57) using (5.17) is equivalent to $\beta > 0$ and

$$4\,N\,\Delta_R N + \beta\left(2\,\Gamma\,J_R N + 2\,N\,J_R\Gamma - N\,\Delta_R J_R^{-1}\Delta_R N\right) \succ 0\;\forall\,R^{eb} \in SO(3). \tag{5.18}$$

It can be noted that there should exist an $\epsilon > 0$ such that $N\,\Delta_R N \succeq \epsilon\,I_3\;\forall\,R^{eb} \in SO(3)$. The matrix in (5.18) multiplied by β might be negative definite $\forall\,R^{eb} \in SO(3)$, for example when $\Gamma = -N$, but should remain bounded. Therefore, there should exist a $\beta^* > 0$ such that for $0 < \beta < \beta^*$ (5.18) is satisfied $\forall\,R^{eb} \in SO(3)$.

The values for β for which both $P_0 \succ 0$ using (5.16) and (4.57) using (5.14) and (5.16) are satisfied can thus be written as $0 < \beta < \eta$, with $\eta = \min(\sqrt{\mu}, \beta^*) > 0$. Therefore, the CQLF can be made positive definite, while its time derivative is negative definite, whose bounds can be used to show local exponential stability. It might be worth investigating in future research which value for β maximizes the predicted local exponential convergence rate, since this might also give some insights into which observer parameters affect this convergence rate the most.

The other equilibria with $\|u\| = \pi$ can be shown to be unstable by using Theorem 1 with the quadratic function from (4.56) with (4.59) for the continuously differentiable function

$$V(x) = x^\top \begin{bmatrix} U\left((\kappa-\lambda_i)\,e_i\,e_i^\top + \frac{4}{\pi^2}(\Omega-\lambda_i\,I_3)\right)U^\top & 0_{3\times3} \\ 0_{3\times3} & -\Gamma^{-1} \end{bmatrix} x, \tag{5.19}$$

with $x = \begin{bmatrix} (u-\bar{u})^\top & {\tilde{L}^e}{}^\top \end{bmatrix}^\top$ and $\bar{u}$ the value of u at each of the considered equilibria with $\|u\| = \pi$. In Section 4.4 it is shown that the upper left three by three block of the matrix in (5.19) has at least one positive eigenvalue and thus set $\mathcal{U} = \{x \in \mathbb{R}^6 \,|\, \|x\| \le r \,|\, V(x) > 0\}$ should be nonempty.

The time derivative of (5.19) can be obtained by using (5.15) and (4.59) in (4.56), resulting in

$$\dot{V}(x) = 2\,x^\top \begin{bmatrix} T^\top\Delta\,T & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix} x, \tag{5.20}$$

with T as defined in (4.54b). It can be shown that T is a full rank matrix, so the upper left three by three matrix from (5.20) is positive definite. It can be noted that any point in the set $\mathcal{U}$ requires that $\|u - \bar{u}\| > 0$. Therefore, by combining this with the positive definiteness of the upper left three by three matrix from (5.20) it follows that $\dot{V}(x) > 0 \; \forall x \in \mathcal{U}$ and according to Theorem 1 this implies that each nonzero equilibrium of the minimal model is unstable.

The error dynamics for the observer dynamics from (5.12a) and (5.12b) have thus been shown to be Lyapunov stable, have no limit cycles, the equilibrium with $\|u\| = 0$ is locally exponentially stable and the equilibria with $\|u\| = \pi$ are unstable. This suggests that nearly all initial conditions should eventually converge to the equilibrium with $\|u\| = 0$, at which the estimation error is zero. The basins of attraction of the unstable equilibria are not empty, but are of a lower dimension than the entire state space. This implies that the basin of attraction of the stable equilibrium spans nearly the entire state space. This nearly global result is similar to that of a damped pendulum, for which the downward and upward positions are a stable and unstable equilibrium respectively.

5.3 Summary

In Section 5.1 it is shown how the observer structure from Chapter 3 can be adapted to minimal model from (5.1). However, when verifying the sufficient conditions, which would show global exponential stability, it is required to assume that rate of rotation, so the angular velocity or angular momentum, are bounded. For the resulting observer in Section 5.2, obtained by applying the observer structure from Chapter 4 to the minimal model, no such assumption is required but only a nearly global result is obtained.

Chapter 6

Biased observers

In this chapter the observer structures from Chapter 3 and 4 are applied to the biased model for the attitude dynamics from Section 2.3.2. Similar to the previous chapter, the observer structure from Chapter 3 is first applied to this model in Section 6.1 and in Section 6.2 the observer structure from Chapter 4. In Section 2.3.2 it is shown how the model can be formulated as linear time varying (LTV) and shown under what condition it satisfies the sufficient conditions. In Section 6.2 the Lyapunov based observer correction terms are derived and analyzed.

The biased model that is considered in this chapter is defined in (2.14) as:

$$
\begin{aligned}
\dot{R}^{eb} &= R^{eb}\, S(J^{-1} R^{eb\top} L^e), && (6.1a)\\
\dot{L}^e &= R^{eb} u, && (6.1b)\\
\dot{b}^b &= 0, && (6.1c)\\
u &= \tau^b, && (6.1d)\\
y_1 &= R^{eb}, && (6.1e)\\
y_2 &= J^{-1} R^{eb\top} L^e + b^b, && (6.1f)
\end{aligned}
$$

with u the known input of the system and, y_1 and y_2 the outputs of the system and J the known mass moment of inertia matrix.

6.1 Biased linear time varying observer

In this section the observer structure from Chapter 3 is applied to the biased model from (6.1). The transformation of the biased model to a LTV model and the resulting observer dynamics are first discussed in Subsection 6.1.1. This is followed by Subsection 6.1.2, in which it is shown under which assumptions the sufficient conditions of the used observer structure are satisfied.

6.1.1 Observer dynamics

Similar to Section 5.1.1 it is required that the considered model can be formulated to fit the LTV model from (3.1):

$$\begin{cases} \dot{x}(t) = A(R^{eb})\, x(t) + B(R^{eb})\, u(t), \\ y(t) = C(R^{eb})\, x(t). \end{cases}$$

This can be done by defining the state vector $x(t)$ and equivalent output $y(t)$ as

$$x(t) := \begin{bmatrix} \rho_r \\ L^e \\ b^b \end{bmatrix} \in \mathbb{R}^{12}, \quad y(t) := \begin{bmatrix} \rho_r \\ J^{-1} {R^{eb}}^\top L^e + b^b \end{bmatrix} \in \mathbb{R}^9,$$

with ρ_r as defined in (3.17). By combining (3.18), (2.14b), (2.14c) and $\omega = J^{-1} {R^{eb}}^\top L^e$ it can be shown that using the following matrices for the LTV model yield a model that is equivalent to the biased model

$$\begin{cases} A(R^{eb}) := \begin{bmatrix} 0_{6\times6} & H\,(I_3 \otimes R^{eb})\,\Gamma\, J^{-1} {R^{eb}}^\top & 0_{6\times3} \\ 0_{3\times6} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times6} & 0_{3\times3} & 0_{3\times3} \end{bmatrix}, \quad B(R^{eb}) := \begin{bmatrix} 0_{6\times3} \\ R^{eb} \\ 0_{3\times3} \end{bmatrix}, \\ C(R^{eb}) := \begin{bmatrix} I_6 & 0_{6\times3} & 0_{6\times3} \\ 0_{3\times6} & J^{-1} {R^{eb}}^\top & I_3 \end{bmatrix}. \end{cases} \tag{6.2}$$

By using (6.2) the observer structure from Chapter 3 can be expressed using (3.2) and (3.3):

$$\dot{\hat{x}}(t) = A(R^{eb})\, \hat{x}(t) + B(R^{eb})\, u(t) - K(t)\, \left[C(R^{eb})\, \hat{x}(t) - y(t)\right],$$

$$\begin{cases} \dot{M}(t) = A(R^{eb})M(t) + M(t)A^\top(R^{eb}) - M(t)C^\top(R^{eb})W^{-1}C(R^{eb})M(t) + V + \delta M(t), \\ M(t_0) = M_0 = M_0^\top \succ 0, \quad W = W^\top \succ 0, \\ K(t) = M(t)\, C^\top(R^{eb})\, W^{-1}, \end{cases}$$

with $\hat{x}(t)$ the estimate of $x(t)$, $M_0, V \in \mathbb{R}^{12\times12}$ and $W \in \mathbb{R}^{9\times9}$.

6.1.2 Sufficient condition verification

Similar to Subsection 5.1.2 global converges of the observer dynamics is shown if the matrices defined in (6.2) can be shown to be bounded and that the associated LTV model is uniformly completely observable (UCO). By using the same reasoning as in Subsection 5.1.2 it can be concluded that the obtained LTV model has bounded matrices. As discussed in Section 5.1.2 the obtained LTV model can be shown to be UCO using the generalized observability matrix obtained by substituting (6.2) in (3.22), which yields

$$Q(t) = \begin{bmatrix} I_6 & 0_{6\times3} & 0_{6\times3} \\ 0_{3\times6} & J^{-1} {R^{eb}}^\top & I_3 \\ 0_{6\times6} & H\,(I_3 \otimes R^{eb})\,\Gamma\, J^{-1} {R^{eb}}^\top & 0_{6\times3} \\ 0_{3\times6} & -J^{-1} S(J^{-1} {R^{eb}}^\top L^e)\, {R^{eb}}^\top & 0_{3\times3} \end{bmatrix}, \tag{6.3}$$

with

$$L_q(t) = \begin{bmatrix} 0_{6\times 6} & H\,(I_3 \otimes R^{eb})\,\Gamma\,J^{-1}{R^{eb}}^\top & 0_{6\times 3} \\ 0_{3\times 6} & -J^{-1}S(J^{-1}{R^{eb}}^\top L^e)\,{R^{eb}}^\top & 0_{3\times 3} \end{bmatrix}. \tag{6.4}$$

From (6.3) and (6.4) it can be concluded that the obtained LTV model is UCO if (3.21) using (6.3) is satisfied and if $L_q(t)$ from (6.4) and its time derivative are bounded.

The first six rows of $L_q(t)$ from (6.4) are a sub-matrix of $A(R^{eb})$ from (6.2), which has been shown to be bounded. When showing that the last three rows and the time derivative the first six rows of (6.4) are also bounded one can use a similar argument as for the boundedness of the time derivative of $L_q(t)$ in Section 5.1.2, which thus requires the assumption that L^e is bounded. The time derivative of the last three rows of (6.4) also requires to take the time derivative of L^e, which also adds τ^b to the resulting expression. Therefore, in order for $L_q(t)$ from (6.4) and its time derivative to be bounded requires the assumptions that L^e and τ^b are bounded.

Substituting (5.3) in (3.21) yields

$$\begin{bmatrix} I_6 & 0_{6\times 3} & 0_{6\times 3} \\ 0_{3\times 6} & R^{eb}J^{-1}(I_3 + \Psi(R^{eb}))\,J^{-1}{R^{eb}}^\top + \Omega^\top\Omega & R^{eb}J^{-1} \\ 0_{3\times 6} & J^{-1}{R^{eb}}^\top & I_3 \end{bmatrix} \succeq \alpha\, I_{12}, \tag{6.5}$$

with $\Omega = J^{-1}S(J^{-1}{R^{eb}}^\top L^e)\,{R^{eb}}^\top$ and $\Psi(R^{eb})$ as defined in (3.23):

$$\Psi(R^{eb}) = \Gamma^\top \left(I_3 \otimes {R^{eb}}^\top\right) H^\top H \left(I_3 \otimes R^{eb}\right) \Gamma.$$

The matrix on the left hand side of the inequality of (6.5) is block diagonal, with the first block the upper six by six matrix on the diagonal and the second and last block the lower six by six matrix on the diagonal. Therefore, the inequality of (6.5) is satisfied if both blocks are positive definite. The upper six by six diagonal block is an identity matrix, which is positive definite. The lower six by six diagonal matrix can also be written as

$$\begin{bmatrix} R^{eb}J^{-1}(I_3 + \Psi(R^{eb}))\,J^{-1}{R^{eb}}^\top + \Omega^\top\Omega & R^{eb}J^{-1} \\ J^{-1}{R^{eb}}^\top & I_3 \end{bmatrix} = \Sigma + \Lambda, \tag{6.6}$$

with

$$\Sigma = \begin{bmatrix} \Omega^\top\Omega & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} \end{bmatrix}, \tag{6.7a}$$

$$\Lambda = T\,\Xi^\top\Theta\,\Xi\,T^\top, \tag{6.7b}$$

$$T = \begin{bmatrix} R^{eb} & 0_{3\times 3} \\ 0_{3\times 3} & I_3 \end{bmatrix}, \tag{6.7c}$$

$$\Xi = \begin{bmatrix} J^{-1} & 0_{3\times 3} \\ J^{-1} & I_3 \end{bmatrix}, \tag{6.7d}$$

$$\Theta = \begin{bmatrix} \Psi(R^{eb}) & 0_{3\times 3} \\ 0_{3\times 3} & I_3 \end{bmatrix}. \tag{6.7e}$$

For the matrix from (6.7c) it holds that $T^\top = T^{-1}$, thus in (6.7b) T can be seen as a similarity transformation. The matrix Ξ from (6.7d) is constant, square and full rank. In Section 3.3 it is shown that $\Psi(R^{eb}$ is positive definite with eigenvalues independent of R^{eb} both when using (3.16) or (3.14) for H. Therefore, Θ from (6.7e) should also be positive definite with constant eigenvalues. By combining the properties of T, Ξ and Θ yields that a lower bound for Λ from (6.7b) can be given as minimum of one and the smallest eigenvalue of J^{-2}, times the minimum of one and smallest eigenvalue of $\Psi(R^{eb})$. Similar to Section 5.1.2 it follows that this lower bound should be positive. The matrix Σ from (6.7a) is positive semi-definite. Therefore, the lower six by six block from (6.5), which can also be expressed using (6.6), is a summation of a positive definite and a positive semi-definite matrix, which should also be positive definite. Additionally, the lower bound of Λ should also be a lower bound of the lower six by six block from (6.5). Therefore, both diagonal blocks of (6.5) have been shown to be positive definite, so there should exists a positive constant α such that $\alpha\, I$ is a lower bound for (6.5).

The two sufficient requirements to show UCO are thus satisfied under the assumption that L^e and τ^b are always bounded. It might be possible to show that the assumption, that τ^b is bounded, can be omitted. Namely, if the last three rows of (6.3) are omitted the resulting inequality equivalent to (6.5) would still always be satisfied. The removal of these rows would then also remove the part of that part of the time derivative of $L_q(t)$ from (6.4) which would contain τ^b. However, more research would be required in order to determine if this is allowed. Even if the assumption on τ^b could be omitted the assumption on L^e would still remain. As discussed at the end of Subsection 5.1.2 this assumption could have stability implications on when combining the observer from this section into output feedback.

6.2 Biased Lyapunov based observer

In this section the observer structure from Chapter 4 is applied to the biased model from (6.1). The proposed correction terms and the resulting observer dynamics are derived in Subsection 6.2.1. The equilibria resulting from this dynamics are analyzed in Subsection 6.2.2.

6.2.1 Observer dynamics

Adapting the the proposed observer dynamics from Section 4.1 to the biased model from (6.1) using (4.4), (4.5), (4.7) and (4.8) yields

$$\dot{\hat{R}}^{eb} = \hat{R}^{eb} S(W\, J^{-1} R^{eb\top} \hat{L}^e + (I_3 - W)\,(z^b - \hat{b}^b) + \delta_R), \tag{6.8a}$$

$$\dot{\hat{L}}^e = R^{eb}\tau^b + \delta_L, \tag{6.8b}$$

$$\dot{\hat{b}}^b = \delta_b, \tag{6.8c}$$

with W any matrix in $\mathbb{R}^{3\times 3}$ and $\delta_R, \delta_L, \delta_b\, \mathbb{R}^3$ the not yet defined correction terms. The proposed Lyapunov function and its time derivative can be used to help choose expressions for these correction terms. Applying the proposed Lyapunov function from Section (4.2) to the biased model, using $\tilde{x} = \begin{bmatrix} \tilde{L}^{e\top} & \tilde{b}^{b\top} \end{bmatrix}^\top$, gives

$$V = \sum_{i=1}^{n} \frac{k_i}{2} \left(\tilde{R}^{eb} v_i - v_i \right)^\top \left(\tilde{R}^{eb} v_i - v_i \right) + \frac{1}{2} \begin{bmatrix} \tilde{L}^e \\ \tilde{b}^b \end{bmatrix}^\top \Gamma^{-1} \begin{bmatrix} \tilde{L}^e \\ \tilde{b}^b \end{bmatrix}, \tag{6.9}$$

with $\Gamma \in \mathbb{R}^{6\times 6}$. The corresponding time derivative of (6.9) can be obtained by substituting (4.10), (4.19) solved for $\tilde{L}^e$ and $\delta_x = \begin{bmatrix} \delta_L^\top & \delta_b^\top \end{bmatrix}^\top$ in (4.18)

$$\dot{V} = \left(W \tilde{z}^b - \tilde{b}^b + \delta_R \right)^\top \Pi + \begin{bmatrix} \tilde{z}^b \\ \tilde{b}^b \end{bmatrix}^\top \begin{bmatrix} J\, R^{eb\top} & 0_{3\times 3} \\ -J\, R^{eb\top} & I_3 \end{bmatrix} \Gamma^{-1} \begin{bmatrix} \delta_L \\ \delta_b \end{bmatrix}, \tag{6.10}$$

with Π as defined in (4.17):

$$\Pi = \sum_{i=1}^{n} k_i \left(\hat{R}^{eb\top} v_i \right) \times \left(R^{eb\top} v_i \right).$$

The biased model has the rotation matrix R^{eb} and biased angular velocity measurement z^b as outputs. Therefore, it is not be possible to use $\tilde{b}^b$ in any of the expressions for the correction terms and thus the remaining linear terms in $\tilde{b}^b$ from (6.10) could make the time derivative indefinite. However, the correction terms can be chosen such that the time derivative of the Lyapunov function becomes negative definite in Π and $\tilde{z}^b$ and thus negative semi-definite in the entire error state. Such choice for the correction terms is given by

$$\delta_R = - \begin{bmatrix} \Phi_{11} & 2\,\Phi_{12} + W + X^\top \end{bmatrix} \begin{bmatrix} \Pi \\ \tilde{z}^b \end{bmatrix}, \tag{6.11a}$$

$$\begin{bmatrix} \delta_L \\ \delta_b \end{bmatrix} = \Gamma \begin{bmatrix} R^{eb} J^{-1} & 0 \\ I_3 & I_3 \end{bmatrix} \begin{bmatrix} X & -\Phi_{22} \\ I_3 & 0_{3\times 3} \end{bmatrix} \begin{bmatrix} \Pi \\ \tilde{z}^b \end{bmatrix}, \tag{6.11b}$$

with X any matrix in $\mathbb{R}^{3\times 3}$ and the matrix

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} = \Theta \in \mathbb{R}^{6\times 6},$$

such that $\Theta = \Theta^\top \succ 0$. Substituting (6.11) in (6.10) gives

$$\dot{V} = - \begin{bmatrix} \Pi \\ \tilde{z}^b \end{bmatrix}^\top \Theta \begin{bmatrix} \Pi \\ \tilde{z}^b \end{bmatrix}, \tag{6.12}$$

which is negative semi-definite in the entire error state independent of X and thus shows that the error dynamics is Lyapunov stable. It can be noted that an equivalent result can be obtained when using (4.19) to solve for $\tilde{b}^b$ instead of $\tilde{L}^e$ in (6.10). Although both the Lyapunov function and the final expression for its derivative are not a function of X it can be noted that the error dynamics does depend on X. Substituting (6.11) in (6.13a), (6.8b) and (6.8c) therefore gives the following proposed observer dynamics for the biased model from (6.1)

$$\dot{\hat{R}}^{eb} = \hat{R}^{eb} S(z^b - \hat{b}^b - \Phi_{11}\, \Pi - (2\,\Phi_{12} + X^\top)\, \tilde{z}^b), \tag{6.13a}$$

$$\begin{bmatrix} \dot{\hat{L}}^e \\ \dot{\hat{b}}^b \end{bmatrix} = \begin{bmatrix} R^{eb}\tau^b \\ 0_{3\times 1} \end{bmatrix} + \Gamma \begin{bmatrix} R^{eb}J^{-1} & 0_{3\times 3} \\ I_3 & I_3 \end{bmatrix} \begin{bmatrix} X & -\Phi_{22} \\ I_3 & 0_{3\times 3} \end{bmatrix} \begin{bmatrix} \Pi \\ \tilde{z}^b \end{bmatrix}, \tag{6.13b}$$

with $\tilde{z}^b = J^{-1} R^{eb\top} \hat{L}^e + \hat{b}^b - z^b$ and for which it can be noted that the weighting matrix W has disappeared. However, the value of the weighting matrix W does still affect the value given by this observer for the estimated angular velocity from (4.5).

6.2.2 Analysis of the equilibria

The equilibria of the error dynamics of this observer can be found by solving the relevant equations from (4.20) using (6.11), yielding

$$\tilde{b}^b + \Phi_{11}\Pi + (2\,\Phi_{12} + X^\top)\, \tilde{z}^b = 0_{3\times 1}, \tag{6.14a}$$

$$\Gamma \begin{bmatrix} R^{eb}J^{-1} & 0_{3\times 3} \\ I_3 & I_3 \end{bmatrix} \begin{bmatrix} X & -\Phi_{22} \\ I_3 & 0_{3\times 3} \end{bmatrix} \begin{bmatrix} \Pi \\ \tilde{z}^b \end{bmatrix} = 0_{6\times 1}. \tag{6.14b}$$

By using the assumption that $\Gamma \succ 0$ and that the next matrix from (6.14b) containing R^{eb} is always square and full rank implies that (6.14b) is equivalent to $X\,\Pi - \Phi_{22}\tilde{z}^b = 0_{3\times 1}$ and $\Pi = 0_{3\times 1}$. The assumption that $\Theta = \Theta^\top \succ 0$ also implies that Φ_{22} needs to be full rank, thus solving (6.14b) yields $\Pi = 0_{3\times 1}$ and $\tilde{z}^b = 0_{3\times 1}$. Substituting this solution to (6.14b) in (6.14a) yields $\tilde{b}^b = 0_{3\times 1}$. Substituting $\tilde{z}^b = \tilde{b}^b = 0_{3\times 1}$ in (4.19) also yields $\tilde{L}^e = 0_{3\times 1}$. Therefore, the assumptions on the equilibria from Section 4.3 are satisfied and the attitude at the equilibria in exponential coordinates u are the origin $\|u\| = 0$ and the eigenvectors of M defined in (4.25) normalized to the length of π.

By using the same arguments as in Subsection 5.2.2 the existence of limit cycles in the error dynamics would for this observer require $\Pi = \tilde{z}^b = 0_{3\times 1}$, which would also make such limit cycles equivalent to the equilibria.

By substituting (6.11) in (4.29) yields that $A_R = -\Phi_{11}$, $B_R = -2\,\Phi_{12} - W - X^\top$ and

$$\begin{bmatrix} A_L & B_L \\ A_b & B_b \end{bmatrix} = \Gamma \begin{bmatrix} R^{eb}J^{-1} & 0_{3\times 3} \\ I_3 & I_3 \end{bmatrix} \begin{bmatrix} X & -\Phi_{22} \\ I_3 & 0_{3\times 3} \end{bmatrix}.$$

Therefore, the resulting expressions for the linearizations at the equilibria from (4.52) and (4.53) for the observer considered in this section can be written as

$$A_0 = \begin{bmatrix} R^{eb}\Phi_{11}R^{eb\top}N & -R^{eb}(2\,\Phi_{12} + X^\top)\, J^{-1}R^{eb\top} & -R^{eb}(2\,\Phi_{12} + X^\top + I_3) \\ -A_L\, R^{eb\top}N & B_L\, J^{-1}R^{eb\top} & B_L \\ -A_b\, R^{eb\top}N & B_b\, J^{-1}R^{eb\top} & B_b \end{bmatrix}, \tag{6.15}$$

with $N = M - \kappa\, I_3$ and

$$A_\pi = \begin{bmatrix} -QR^{eb}\Phi_{11}T & -QR^{eb}(2\,\Phi_{12} + X^\top)\,J^{-1}R^{eb^\top} & -QR^{eb}(2\,\Phi_{12} + X^\top + I_3) \\ A_L\,T & B_L\,J^{-1}R^{eb^\top} & B_L \\ A_b\,T & B_b\,J^{-1}R^{eb^\top} & B_b \end{bmatrix}, \tag{6.16}$$

with Q and T as defined in (4.54) and (4.54b) respectively.

When using P_0 from (4.58) for the initial guess for the common quadratic Lyapunov function (CQLF) together with (6.15) yields the following expression for the left hand side of (4.57)

$$A_0^\top P_0 + P_0\,A_0 = -2 \begin{bmatrix} N\,R^{eb} & 0_{3\times3} \\ 0_{3\times3} & -R^{eb}J^{-1} \\ 0_{3\times3} & -I_3 \end{bmatrix} \Theta \begin{bmatrix} N\,R^{eb} & 0_{3\times3} \\ 0_{3\times3} & -R^{eb}J^{-1} \\ 0_{3\times3} & -I_3 \end{bmatrix}^\top, \tag{6.17}$$

which is only negative semi-definite because $\Theta \in \mathbb{R}^{6\times6}$, while $A_0^\top P_0 + P_0\,A_0 \in \mathbb{R}^{9\times9}$. The candidate CQLF obtained from solving a set of linear matrix inequalities, as described in Section 4.4, yielded a valid solution every attempt. However, it was not possible to deduce a pattern in those solutions. The numerical results shown in Chapter 8 also reinforces the hypothesis that the equilibrium with $\|u\| = 0$ of the observer considered in this section is locally exponentially stable. Therefore, finding a CQLF in order to show local exponential stability near the equilibrium with $\|u\| = 0$ of the observer from this section could be a topic in future research.

Similar to Subsection 5.2.2 the other equilibria with $\|u\| = \pi$ can be shown to be unstable by using Theorem 1, (4.56) and (4.59) resulting in

$$V(x) = x^\top \begin{bmatrix} U\left((\kappa - \lambda_i)\,e_i\,e_i^\top + \frac{4}{\pi^2}(\Omega - \lambda_i\,I_3)\right)U^\top & 0_{3\times6} \\ 0_{6\times3} & -\Gamma^{-1} \end{bmatrix} x, \tag{6.18}$$

as the continuously differentiable function with $x = \begin{bmatrix} (u - \bar{u})^\top & \tilde{L}^{e^\top} & \tilde{b}^{b^\top} \end{bmatrix}^\top$ and $\bar{u}$ the value of u at each of the considered equilibria with $\|u\| = \pi$. In Section 4.4 it is shown that the upper left three by three block of the matrix in (6.18) has at least one positive eigenvalue and thus the set $\mathcal{U} = \{x \in \mathbb{R}^9 \,|\, \|x\| \leq r \,|\, V(x) > 0\}$ should be nonempty, with all elements in $\mathcal{U}$ satisfying that $\|u - \bar{u}\| > 0$.

The time derivative of (6.18) can be obtained by using

$$U\left((\kappa - \lambda_i)\,e_i\,e_i^\top + \frac{4}{\pi^2}(\Omega - \lambda_i\,I_3)\right)U^\top Q = -T^\top R^{eb^\top},$$

(6.16) and (4.59) in (4.56), resulting in

$$\dot{V}(x) = 2\,x^\top \mathcal{M}^\top \Theta\,\mathcal{M} x, \tag{6.19}$$

$$\mathcal{M} = \begin{bmatrix} T & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & J^{-1}R^{eb^\top} & I_3 \end{bmatrix}. \tag{6.20}$$

The null space of (6.20), such that $\mathcal{M}\,x = 0_{6\times1}$, is the set of all point such that $u = \bar{u}$ and $J^{-1}R^{eb^\top}\tilde{L}^e + \tilde{b}^b = \tilde{z}^b = 0_{3\times1}$. Therefore, $\dot{V}(x) > 0$ whenever

$\|u - \bar{u}\| > 0$ or $\|\tilde{z}^b\| > 0$. Combining this with the non-empty set $\mathcal{U}$ it follows that $\dot{V}(x) > 0 \; \forall \, x \in \mathcal{U}$ and according to Theorem 1 this implies that each nonzero equilibrium of the biased model is unstable.

The error dynamics for the observer dynamics from (6.13a) and (6.13b) have thus been shown to be Lyapunov stable, have no limit cycles and the equilibria with $\|u\| = \pi$ are unstable. Therefore, if it could also be shown that the equilibrium with $\|u\| = 0$ is locally exponentially stable a similar conclusion could be drawn about the basin of attraction of the equilibrium with $\|u\| = 0$ as in Subsection 5.2.2.

6.3 Summary

In Section 6.1 it is shown how the observer structure from Chapter 3 can be adapted to biased model from (6.1). However, when verifying the sufficient conditions, which would show global exponential stability, it is required to assume that rate of rotation, so the angular velocity or angular momentum, and the applied torque are bounded. For the resulting observer in Section 6.2, obtained by applying the observer structure from Chapter 4 to the biased model, no such assumption is required. However, the attempts to show that the equilibrium of the error dynamics with zero estimation error is locally exponentially stable have not yet succeeded. And thus one can also not yet draw the same conclusion as in Section 5.2 that there is a nearly global basin of attraction.

Chapter 7

Kinematic observers

In this chapter the observer structures from Chapter 3 and 4 are applied to the kinematic model for the attitude dynamics from Section 2.3.3. First, the observer structure from Chapter 3 is applied to this model in Section 7.1, in which it is shown how the model can be formulated as linear time varying and shown under what condition it satisfied the sufficient conditions. This is followed by Section 7.2, in which the observer structure from Chapter 4 is applied to the kinematic model and analyzed.

The kinematic model that is considered in this chapter is defined in (2.15) as:

$$\dot{R}^{eb} = R^{eb}\, S(u - b^b), \tag{7.1a}$$

$$\dot{b}^b = 0, \tag{7.1b}$$

$$u = z^b, \tag{7.1c}$$

$$y = R^{eb}, \tag{7.1d}$$

where the biased angular velocity measurement u is seen as the known input of the system and y the output of the system.

7.1 Kinematic linear time varying observer

In this section the observer structure from Chapter 3 is applied to the kinematic model from (7.1). The transformation of the kinematic model to a LTV model and the resulting observer dynamics are first discussed in Subsection 7.1.1. This is followed by Subsection 7.1.2, in which it is shown under which assumptions the sufficient conditions of the used observer structure are satisfied.

7.1.1 Observer dynamics

Similar to Section 5.1.1 and 6.1.1 it is required that the considered model can be formulated to fit the LTV model from (3.1):

$$\begin{cases} \dot{x}(t) = A(R^{eb})\, x(t) + B(R^{eb})\, u(t), \\ y(t) = C(R^{eb})\, x(t). \end{cases}$$

This can be done by defining the state vector $x(t)$ and equivalent output $y(t)$ as

$$x(t) := \begin{bmatrix} \rho_r \\ b^b \end{bmatrix} \in \mathbb{R}^9, \quad y(t) := \rho_r \in \mathbb{R}^6,$$

with ρ_r as defined in (3.17). By combining (3.18), (2.15b) and $\omega = u - b^b$ it can be shown that using the following matrices for the LTV model yield a model that is equivalent to the kinematic model

$$\begin{cases} A(R^{eb}) := \begin{bmatrix} 0_{6\times6} & -H\,(I_3 \otimes R^{eb})\,\Gamma \\ 0_{3\times6} & 0_{3\times3} \end{bmatrix}, \quad B(R^{eb}) := \begin{bmatrix} H\,(I_3 \otimes R^{eb})\,\Gamma \\ 0_{3\times3} \end{bmatrix}, \\ C(R^{eb}) := \begin{bmatrix} I_6 & 0_{6\times3} \end{bmatrix}. \end{cases} \tag{7.2}$$

By using (7.2) the observer structure from Chapter 3 can be expressed using (3.2) and (3.3):

$$\dot{\hat{x}}(t) = A(R^{eb})\,\hat{x}(t) + B(R^{eb})\,u(t) - K(t)\left[C(R^{eb})\,\hat{x}(t) - y(t)\right],$$

$$\begin{cases} \dot{M}(t) = A(R^{eb})M(t) + M(t)A^\top(R^{eb}) - M(t)C^\top(R^{eb})W^{-1}C(R^{eb})M(t) + V + \delta M(t), \\ M(t_0) = M_0 = M_0^\top \succ 0, \quad W = W^\top \succ 0, \\ K(t) = M(t)\,C^\top(R^{eb})\,W^{-1}, \end{cases}$$

with $\hat{x}(t)$ the estimate of $x(t)$, $M_0, V \in \mathbb{R}^{9\times9}$ and $W \in \mathbb{R}^{6\times6}$.

7.1.2 Sufficient condition verification

Similar to Subsection 5.1.2 and 6.1.2, global converges of the observer dynamics is shown if the matrices defined in (7.2) can be shown to be bounded and that the associated LTV model is uniformly completely observable (UCO). By using the same reasoning as in Subsection 5.1.2 it can be concluded that the obtained LTV model has bounded matrices. As discussed in more detail in Subsection 5.1.2 the obtained LTV model can be shown to be UCO using the generalized observability matrix obtained by substituting (7.2) in (3.22), which yields

$$Q(t) = \begin{bmatrix} I_6 & 0_{6\times3} \\ 0_{6\times6} & -H\,(I_3 \otimes R^{eb})\,\Gamma \end{bmatrix}, \tag{7.3}$$

with

$$L_q(t) = \begin{bmatrix} 0_{6\times6} & -H\,(I_3 \otimes R^{eb})\,\Gamma \end{bmatrix}. \tag{7.4}$$

From (7.3) and (7.4) it can be concluded that the obtained LTV model is UCO if (3.21) using (7.3) is satisfied and if $L_q(t)$ from (7.4) and its time derivative are bounded.

The entirety of $L_q(t)$ from (7.4) is a sub-matrix of $A(R^{eb})$ from (7.2), which has been shown to be bounded. However, similar to Subsection 5.1.2 the time derivative of $L_q(t)$ is a function of the rate of rotation $\omega = z^b - b^b$. Therefore, in order for the time derivative of $L_q(t)$ from (7.4) to also be bounded requires the assumptions that ω is bounded.

Substituting (7.3) in (3.21) yields

$$\begin{bmatrix} I_6 & 0_{6\times 3} \\ 0_{3\times 6} & \Psi(R^{eb}) \end{bmatrix} \succeq \alpha\, I_9, \tag{7.5}$$

with $\Psi(R^{eb})$ as defined in (3.23):

$$\Psi(R^{eb}) = \Gamma^\top \left(I_3 \otimes R^{eb\top}\right) H^\top H \left(I_3 \otimes R^{eb}\right) \Gamma.$$

Similar to Subsection 5.1.2 and 6.1.2 the matrix from (7.5) is block diagonal, with the first block the upper six by six matrix on the diagonal and the second and last block the lower three by three matrix on the diagonal. Therefore, the inequality of (7.5) is satisfied if both blocks are positive definite. The upper diagonal six by six block is an identity matrix, which is positive definite. In Section 3.3 it is shown that $\Psi(R^{eb})$ is positive definite with eigenvalues independent of R^{eb} both when using (3.16) or (3.14) for H. Therefore, there has to exist an $\alpha > 0$ which satisfies (7.5).

The two sufficient requirements to show UCO are thus satisfied under the assumption that $\omega = z^b - b^b$ is always bounded. As discussed at the end of Subsection 5.1.2 this assumption could have stability implications when combining the observer from this section into output feedback.

7.2 Kinematic Lyapunov based observer

In this section the observer structure from Chapter 4 is applied to the kinematic model from (7.1). The proposed correction terms and the resulting observer dynamics are derived in Subsection 7.2.1. The equilibria resulting from this dynamics are analyzed in Subsection 7.2.2.

7.2.1 Observer dynamics

Adapting the proposed observer dynamics from Section 4.1 to the kinematic model from (7.1) using (4.4), (4.5), $W = 0_{3\times 3}$ and (4.8) yields

$$\dot{\hat{R}}^{eb} = \hat{R}^{eb} S(z^b - \hat{b}^b + \delta_R), \tag{7.6a}$$

$$\dot{\hat{b}}^b = \delta_b, \tag{7.6b}$$

with $\delta_R, \delta_b\, \mathbb{R}^3$ the not yet defined correction terms. The proposed Lyapunov function and its time derivative can be used to help choose expressions for these correction terms. Applying the proposed Lyapunov function from Section (4.2) to the kinematic model, using $\tilde{x} = \tilde{b}^b$, gives

$$V = \sum_{i=1}^{n} \frac{k_i}{2} \left(\tilde{R}^{eb}\, v_i - v_i\right)^\top \left(\tilde{R}^{eb}\, v_i - v_i\right) + \frac{1}{2} \tilde{b}^{b\top} \Gamma^{-1} \tilde{b}^b, \tag{7.7}$$

with $\Gamma \in \mathbb{R}^{3\times 3}$. The corresponding time derivative of (7.7) can be obtained by substituting (4.10) with $W = 0_{3\times 3}$ and $\delta_x = \delta_b$ in (4.18)

$$\dot{V} = \left(\delta_R - \tilde{b}^b\right)^\top \Pi + \tilde{b}^{b\top}\Gamma^{-1}\delta_L, \tag{7.8}$$

with Π as defined in (4.17):

$$\Pi = \sum_{i=1}^{n} k_i \left(\hat{R}^{eb\top} v_i\right) \times \left(R^{eb\top} v_i\right).$$

The kinematic model only has the rotation matrix R^{eb} as output, since the biased angular velocity measurement is interpreted as the input. Therefore, it is not be possible to use $\tilde{b}^b$ in the expressions for the correction terms and thus the remaining linear terms in $\tilde{b}^b$ from (7.8) could make the time derivative indefinite. However, the correction terms can be chosen such that the time derivative of the Lyapunov function becomes negative definite in Π and thus negative semi-definite in the entire error state. Such choice for the correction terms is given by

$$\delta_R = -\Delta\,\Pi, \tag{7.9a}$$
$$\delta_b = \Gamma\,\Pi, \tag{7.9b}$$

with $\Delta \in \mathbb{R}^{3\times 3}$ and $\Delta = \Delta^\top \succ 0$. Substituting (7.9) in (7.8) gives

$$\dot{V} = -\Pi^\top \Delta\,\Pi, \tag{7.10}$$

which is negative semi-definite in the entire error state and thus shows that the error dynamics is Lyapunov stable. Substituting (7.9) into (7.6a) and (7.6b) therefore gives the following proposed observer dynamics for the kinematic model from (7.1)

$$\dot{\hat{R}}^{eb} = \hat{R}^{eb} S(z_b - \hat{b}^b - \Delta\,\Pi), \tag{7.11a}$$
$$\dot{\hat{b}}^b = \Gamma\,\Pi. \tag{7.11b}$$

7.2.2 Analysis of the equilibria

The equilibria of the error dynamics of this observer can be found by solving the relevant equations from (4.20) using (7.9) and $W = 0_{3\times 3}$, yielding

$$\tilde{b}^b + \Delta\,\Pi = 0_{3\times 1}, \tag{7.12a}$$
$$\Gamma\,\Pi = 0_{3\times 1}. \tag{7.12b}$$

By using the assumption that $\Gamma \succ 0$ it follows from (7.12b) that $\Pi = 0_{3\times 1}$. Substituting this in (7.12a) yields $\tilde{b}^b = 0_{3\times 1}$. Therefore, the assumptions on the equilibria from Section 4.3 are satisfied and the attitude at the equilibria in exponential coordinates u are the origin $\|u\| = 0$ and the eigenvectors of M defined in (4.25) normalized to the length of π.

By using the same arguments as in Subsection 5.2.2 the existence of limit cycles in the error dynamics would for this observer require $\Pi = 0_{3\times 1}$, which would also make such limit cycles equivalent to the equilibria.

By substituting (7.9) in (4.29) yields that $A_R = -\Delta$, $B_R = 0_{3\times 3}$, $A_b = \Gamma$ and $B_b = 0_{3\times 3}$. Therefore, the resulting expressions for the linearizations at the equilibria from (4.52) and (4.53) for the observer considered in this section can be written as

$$A_0 = \begin{bmatrix} R^{eb}\Delta\, {R^{eb}}^\top N & -R^{eb} \\ -\Gamma\, {R^{eb}}^\top N & 0_{3\times 3} \end{bmatrix}, \tag{7.13}$$

with $N = M - \kappa\, I_3$ and

$$A_\pi = \begin{bmatrix} -Q\, R^{eb}\Delta\, T & -Q\, R^{eb} \\ \Gamma\, T & 0_{3\times 3} \end{bmatrix}, \tag{7.14}$$

with Q and T as defined in (4.54) and (4.54b) respectively.

When using P_0 from (4.58) for the initial guess for the common quadratic Lyapunov function (CQLF) together with (7.13) yields the following expression for the left hand side of (4.57)

$$A_0^\top P_0 + P_0\, A_0 = -2 \begin{bmatrix} N\, R^{eb}\Delta\, {R^{eb}}^\top N & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} \end{bmatrix}, \tag{7.15}$$

which is only negative semi-definite. In order for a matrix to be positive definite it is required that any sub-matrix on its diagonal is also positive definite. When using a different constant matrix for P_0 the resulting lower right three by three sub-matrix on the diagonal of $A_0^\top P_0 + P_0\, A_0$ is always of the form $\mathcal{Q} = W^\top R^{eb} + {R^{eb}}^\top W$ with W a sub-matrix from P_0. However, for every $W \neq 0_{3\times 3}$ this matrix is not negative definite when considering all $R^{eb} \in SO(3)$, since $y^\top \mathcal{Q}\, y = 2\,(W\, y) \cdot (R^{eb} y)$ and there always exists a $R^{eb} \in SO(3)$ such that for $\|y\| \neq 0$ the vector $R^{eb} y$ points partially in the same direction as the vector $W\, y$.

An important difference between the linearizations of the minimal model from Subsection 5.2.2, for which local exponential stability is shown, and the kinematic model is that in linearizations of the minimal model the rotation matrix always appears in pairs, such as $R^{eb}\Delta\, {R^{eb}}^\top$ and $R^{eb} J^{-1} {R^{eb}}^\top$, while for the kinematic model the rotation matrix also appears alone. Therefore, it is attempted to use a time varying coordinate transformation $\tilde{x}' = \mathcal{T}\, \tilde{x}$, with

$$\mathcal{T} = \begin{bmatrix} I_3 & 0_{3\times 3} \\ 0_{3\times 3} & R^{eb} \end{bmatrix} \tag{7.16}$$

such that the transformed linearization is characterized with

$$A_0' = \begin{bmatrix} R^{eb}\Delta {R^{eb}}^\top N & -I_3 \\ -R^{eb}\Gamma\, {R^{eb}}^\top N & S({}^{be}\omega^e) \end{bmatrix}, \tag{7.17}$$

which introduces a potentially unbounded term ${}^{be}\omega^e$. Therefore, it might require assuming some upper bound on ${}^{be}\omega^e$, similar to the local stability analysis of the explicit complementary filter with bias correction [23]. Using this time varying coordinate transformation does mean that one no longer can use the initial guess for a CQLF. Therefore, similar to Subsection 6.2.2 it is also attempted to find a candidate CQLF by solving a set of linear matrix inequalities using bounded values for ${}^{be}\omega^{e}$. These sets of linear matrix inequalities yielded a valid solution every attempt. However, it was not possible to deduce a pattern in those solutions. The numerical results shown in Chapter 8 also reinforces the hypothesis that the equilibrium with $\|u\| = 0$ of the observer considered in this section is locally exponentially stable. Therefore, finding a CQLF in order to show local exponential stability near the equilibrium with $\|u\| = 0$ could also be a topic in future research for the observer considered in this section.

The other equilibria of the kinematic model can be shown to be unstable using Theorem 1 and (5.19) as the continuously differentiable function with $x = \begin{bmatrix}(u-\bar{u})^\top & \tilde{b}^{b\top}\end{bmatrix}^\top$. Similar to Subsection 5.2.2 and 6.2.2 it holds that the set $\mathcal{U} = \{x \in \mathbb{R}^9 \,|\, \|x\| \leq r \,|\, V(x) > 0\}$ should be nonempty, with all elements in $\mathcal{U}$ satisfying that $\|u - \bar{u}\| > 0$. The corresponding time derivative of the continuously differentiable function can be obtained by using (7.14) in (4.56), which results in the same expression as (5.20). Therefore, it also follows that each nonzero equilibrium of the kinematic model is unstable.

The error dynamics for the observer dynamics from (7.11a) and (7.6b) have thus been shown to be Lyapunov stable, have no limit cycles and the equilibria with $\|u\| = \pi$ are unstable. Therefore, if it could also be shown that the equilibrium with $\|u\| = 0$ is locally exponentially stable a similar conclusion could be drawn about the basin of attraction of the equilibrium with $\|u\| = 0$ as in Subsection 5.2.2.

7.3 Summary

In Section 7.1 it is shown how the observer structure from Chapter 3 can be adapted to kinematic model from (7.1). However, when verifying the sufficient conditions, which would show global exponential stability, it is required to assume that rate of rotation, so the angular velocity or angular momentum, is bounded. For the resulting observer in Section 7.2, obtained by applying the observer structure from Chapter 4 to the kinematic model, no such assumption is required. However, the attempts to show that the equilibrium of the error dynamics with zero estimation error is locally exponentially stable have not yet succeeded. And thus one can also not yet draw the same conclusion as in Section 5.2 that there is a nearly global basin of attraction.

Chapter 8

Numerical analysis

In this chapter the numerical side of the the ordinary differential equations from the models from Chapter 2 and the observers from Chapter 5, 6 and 7 is discussed. Firstly, in Section 8.1 it is discussed how the attitude part of the dynamics can be modified, such that it can be assured that coordinates used to represent the attitude remain well behaved. Secondly, in Section 8.2 a performance measure, in the form of a cost function, for the observers is proposed. That cost function can be used to choose the observer parameters and as a performance measure between the different observers. And lastly, in Section 8.3 some numerical simulation results are shown and discussed to demonstrate convergence of the estimation error to zero of the proposed observers.

8.1 Numerical implementation

It is difficult to ensure that rotation matrices remain in or close to $SO(3)$ when performing numerical simulations with rotation matrices for the attitude representation, whose dynamics is given in (2.4). Instead unit quaternions can be used. A quaternion is defined as $q \coloneqq q_w + q_x\, i + q_y\, j + q_z\, k \in \mathbb{H}$ with $q_w, q_x, q_y, q_z \in \mathbb{R}$, i, j, k the generalized imaginary numbers, such that $i^2 = j^2 = k^2 = i\,j\,k = -1$, and $\mathbb{H}$ the set of all quaternions [16, p. 22]. It can be noted that i, j and k do not commute with each other, so in general $q_1\, q_2 \neq q_2\, q_1$. A unit quaternion is a quaternion of unit length, where the length is defined as $\sqrt{q_w^2 + q_x^2 + q_y^2 + q_z^2}$. A quaternion can also be represented as a scalar-vector pair (r, v), with $r = q_w \in \mathbb{R}$ and $v = \begin{bmatrix} q_x & q_y & q_z \end{bmatrix}^\top \in \mathbb{R}^3$.

Vector coordinates $x \in \mathbb{R}^3$ can be rotated using a quaternion q by taking the vector component of the quaternion resulting from $q\,(0, x)\, q^{-1}$, with $q^{-1} = (r, -\vec{v})/\|q\|$ [16, p. 45]. For convenience this vector rotation is denoted with $q(x)$. From this definition of rotating x using unit quaternions it can be observed that both q and $-q$ yield the same rotated vector. So there is a double mapping from an attitude representation to a unit quaternion. However, there always is a unique mapping from a unit quaternion to a rotation matrix and is defined as [16, p. 45]

$$R = \begin{bmatrix} q_w^2 + q_x^2 - q_y^2 - q_z^2 & 2(q_x\, q_y - q_w\, q_z) & 2(q_x\, q_z + q_w\, q_y) \\ 2(q_x\, q_y + q_w\, q_z) & q_w^2 - q_x^2 + q_y^2 - q_z^2 & 2(q_y\, q_z - q_w\, q_x) \\ 2(q_x\, q_z - q_w\, q_y) & 2(q_y\, q_z + q_w\, q_x) & q_w^2 - q_x^2 - q_y^2 + q_z^2 \end{bmatrix}. \tag{8.1}$$

The notation q^{eb} is used to represent a unit quaternion which is equivalent to the rotation matrix R^{eb} and thus can transform vector coordinates expressed in the body frame b into vector coordinates expressed using the inertial frame e, according to $v^e = q^{eb}(v^b)$. The time derivative of the attitude of a rigid body, expressed as a unit quaternion, is given by [33]

$$\dot{q}^{eb} = \frac{1}{2}\, q^{eb}\, \left(0, {}^{be}\omega^b\right). \tag{8.2}$$

When implementing the product between quaternions, used for rotating vector coordinates and when evaluating (8.2), it is more convenient to treat each quaternion as a vector of four real numbers instead of generalized imaginary numbers. The real vector representation is denoted with $\overline{q^{eb}} = \begin{bmatrix} q_w^{eb} & q_x^{eb} & q_y^{eb} & q_z^{eb} \end{bmatrix}^\top \in \mathbb{R}^4$. The product of two quaternions, $q_1\, q_2$, can be shown to be equivalent to [16, p. 23]

$$\overline{q_1\, q_2} = \begin{bmatrix} r_1 & -\vec{v}_1^\top \\ \vec{v}_1 & r_1\, I_3 + S(\vec{v}_1) \end{bmatrix} \overline{q_2} = \begin{bmatrix} r_2 & -\vec{v}_2^\top \\ \vec{v}_2 & r_2\, I_3 - S(\vec{v}_2) \end{bmatrix} \overline{q_1}. \tag{8.3}$$

Therefore, the time derivative of $\overline{q^{eb}}$ can be obtained by combining (8.2) with (8.3), resulting in

$$\dot{\overline{q^{eb}}} = \mathcal{H}({}^{be}\omega^b)\, \overline{q^{eb}}, \tag{8.4}$$

with

$$\mathcal{H}(x) = \frac{1}{2} \begin{bmatrix} 0 & -x^\top \\ x & -S(x) \end{bmatrix}. \tag{8.5}$$

Numerically solving the dynamics of unit quaternions using (8.4) does not ensure that the quaternions remain of unit length. Instead, each quaternion could be normalized whenever it is used in other calculations. To ensure that the simulated quaternion remains close to unit length, in order to prevent the singularity at the origin, one could use specialized integration methods [34]. However, this would require a custom ordinary differential equations solver, which is hard to combine with common variable-time-step solvers, such as ode45 from Matlab. Instead a correction term can be added of the form $f(\rho)\,\overline{q^{eb}}$ with $\rho = \|\overline{q^{eb}}\|^2$, similar to Baumgarte stabilization of dynamical systems with constraints [3]. By combining this with (8.4) the modified dynamics of the quaternion can be written as

$$\dot{\overline{q^{eb}}} = \left(\mathcal{H}({}^{be}\omega^b) + f(\rho)\, I_4\right)\, \overline{q^{eb}}. \tag{8.6}$$

By using (8.6) it can be shown that time derivative of ρ can be written as

$$\begin{aligned} \dot{\rho} &= 2\, \overline{q^{eb}}^\top \dot{\overline{q^{eb}}}, && (8.7a) \\ &= 2\, \overline{q^{eb}}^\top \left(\mathcal{H}({}^{be}\omega^b) + f(\rho)\, I_4\right)\, \overline{q^{eb}}, && (8.7b) \\ &= 2\, \overline{q^{eb}}^\top \mathcal{H}({}^{be}\omega^b)\, \overline{q^{eb}} + 2\, f(\rho)\, \rho. && (8.7c) \end{aligned}$$

Combining (8.7c) with the fact that $\overline{q^{eb}}^\top \mathcal{H}({}^{be}\omega^b)\,\overline{q^{eb}} = 0$, it can be shown that (8.7) can be simplified to

$$\dot{\rho} = 2\,\rho\, f(\rho). \tag{8.8}$$

By defining the error from unit length as $e \coloneqq \rho - 1$ and choosing the correction term as one of the following expressions with $k > 0$

$$f(\rho) = -k\,\rho\,(\rho - 1), \tag{8.9a}$$
$$f(\rho) = -k\,(\rho - 1), \tag{8.9b}$$
$$f(\rho) = -k\,(\rho - 1)/\rho, \tag{8.9c}$$

it can be shown that when combining each expression from (8.9) with (8.8) yields

$$\dot{e} = -2\,k\,(e + 1)^2\, e, \tag{8.10a}$$
$$\dot{e} = -2\,k\,(e + 1)\, e, \tag{8.10b}$$
$$\dot{e} = -2\,k\, e, \tag{8.10c}$$

respectively. For each error dynamics from (8.10) it can be shown that it has an equilibrium at $e = 0$, which is locally exponentially stable with the same linearization $\dot{e} = -2\,k\,e$. The first two error dynamics from (8.10) also have another equilibrium at $e = -1$, which can be shown to be unstable, while the third has a singularity at $e = -1$. The third proposed correction term (8.9c) has been proposed before [13].

The normalized quaternion, defined by $q_n \coloneqq \overline{q^{eb}}/\sqrt{\rho}$, can be shown, by using (8.6) and (8.8), to have the following dynamics

$$\dot{q}_n = \frac{\dot{q}}{\sqrt{\rho}} - \frac{q}{2\,\rho\sqrt{\rho}}\dot{\rho} = \mathcal{H}(\omega)\, q_n,$$

as long as $\rho \neq 0$, which is equal to the attitude dynamics with q_n as unit quaternion. Therefore, the normalized quaternion q_n can be used for all calculations requiring the unit quaternion instead of $\overline{q^{eb}}$.

8.2 Measuring observer performance

In order to quantify the performance of an observer one could use a cost function. Such cost function can be defined as the integral of some positive definite function of the estimation error over some time interval. This cost function can also be used to choose the observer parameters, by minimizing the cost with respect to the parameters. However, when only penalizing the estimation error such minimization might not be well defined, since the estimation error might always go faster to zero the bigger some observer parameters are. In order to make the minimization of the cost function well defined one could also include the time derivative of the estimation error or the observer parameter directly in the cost function.

If the parameters of each observer from Chapter 5, 6 and 7 are found by minimizing the same cost function, then that cost function could also be used to compare the performance of each observer with each other. All these observers are defined for the three different models of the attitude dynamics from Chapter 2 and each model uses different states, so in order to be able to use the same cost function for each observer it would be required that the used estimation error can be defined for each observer. One way the same combined estimation error could be defined for each observer is by using the attitude estimation error $\tilde{R}^{eb} = \hat{R}^{eb} R^{eb\top}$ and the angular velocity estimation error $\tilde{\omega}$ as defined by (4.10). The positive definite function of this estimation error could for example be defined as

$$g(\tilde{R}^{eb}, \tilde{\omega}) = \Phi(\tilde{R}^{eb}) + \tilde{\omega}^\top \Lambda\, \tilde{\omega}, \tag{8.11}$$

with $\Lambda = \Lambda^\top \succ 0$ and $\Phi(\tilde{R}^{eb})$ a metric for 3D rotations [17]. One example of such a metric would be $\| \log(\tilde{R}^{eb})\|$ which is equivalent to $\|u\|$, with u the attitude estimation error expressed in exponential coordinates defined in Section 4.3.

The time derivative of the attitude estimation error can in general not be defined for the observers that use the structure from Chapter 3. Namely, the attitude estimation for those observers might be discontinuous and thus the time derivative of the attitude estimation error might not be well defined. Therefore, if this time derivative is included in the cost function its corresponding metric should be chosen such that the integral of this metric is always well defined. Each observer has a set of different parameters, so including those in the cost function would yield a different cost function for each observer. Therefore, in order to have the same cost function for all observers which can be minimized, one could include the time derivative of the estimation error in the cost function and use an appropriately chosen metric or assume that estimation error is sufficiently small such that the projected attitude estimations remains continuous for the observers that use the structure from Chapter 3.

The initial conditions of the models from Chapter 2 and observers from Chapter 5, 6 and 7 also affect the resulting values of the cost function and thus the optimized observer parameters. In practice all initial conditions are not known before hand and it is desired that an observer has certain performance guarantees regardless of the actual initial conditions. Therefore, it would be a bad idea to optimize the parameters of the observer with respect to only one initial condition. Ideally the parameters would be obtained by minimizing the cost function for all initial conditions such that estimation error is sufficiently small. However, this is difficult to do, especially when considering nonlinear dynamics. Instead, a finite number of initial conditions could be used, such that the cost function could be defined using

$$\begin{cases} J(\theta) = \dfrac{1}{N} \displaystyle\sum_{i=1}^{N} \int_0^T g(t, x^i(t))\, dt, \\ \qquad \text{s.t. } \dot{x}^i(t) = f(t, x^i(t), \theta), \quad x^i(0) = x_0^i, \end{cases} \tag{8.12}$$

with θ the parameters of the considered observer, N the number of used initial conditions, T the integral horizon, $f(t, x^i(t), \theta)$ the dynamics of the considered model and observer, $g(t, x^i(t))$ the positive definite function of the estimation error and its derivative, and x_0^i the ith considered initial condition of the model and observer. However, very different θ are obtained when numerically solving for θ by minimizing (8.12), while using different randomly sampled initial conditions with $N = 100$. Due to increasing computation time it is not practical to use even larger N. Instead, it might be possible to solve the minimization analytically, with potentially additional assumptions and simplifications such as only considering the performance close to zero estimation error, such that the linearization of each observer can be used. However, this would have to be investigated further in future research.

The necessity of the addition of the time derivative of the estimation error or the observer parameters directly in the cost function, in order to make the minimization of (8.12) well defined, might disappear when stochastic noise is acting on the inputs and outputs of the system. Namely, those noises might make it impossible to maintain zero estimation error, even if certain observer parameters become very large. However, this would make solving the minimization problem even harder. This would therefore also have to be investigated further in future research.

8.3 Numerical results

In Figure 8.1 the norm of the estimation error of the attitude and angular velocity of a numerical simulation of all proposed observers are shown. It can be noted that the estimation error of the attitude is represented in exponential coordinates, such that $\|u\|$ is equivalent to the angle in radians of the smallest rotation, such that $\|u\| = 0$ represents zero attitude estimation error. For the simulation the observer parameters are chosen such that the error dynamics of each observer has roughly the same rate of convergence. Both y-axes in Figure 8.1 have a logarithmic scale and all errors eventually seem to follow a linear line until they hit the limits of the machine precision. This suggests that all observers are locally exponentially stable. This was already shown to be true for the observers that use the structure from Chapter 3 and the observer for the minimal model from Section 5.2. However, this does strengthen the hypothesis that the remaining two observers from Section 6.2 and 7.2 are also locally exponentially stable.

It is worth pointing out that both the attitude and angular velocity errors from the observer for the minimal model from Section 5.2 in Figure 8.1 do take significantly longer before they seem to follow a linear downward trend, which for the logarithmic axis implies exponential decay. This behavior can be explained by looking at its Lyapunov function and corresponding time derivative from (5.8) and (5.11) respectively, which when substituting in (4.17a) can be written as

$$\begin{aligned} V &= \sum_{i=1}^{n} \frac{k_i}{2} \left(\tilde{R}^{eb}\, v_i - v_i\right)^\top \left(\tilde{R}^{eb}\, v_i - v_i\right) + \frac{1}{2} \tilde{L}^{e\top} \Gamma^{-1} \tilde{L}^{e}, \\ \dot{V} &= -\left(\sum_{i=1}^{n} k_i\, S(\tilde{R}^{eb\top} v_i)\, v_i\right)^\top R^{eb} \Delta\, R^{eb\top} \left(\sum_{i=1}^{n} k_i\, S(\tilde{R}^{eb\top} v_i)\, v_i\right). \end{aligned}$$

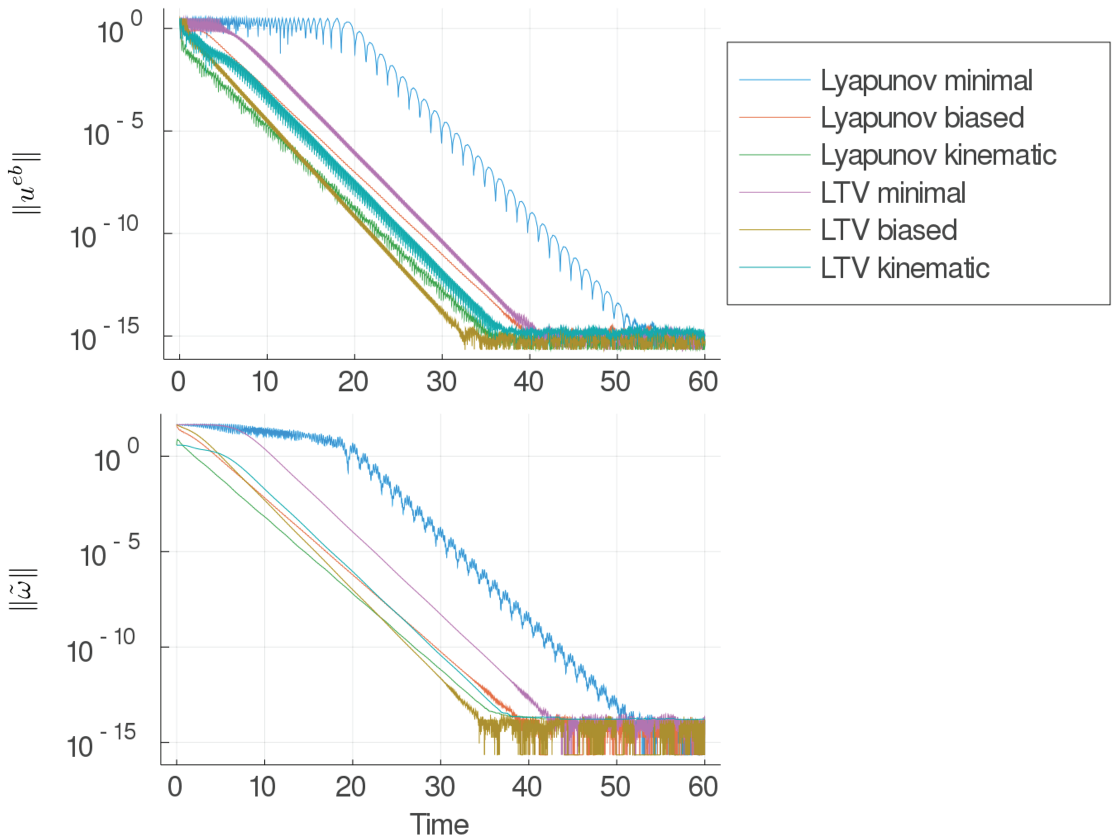

Figure 8.1: *Estimation errors of each of the proposed observers, with large* $\|\tilde{\omega}(0)\|$.

Namely, in the limit of $\|\tilde{L}^e\| \to \infty$ the value of that Lyapunov function also goes to infinity, while the time derivative of that Lyapunov function is only a function of rotation matrices, which belong to the bounded set $SO(3)$, and thus there exists a positive constant $\gamma > 0$ such that $\dot{V} \geq -\gamma$. Therefore, if the initial condition of $\tilde{L}^e(0)$ is such that the value of $\|\tilde{L}^e(0)\|$ and $V(0)$ are sufficiently large then $V(t)$ can at best decrease in time at a constant rate. This bound on the rate at which $V(t)$ decreases should initially limit the exponential decay of the estimation errors and thus give a possible explanation for the delayed linear downward trend in Figure 8.1.

The Lyapunov function and corresponding time derivative of observer for the kinematic model from Section 7.2 from (7.7) and (7.10) respectively also have a similar property as the minimal model from Section 5.2. However, instead of a large $\|\tilde{L}^e(0)\|$ the observer for the kinematic model from Section 7.2 has initially a slow decay if $\|\tilde{b}^b(0)\|$ is sufficiently large. This is demonstrated by the numerical simulation results from Figure 8.2, for which the same observer parameters are used as for the numerical simulation from Figure 8.1 and only the initial conditions are altered. The code used to generate figures 8.1 and 8.2 can be found in [35].

Lastly, it can be noted that the results from the observers for the minimal model from Section 5.1 and the kinematic model from Section 7.1 in Figure 8.1 and 8.2 respectively also have a delayed linear downward trend. However, those delays are not as big as those of the observers that use the structure from Chapter 4.

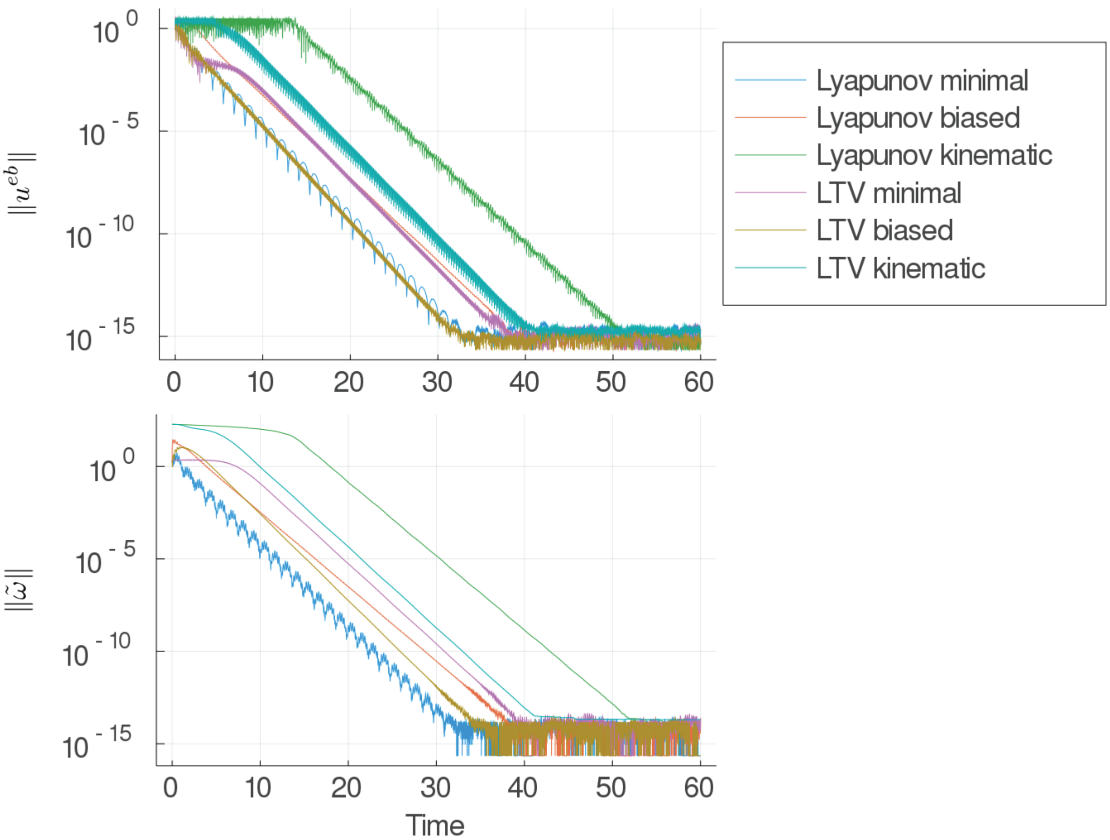

Figure 8.2: *Estimation errors of each of the proposed observers, with large* $\|\tilde{b}^b(0)\|$.

This behavior of the observers from Section 5.1 and 7.1 cannot be explained in a similar way as for the observers based on Chapter 4, and might be worth exploring in future research.

Chapter 9

Conclusions and recommendations

In this chapter the conclusions that can be drawn from this thesis and recommendations for future research are discussed. First, a brief summary together with the main conclusions of this thesis are given in Section 9.1. Lastly, the suggestions for future research made throughout this thesis together with a discussion of additional suggestions are given in Section 9.2.

9.1 Conclusions

In this thesis two different structures of constructing observers for three different models of the nonlinear attitude dynamics of a rigid body are explored. The three model representations of the attitude dynamics described in Chapter 2 are a combination of known approaches from the literature and these models differ mainly in which sensor information is used.

The first observer structure is discussed in Chapter 3, in which it is assumed that the considered model can be transformed to fit a linear time varying (LTV) state space model, for which a known observer structure from the literature can be used if the LTV model is uniformly completely observable. Each of the three models from Chapter 2 are transformed into such LTV model in Section 5.1, 6.1 and 7.1 respectively. Additionally, for each obtained LTV model it is also shown that they satisfy the sufficient conditions for uniform complete observability. However, these sufficient conditions are only satisfied under the assumption that the angular velocity remains bounded. Additionally, the observer for the biased model also requires the assumption that the applied torque remains bounded. If those assumptions are satisfied it is guaranteed that the proposed observer structure yields an exponential convergence of the estimated state to the true state. The part of this estimated state associated with the attitude is not guaranteed to represent the intended attitude representation. Therefore, that part of the estimated state need to be projected to the nearest attitude representation, which should always be possible. However, it is shown that this projection might be discontinuous in time.

The second observer structure is discussed in Chapter 4, which uses a Lyapunov function as starting point. Each of the three models from Chapter 2 are adapted to this observer structure and in Section 5.2, 6.2 and 7.2 it is shown that the time derivative of a proposed Lyapunov function can be made negative semi-definite. All three resulting observers are shown to have three additional equilibria, besides the one with zero estimation error. By linearizing the error dynamics of each of those proposed observers at the equilibria it is shown that each of the additional equilibria is unstable. The linearization at the equilibrium with zero estimation error can be used to show that the observer for the minimal model is locally exponentially stable around that equilibrium. A similar result of local exponential stability has not been obtained for the other two observers, but it is hypothesized that such results do exist. Furthermore, it is also shown that no limit cycles can exist in the remainder of the state space of the error dynamics of each observer.

Lastly, in Chapter 8 it is discussed how the constrained attitude dynamics can be implemented in numerical simulations, how the performance of observers can be quantified by a cost function and what behavior of the error dynamics can be observed from numerical simulations. The proposed class of cost functions, if chosen appropriately, can also be used to compare observers with each other, provided that the parameters of each observer are chosen such that the cost function is minimized. The numerical simulations suggest that all of the proposed observers are locally exponentially stable. However, for certain initial conditions the observers for the minimal and kinematic models could take much longer before they start converging exponentially. Though, the minimal and kinematic observers using the structure from Chapter 3 do not take as long as the minimal and kinematic observers using the structure from Chapter 4.

9.2 Recommendations

For the observers using the structure from Chapter 3 the attitude is represented by two linearly independent vectors in order to reduce the size of the combined state vector, while still ensuring a unique attitude representation. It can be noted that from these vectors a rotation matrix can be recovered using the projection obtained when solving Wahba's problem. Even though the estimated values of these vectors by the observers vary continuously in time, the projection of the estimated vectors to a rotation matrix might not always vary continuously in time. It is hypothesized that the discontinuities only happen when there is a large difference between the true and estimated values of these vectors, such that the attitude estimation error is also exponentially stable. However, this hypothesis would have to be investigated in future research.

These two vectors use six coordinates to represent the attitude. However, there exist also five-dimensional attitude parametrizations, which are unique and non-singular. Using such a five-dimensional parametrization instead of the two vectors would reduce the size of the total state of each observers that use the structure from Chapter 3, lowering the computational resources required to use them. Similar to the two vectors, the estimated five-dimensional parametrization would have to be projected. However, such projection is not trivial and would require more research.

In Subsection 6.1.2 it is shown that the linear time varying model for the biased model is uniformly completely observable if both the angular velocity and torque are bounded. It is hypothesized that this assumption on the torque could be omitted. However, additional research would be required to validate this hypothesis.

In Subsection 5.2.2 it is shown that the Lyapunov based observer for the minimal model is locally exponentially stable with the aid of a common quadratic Lyapunov function (CQLF). This CQLF is parameterized by the variable β, for which a range of possible values exists. The value used for β does affect the decay rate of the upper bound of local exponential stability. Therefore, choosing β such that the decay rate of this upper bound is maximized might also give more insights into the influence each observer parameter has on this exponential decay rate.

For the Lyapunov based observers for biased and kinematic model no CQLFs have been found which would prove their local exponential stability. However, the numerical results of the linear matrix inequalities, as discussed in Subsection 6.2.2 and 7.2.2, and the simulations from Section 8.3 do suggest that both models are locally exponentially stable. However, proving this might require a different approach than trying to find a CQLF.

For the Lyapunov based observer for the minimal model the only local difference between the proposed Lyapunov function from (5.8) and its CQLF using (5.16) is a cross term between the attitude and angular momentum estimation errors. Therefore, it might also be possible to achieve (local) negative definite instead of just negative semi-definite time derivatives of the Lyapunov functions for all three observers using the structure from Chapter 4 by also including cross terms. This could for example be achieved by instead of (4.13) to use

$$V = \frac{1}{2}\tilde{z}^\top \Gamma^{-1} \tilde{z}, \tag{9.1}$$

with $\Gamma = \Gamma^\top \succ 0$ and the vector $\tilde{z}$ containing each $\tilde{R}^{eb} v_i - v_i$ and $\tilde{x}$. Some numerical testing does seem to suggest that (9.1) could cause an increase in the number equilibria compared to using (4.13). However, the potential benefits or downsides of using (9.1) could be explored further in future research.

A locally stable equilibrium near zero estimation error, instability of the other equilibria and lack of limit cycles does not yet define the size of the basin of attraction of the stable equilibrium. Showing what those basins of attractions are for the observers using the structure from Chapter 4 would therefore require additional research.

Instead of having to deal with unstable equilibria in the observers using the structure from Chapter 4 it might be possible to vary some of the observer parameter with time. For example by changing the values of k_i and v_i once the estimation error gets close to any of the corresponding unstable equilibria. However, it would have to be assured that the corresponding Laypunov function remains piecewise continuously differentiable in time and positive definite.

It is proposed that the cost function, described in Section 8.2, could be minimized to find observer parameters, which are optimal with respect to that cost function. However, defining such a cost function, which is not biased towards initial conditions and has a well defined minimum, is not straightforward. Therefore, it might be good to try to simplify this problem, for example by linearizing the error dynamics.

In practice most systems are also subjected to stochastic disturbances on their inputs and outputs. Furthermore, the model of the dynamics might also have uncertainties. Therefore, it would be worth researching how robust each of the proposed observers are regarding those disturbances and uncertainties. The proposed observers using the structure from Chapter 3 are similar to a Kalman filter, which might help with analysis regarding the stochastic disturbances. However, that analysis is not equivalent to that of a Kalman filter if the measured rotation is subjected to a disturbance. It can be noted that the only sources of deterministic modeling errors for the proposed models in this thesis are uncertainties in the mass moment of inertia matrix and applied torque. Both uncertainties do not affect the observers for the kinematic model. Though, the kinematic model might be more susceptible to stochastic disturbances.

The sensors, that are assumed to be available, can in practice not continuously provide measurement information and instead only provide that information at discrete moments in time. Therefore, one could also investigate how the proposed observers could be modified such that they can handle such discrete measurements. Additionally, it could also be investigated how the continuous dynamics of the observers could be discretized, instead of having to rely on solvers for ordinary differential equations.

Besides the attitude dynamics discussed in this thesis, rigid bodies also have translational dynamics. Therefore, future research could also consider observers for the combined dynamics. A potential application of the proposed observers or the observers for the combined dynamics would be output feedback control. For linear systems the state estimation and state feedback can be decoupled from each other due to certainty equivalence. However, the dynamics considered by the observer in this thesis is nonlinear and thus the stability analysis of such output feedback would have to consider the combined dynamics.

Bibliography

[1] Y. Bar-Shalom and E. Tse. Dual effect, certainty equivalence, and separation in stochastic control. *IEEE Transactions on Automatic Control*, 19(5):494–500, 1974.

[2] P. Batista, N. Petit, C. Silvestre, and P. Oliveira. Relaxed conditions for uniform complete observability and controllability of ltv systems with bounded realizations. *IFAC-PapersOnLine*, 50(1):3598–3605, 2017.

[3] J. Baumgarte. Stabilization of constraints and integrals of motion in dynamical systems. *Computer methods in applied mechanics and engineering*, 1(1):1–16, 1972.

[4] K. Benzaid, N. Mansouri, and O. Labbani-Igbida. Robust trajectory tracking control of a quadrotor uav. In *Systems and Control (ICSC), 2013 3rd International Conference on*, pages 206–211. IEEE, 2013.

[5] S. Berkane and A. Tayebi. On the design of attitude complementary filters on so(3). *IEEE Transactions on Automatic Control*, 63(3):880–887, 2018.

[6] G. Besançon. *Nonlinear observers and applications*, volume 363. Springer, 2007.

[7] G. Besançon, G. Bornard, and H. Hammouri. Observer synthesis for a class of nonlinear control systems. *European Journal of control*, 2(3):176–192, 1996.

[8] J. Bohn and A. K. Sanyal. Almost global finite-time stable observer for rigid body attitude dynamics. In *2014 American Control Conference*, pages 4949–4954. IEEE, 2014.

[9] S. Bonnabel and P. Rouchon. On invariant observers. In *Control and observer design for nonlinear finite and infinite dimensional systems*, pages 62–64. Springer, 2005.

[10] G. Bornard, N. Couenne, and F. Celle. Regularly persistent observers for bilinear systems. In *New trends in nonlinear control theory*, pages 130–140. Springer, 1989.

[11] S. Bouabdallah and R. Siegwart. Backstepping and sliding-mode techniques applied to an indoor micro quadrotor. pages 2247–2252, 2005.

[12] H. N. Dang, B. Mohamed, and H. Rafaralahy. Trajectory-tracking control design for an under-actuated quadrotor. In *Control Conference (ECC), 2014 European*, pages 1765–1770. IEEE, 2014.

[13] S. Gros, M. Zanon, and M. Diehl. Baumgarte stabilisation over the so(3) rotation group for control. In *2015 54th IEEE Conference on Decision and Control (CDC)*, pages 620–625. IEEE, 2015.

[14] H. Gui and G. Vukovich. Finite-time angular velocity observers for rigid-body attitude tracking with bounded inputs. *International Journal of Robust and Nonlinear Control*, 27(1):15–38, 2017.

[15] H. Hahn. *Rigid Body Dynamics of Mechanisms: 1 Theoretical Basis*, volume 1. Springer Science & Business Media, 2002.

[16] L. Huang. *A concise introduction to mechanics of rigid bodies: Multidisciplinary engineering*. Springer, 2nd edition, 2017.

[17] D. Q. Huynh. Metrics for 3d rotations: Comparison and analysis. *Journal of Mathematical Imaging and Vision*, 35(2):155–164, 2009.

[18] H. K. Khalil. *Nonlinear systems*. Prentice Hall, 3rd edition, 2003.

[19] J. Liesen and V. Mehrmann. *Linear algebra*. Springer, 2015.

[20] D. Luenberger. An introduction to observers. *IEEE Transactions on automatic control*, 16(6):596–602, 1971.

[21] T. Madani and A. Benallegue. Backstepping control for a quadrotor helicopter. In *Intelligent Robots and Systems, 2006 IEEE/RSJ International Conference on*, pages 3255–3260. IEEE, 2006.

[22] L. Magnis and N. Petit. Angular velocity nonlinear observer from vector measurements. *Automatica*, 75:46–53, 2017.

[23] R. Mahony, T. Hamel, and J.-M. Pflimlin. Nonlinear complementary filters on the special orthogonal group. *IEEE Transactions on automatic control*, 53(5):1203–1218, 2008.

[24] R. Mahony, V. Kumar, and P. Corke. Multirotor aerial vehicles. *IEEE Robotics and Automation magazine*, 20(32), 2012.

[25] F. L. Markley. Multiplicative vs. additive filtering for spacecraft attitude determination. *Dynamics and Control of Systems and Structures in Space*, 2004.

[26] L. Markley. Humble problems. *Advances in the Astronautical Sciences*, 124:2205–2222, 2006.

[27] G. V. Raffo, M. G. Ortega, and F. R. Rubio. Robust nonlinear control for path tracking of a quad-rotor helicopter. *Asian Journal of Control*, 17(1):142–156, 2015.

[28] S. Salcudean. A globally convergent angular velocity observer for rigid body motion. *IEEE transactions on Automatic Control*, 36(12):1493–1497, 1991.

[29] R. Schwertassek and R. Roberson. *Dynamics of multibody systems*. Springer-Verlag, Berlin, 1988.

[30] B. Shafai and R. Carroll. Minimal-order observer designs for linear time-varying multivariable systems. *IEEE transactions on automatic control*, 31(8):757–761, 1986.

[31] S. Skogestad and I. Postlethwaite. *Multivariable Feedback Control: Analysis and Design*. Wiley, 2 edition, 2007.

[32] M. A. Smith, H. F. King, W. T. Guston, and R. Casey. Convertawings, inc. *Flight: and Aircraft Engineer*, 70(2493):722–723, Nov 1956.

[33] J. Stuelpnagel. On the parametrization of the three-dimensional rotation group. *SIAM review*, 6(4):422–430, 1964.

[34] Z. Terze, A. Müller, and D. Zlatar. Singularity-free time integration of rotational quaternions using non-redundant ordinary differential equations. *Multibody system dynamics*, 38(3):201–225, 2016.

[35] K. van der Veen. fibonatic/Thesis-simulations-in-Julia: Thesis- simulations. `https://doi.org/10.5281/zenodo.3754579`, Apr. 2020.

[36] G. Wahba. A least squares estimate of satellite attitude. *SIAM review*, 7(3):409, 1965.

[37] W. R. Young. *The Helicopters: The Epic of Flight*. Time-Life Books, 1982.

Appendix A

Reduced order observer

The following linear time varying model is considered for a reduced order observer

$$\dot{x}_1 = A_{11}\,x_1 + A_{12}\,x_2 + B_1\,u, \tag{A.1a}$$
$$\dot{x}_2 = A_{21}\,x_1 + A_{22}\,x_2 + B_2\,u, \tag{A.1b}$$
$$y = x_1, \tag{A.1c}$$

where A_{11}, A_{12}, A_{21}, A_{22}, B_1 and B_2 are possibly time-varying matrices. The reduced order observer only has to find an estimate for x_2, since x_1 is already directly measured. The general structure for a reduced order observer for estimating x_2 from (A.1) is given by

$$\hat{x}_2 = z + K\,y, \tag{A.2a}$$
$$\dot{z} = Q\,z + R\,y + S\,u, \tag{A.2b}$$

with K, Q, R and S also potentially time-varying matrices. By combining (A.1a) and (A.2) the time derivative of $\hat{x}_2$ can be shown to be

$$\dot{\hat{x}}_2 = Q\,z + R\,x_1 + S\,u + \dot{K}\,x_1 + K\left(A_{11}\,x_1 + A_{12}\,x_2 + B_1\,u\right). \tag{A.3}$$

By defining the error as $e \coloneqq x_2 - \hat{x}_2$, using (A.1b), (A.3) and $z = \hat{x}_2 - K\,x_1$ the time derivative of e can be written as

$$\dot{e} = (A_{21} + QK - R - \dot{K} - K A_{11})x_1 + (A_{22} - K A_{12})x_2 + (B_2 - S - K B_1)u - Q\,\hat{x}_2. \tag{A.4}$$

The dynamics of the error in (A.4) can be made only linearly dependent on e by choosing

$$Q = A_{22} - K\,A_{12}, \tag{A.5a}$$
$$R = A_{21} + Q\,K - \dot{K} - K\,A_{11}, \tag{A.5b}$$
$$S = B_2 - K\,B_1, \tag{A.5c}$$

which simplifies (A.4) to $\dot{e} = Q\,e = (A_{22} - K\,A_{12})\,e$. If there always exists a left inverse $A_{12,L}^{-1}$, such that $A_{12,L}^{-1} A_{12} = I$, the error can be driven to zero by using

$$K = (A_{22} - Q)\,A_{12,L}^{-1}, \tag{A.6}$$

with Q some constant Hurwitz matrix. Another option for assuring that the error can be driven to zero would be to use the observer dynamics from Section 3.1, with $A(t) = A_{22}$ and $C(t) = A_{12}$ such that

$$K = M(t)\, A_{12}^{\top} W^{-1}, \tag{A.7}$$

with $M(t)$ and W as defined for (3.2) and (3.3). This observer dynamics does require that (A_{22}, A_{12}) is UCO, but this should be true if (A.1) is UCO. Combining (A.5) with either (A.6) or (A.7) would define all matrices for the reduced order observer defined in (A.2). However, the expression for R in (A.5b) also requires knowing $\dot{K}$. For (A.6) and (A.7) this means that the time derivatives of A_{22}, A_{12} and A_{12} need to be known respectively.

The first problem formulation, defined in (2.13), using (3.11) gives

$$\dot{\rho} = (I_3 \otimes R^{eb})\, \Gamma\, J^{-1} {R^{eb}}^{\top} L^e, \tag{A.8a}$$

$$\dot{L}^e = R^{eb} \tau^b. \tag{A.8b}$$

Applying the reduced order observer to (A.8) with $y = \rho$ and $u = \tau^b$ means that the matrices from (A.1) can be defined as $A_{11}, A_{21}, A_{22}, B_1 = 0$, $A_{12} = (I_3 \otimes R^{eb})\, \Gamma\, J^{-1} {R^{eb}}^{\top}$ and $B_2 = R^{eb}$. Since A_{22} is constant, its derivative is known. The matrix A_{12} is a function of the rotation matrix R^{eb}. Therefore, the time derivative of A_{12} would also require knowing the time derivative of R^{eb}. However, the time derivative of R^{eb} is a function of L^e, which is not known. So this method of reduced order observer cannot be applied to (A.8).