TWO OBSERVER-BASED TRACKING ALGORITHMS FOR A UNICYCLE MOBILE ROBOT

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A trajectory tracking problem for the three-dimensional kinematic model of a unicycle-type mobile robot is considered. It is assumed that only two of the tracking error coordinates are measurable. By means of cascaded systems theory we develop observers for each of the error coordinates and show the $\mathcal{K}$-exponential convergence of the tracking error in combined closed-loop observer-controller systems. The results are illustrated with computer simulations.

Keywords: observer, trajectory tracking, mobile robot

1. Introduction

In recent years the stabilization problem of non-holonomic systems has received considerable attention. One of the reasons is that for these systems Brockett's necessary condition for smooth stabilization is not met (Brockett, 1983) and no smooth time-invariant stabilizing control law exists. For an overview, we refer the reader to the paper (Kolmanovsky and McClamroch, 1995) and references cited therein. The tracking problem has received less attention. In (Fierro and Lewis, 1995; Kanayama *et al.*, 1990; Micaelli and Samson, 1993; Murray *et al.*, 1992; Walsh *et al.*, 1994) a linearization-based tracking control scheme was derived. The idea of input-output linearization was used in (Oelen and van Amerongen, 1994). In (Fliess *et al.*, 1995) the trajectory stabilization problem was dealt with by means of a differentially flat system approach. A dynamic feedback linearization technique for a wheeled mobile robot was presented in (Canudas de Wit *et al.*, 1996). All these publications solve the local tracking problem. The first global tracking control law that we are aware of was proposed in (Samson and Ait-Abderrahim, 1991). Another global tracking result was derived in (Jiang and Nijmeijer, 1997) using integrator backstepping. Global tracking results yielding exponential convergence were presented in (Dixon *et al.*, 1999; Panteley *et al.*, 1998) under a persistence-of-excitation assumption on the reference trajectory. A fuzzy PD controller using look-up tables for the unicycle robot is given in (Ulyanov *et al.*, 1998).

In the paper (Panteley *et al.*, 1998) a state feedback controller for the unicycle-type mobile robot was proposed. Here we adapt this result to develop an output-feedback trajectory tracking controller under the assumption that one of the tracking error coordinates is unknown. Our solution to this problem employs tools of cascaded systems and linear systems theory. By constructing reduced-order observers we have achieved global $\mathcal{K}$-exponential stability in the case of uncertain position error, and local exponential stability in the case of unmeasurable orientation. Our stability analysis is based on the results of cascaded systems. A similar problem of motion planning with measurements of the position coordinates was solved in (Guillaume and Rouchon, 1998; Jiang and Nijmeijer, 1999). A part of the results included in this paper, concerning the position error observer, was presented in (Lefeber, 2000; Lefeber *et al.*, 2001).

The organization of the paper is as follows. In Section 2 we recall definitions and theorems from stability theory and formulate the tracking problem. In Section 3 we present an observer for one of the position-error coordinates and the observer-based controller. In Section 4 the case of an unmeasured orientation angle is considered and an appropriate controller is proposed. Computer simulations illustrating the behaviour of both controllers are presented in Section 5. Section 6 concludes the paper.

2. Preliminaries and Problem Formulation

Below we recall some standard concepts of stability theory (Krstić *et al.*, 1995).

2.1. Preliminaries

Definition 1. A continuous function $\alpha : [0,a) \to [0,\infty)$ is said to belong to *class* $\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to *class* $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$.

Definition 2. A continuous function $\beta : [0,a)\times[0,\infty) \to [0,\infty)$ is said to belong to *class* $\mathcal{KL}$ ($\beta \in \mathcal{KL}$) if for each fixed s the mapping $\beta(r,s)$ belongs to class $\mathcal{K}$ with respect to r, and if for each fixed r the mapping $\beta(r,s)$ is decreasing with respect to s and $\beta(r,s) \to 0$ as $s \to \infty$. It is said to belong to *class* $\mathcal{KL}_\infty$ if, in addition, for each fixed s the mapping $\beta(r,s)$ belongs to class $\mathcal{K}_\infty$ with respect to r.

Definition 3. The equilibrium point $x = 0$ of a non-autonomous system $\dot{x} = f(t,x)$ is

- *locally uniformly asymptotically stable (LUAS)* if there exist a function $\beta \in \mathcal{KL}$ and a positive constant c such that for all $t > t_0 > 0$ and for all initial states $\|x(t_0)\| < c$
$$\|x(t)\| \le \beta\big(\|x(t_0)\|, t - t_0\big); \tag{1}$$
- *globally uniformly asymptotically stable (GUAS)* if (1) is satisfied with $\beta \in \mathcal{KL}_\infty$ for any initial state $x(t_0)$;
- *locally exponentially stable (LES)* if (1) is satisfied with $\beta(r,s) = kre^{-\gamma s}$, $k > 0$, $\gamma > 0$ for $\|x(t_0)\| < c$;
- *globally exponentially stable (GES)* if (1) is satisfied with $\beta(r,s) = kre^{-\gamma s}$, $k > 0$, $\gamma > 0$ for any initial state $x(t_0)$.

Definition 4. (Sørdalen and Egeland, 1995, Def. 2) The equilibrium point $x = 0$ of a non-autonomous system $\dot{x} = f(t,x)$ is said to be *globally $\mathcal{K}$-exponentially stable* if there exist a function $\kappa \in \mathcal{K}$ and a constant $\gamma > 0$ such that for all $(t_0, x(t_0)) \in \mathbb{R}^+ \times \mathbb{R}^n$ we have
$$\|x(t)\| \le \kappa\big(\|x(t_0)\|\big)e^{-\gamma(t-t_0)}, \qquad \forall t \ge t_0 \ge 0.$$

Definition 5. A continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}$ is said to be *persistently exciting (PE)* if there exist constants ϵ_1, ϵ_2, $\delta > 0$ such that for all $t \ge 0$ we have
$$\epsilon_1 \le \int_t^{t+\delta} \phi^2(\tau)\,\mathrm{d}\tau \le \epsilon_2.$$

Lemma 1. (Khalil, 1996) *Consider the system,* $x \in \mathbb{R}^2$,
$$\dot{x} = \begin{bmatrix} -c_1 & -c_2\phi(t) \\ c_3\phi(t) & 0 \end{bmatrix} x. \tag{2}$$
If $c_1 > 0$, $c_2c_3 > 0$ and $\phi(t)$ is PE, then the system (2) is GES.

Theorem 1. (Lefeber *et al.*, 2000) *Consider the system,* $x \in \mathbb{R}^4$,
$$\dot{x} = \begin{bmatrix} -c_1 & -c_2\phi(t) & d_1 & d_2\phi(t) \\ \phi(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & -l_2\phi(t) \\ 0 & 0 & \phi(t) & -l_1 \end{bmatrix} x. \tag{3}$$
When $\phi(t)$ is PE, $c_1 > 0$, $c_2 > 0$, $l_1 > 0$, $l_2 > 0$, then the system (3) is GES.

Theorem 2. (Ioannou and Sun, 1996, Thm. 3.4.6 (v)) *The system $\dot{x} = A(t)x$ is GES if and only if it is GUAS.*

Theorem 3. (Krstić *et al.*, 1995, Thm. A.5) *Let $x = 0$ be an equilibrium point of a non-autonomous system $\dot{x} = f(t,x)$ and $D = \{x \in \mathbb{R}^n | \, \|x\| < c\}$. Let $V\colon\ D \times \mathbb{R}^n \to \mathbb{R}^+$ be a continuously differentiable function such that $\forall t \ge 0$, $\forall x \in D$,*
$$\alpha_1\big(\|x\|\big) \le V(t,x) \le \alpha_2\big(\|x\|\big), \tag{4}$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -\alpha_3\big(\|x\|\big). \tag{5}$$
Then the equilibrium point $x = 0$ is

- locally uniformly asymptotically stable *if α_1, α_2 and α_3 are $\mathcal{K}$ functions on $[0,c)$;*
- globally uniformly asymptotically stable *if $D = \mathbb{R}^n$, α_1, α_2 are $\mathcal{K}_\infty$ functions, and α_3 is a $\mathcal{K}$ function on $\mathbb{R}^+$;*
- locally exponentially stable *if $\alpha_i(\rho) = k_i\rho^\gamma$ on $[0,c)$, $\gamma > 0$, $k_i > 0$, $i = 1,2,3$;*
- globally exponentially stable *if $D = \mathbb{R}^n$, and $\alpha_i(\rho) = k_i\rho^\gamma$ on $\mathbb{R}_+$, $\gamma > 0$, $k_i > 0$, $i = 1,2,3$.*

2.2. Cascaded Systems

Consider a system $\dot{z} = f(t,z)$ that can be written as
$$\begin{aligned} \dot{z}_1 &= f_1(t,z_1) + g(t,z_1,z_2)z_2, \\ \dot{z}_2 &= f_2(t,z_2), \end{aligned} \tag{6}$$

where $z_1 \in \mathbb{R}^n$, $z_2 \in \mathbb{R}^m$, $(z_1, z_2) = (0, 0)$ is an equilibrium point of (6), $f_1(t, z_1)$ is continuously differentiable in (t, z_1) and $f_2(t, z_2)$, $g(t, z_1, z_2)$ are continuous in their arguments, as well as locally Lipschitz in z_2 and (z_1, z_2), respectively.

Assumption 1. Assume that there exist continuous functions $k_1 \colon \mathbb{R}^+ \to \mathbb{R}$ and $k_2 \colon \mathbb{R}^+ \to \mathbb{R}$ such that

$$\|g(t, z_1, z_2)\| \le k_1\big(\|z_2\|\big) + k_2\big(\|z_2\|\big)\,\|z_1\|, \qquad (7)$$

where $\|g(t, z_1, z_2)\|$ denotes the Frobenius norm of the matrix $g(t, z_1, z_2)$.

Then we can formulate the following corollary from a result presented in (Panteley and Loría, 1998), see also (Panteley *et al.*, 1998):

Corollary 1. *Assume that the subsystem $\dot{z}_1 = f_1(t, z_1)$ of (6) is GES, the subsystem $\dot{z}_2 = f_2(t, z_2)$ is globally $\mathcal{K}$-exponentially stable and $g(t, z_1, z_2)$ satisfies (7). Then the cascaded system (6) is globally $\mathcal{K}$-exponentially stable.*

2.3. Problem Formulation

A kinematic model of the unicycle-type mobile robot is given by the following equations:

$$\begin{cases} \dot{x} = v\cos\theta, \\ \dot{y} = v\sin\theta, \\ \dot{\theta} = \omega. \end{cases}$$

The geometric interpretation of coordinates $\mathbf{x} = (x, y, \theta)$ is shown in Fig. 1. The forward velocity v and the angular velocity ω serve as the system controls.

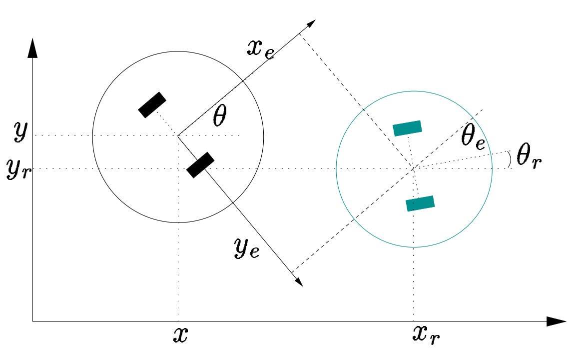

Fig. 1. The unicycle coordinates (x, y, θ), reference coordinates (x_r, y_r, θ_r) and moving frame coordinates (x_e, y_e, θ_e).

We consider the problem of tracking a reference trajectory $\mathbf{x_r} = (x_r, y_r, \theta_r)$ generated by the reference system

$$\begin{cases} \dot{x}_r = v_r\cos\theta_r, \\ \dot{y}_r = v_r\sin\theta_r, \\ \dot{\theta}_r = \omega_r, \end{cases}$$

where v_r and ω_r are continuous functions of time.

Following (Kanayama *et al.*, 1990), we express the error coordinates in the moving frame in the form

$$\begin{bmatrix} x_e \\ y_e \\ \theta_e \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_r - x \\ y_r - y \\ \theta_r - \theta \end{bmatrix},$$

and compute the error dynamics as

$$\dot{\mathbf{x}}_\mathbf{e} = \begin{bmatrix} \dot{x}_e \\ \dot{y}_e \\ \dot{\theta}_e \end{bmatrix} = \begin{bmatrix} \omega y_e - v + v_r\cos\theta_e \\ -\omega x_e + v_r\sin\theta_e \\ \omega_r - \omega \end{bmatrix}. \qquad (8)$$

We shall assume that in the dynamic system (8) only two error coordinates are measured while the remaining one is unknown. To this end, we define the output function $\mathbf{y}$

$$\mathbf{y} = f(\mathbf{x_e}), \quad \dim\mathbf{y} = 2. \qquad (9)$$

Upon defining the output $\mathbf{y}$, the dynamic output-feedback state-tracking control problem can be formulated as follows:

Find velocity control laws v and ω of the form

$$v = v(t, \mathbf{y}, \mathbf{z}), \quad \omega = \omega(t, \mathbf{y}, \mathbf{z}), \qquad (10)$$

where $\mathbf{z}$ is generated by the observer

$$\dot{\mathbf{z}} = g(t, \mathbf{y}, \mathbf{z}), \qquad (11)$$

such that the closed-loop error system of (8), (10) and (11) is globally $\mathcal{K}$-exponentially stable.

The scheme of the closed-loop robot-observer-controller system is depicted in Fig. 2.

3. Position-Error Observer

In this section we address the problem of unmeasurable one of position error coordinates x_e or y_e. For the purpose of designing an observer-based controller we choose a control law proposed in (Panteley *et al.*, 1998):

$$\omega = \omega_r + c_1\theta_e, \qquad c_1 > 0, \qquad (12a)$$

$$v = v_r + c_2 x_e - c_3\omega_r y_e, \quad c_2 > 0, \quad c_3 > -1. \qquad (12b)$$

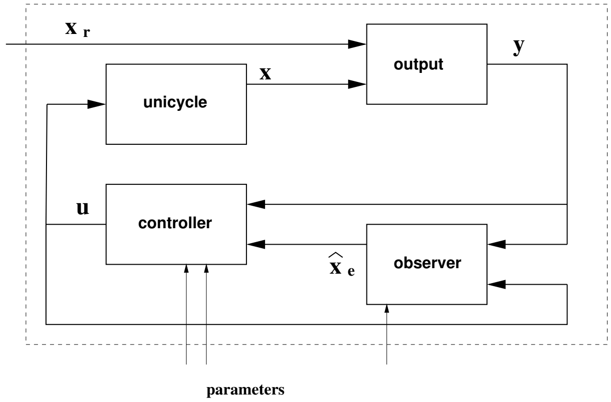

Fig. 2. Scheme of the observer-based controller for the unicycle-type robot.

In this case we obtain, in combination with the error dynamics (8), the cascaded structure

$$\underbrace{\begin{bmatrix}\dot{x}_e\\ \dot{y}_e\end{bmatrix} = \begin{bmatrix}-c_2 & (1+c_3)\omega_r\\ -\omega_r & 0\end{bmatrix}\begin{bmatrix}x_e\\ y_e\end{bmatrix}}_{\dot{z}_1=f_1(t,z_1)} + \underbrace{\begin{bmatrix} c_1 y_e + v_r\dfrac{\cos\theta_e - 1}{\theta_e}\\ -c_1 x_e + v_r\dfrac{\sin\theta_e}{\theta_e}\end{bmatrix}\theta_e}_{g(t,z_1,z_2)z_2}, \tag{13a}$$

$$\underbrace{\dot{\theta}_e = -c_1\theta_e.}_{\dot{z}_2=f_2(t,z_2)} \tag{13b}$$

The subsystem $\dot{z}_2 = f_2(t,z_2)$ of (13b) is GES. Assume that v_r is bounded and ω_r is persistently exciting. This being so, from Lemma 1 we obtain that the subsystem $\dot{z}_1 = f_1(t,z_1)$ is also GES and the interconnection term $g(t,z_1,z_2)$ satisfies Assumption 1. Hence, by means of Corollary 1, we conclude that the overall closed-loop system (13) is globally $\mathcal{K}$-exponentially stable.

Now we assume that we are unable to measure the forward-error x_e, so only the values of y_e and θ_e are available, i.e.

$$\mathbf{y} = [y_1\ y_2]^T = [y_e\ \theta_e]. \tag{14}$$

The case of unmeasured y_e can be addressed analogously.

We notice that the control ω in (12a) depends only on the available output $y_2(\theta_e)$ and therefore it can be directly used in the observer-based controller; in the control v the unmeasurable state x_e must be replaced by its estimate. To find an estimate of x_e, we first consider the subsystem $\dot{z}_1 = f_1(t,z_1)$ of (13a) without the substitution of the control v (12b), which corresponds to the case of $\theta_e = 0$. Further, in Proposition 1, we shall show that the same observer can be used for an arbitrary θ_e. We have

$$\begin{bmatrix}\dot{x}_e\\ \dot{y}_e\end{bmatrix} = \begin{bmatrix}0 & \omega_r\\ -\omega_r & 0\end{bmatrix}\begin{bmatrix}x_e\\ y_e\end{bmatrix} + \begin{bmatrix}v_r - v\\ 0\end{bmatrix}. \tag{15}$$

We define a new variable z as a linear combination of the measured and unknown states

$$z = x_e - b y_e,$$

where b is a function of time, still to be determined in order to guarantee the asymptotic stability of the reduced-order observer. Differentiating z with respect to time along the dynamics (15) yields

$$\begin{aligned}\dot{z} &= \omega_r y_e + (v_r - v) - \dot{b} y_e + b\omega_r x_e\\ &= b\omega_r(x_e - b y_e) + b^2\omega_r y_e + \omega_r y_e + (v_r - v) - \dot{b} y_e\\ &= b\omega_r z + \left(b^2\omega_r + \omega_r - \dot{b}\right) y_e + (v_r - v).\end{aligned}$$

Defining the reduced-order observer dynamics as

$$\dot{\hat{z}} = b\omega_r\hat{z} + \left(b^2\omega_r + \omega_r - \dot{b}\right) y_e + (v_r - v),$$

we obtain for the observation-error $\tilde{z} = z - \hat{z}$

$$\dot{\tilde{z}} = b\omega_r\tilde{z}. \tag{16}$$

Solutions of (16) satisfy

$$\tilde{z}(t) = \tilde{z}(t_0) e^{\int_{t_0}^{t} b(\tau)\omega_r(\tau)\,\mathrm{d}\tau}.$$

If we now take $b = -l\omega_r$ with l as a positive constant and assume furthermore that ω_r is PE, we have the existence of $\epsilon_1 > 0$, $\epsilon_2 > 0$, and $\delta > 0$ such that

$$\frac{\epsilon_1}{\delta}(t - t_0) < \int_{t_0}^{t} \omega_r^2(\tau)\,\mathrm{d}\tau < \frac{\epsilon_2}{\delta}(t - t_0),$$

which enables us to conclude that (16) is GES and the estimate $\hat{z}$ tends to z. The estimate of x_e for the subsystem $\dot{z}_1 = f_1(t,z_1)$, defined as

$$\hat{x}_e = \hat{z} - l\omega_r y_e,$$

converges exponentialy to the original state x_e.

Now we plug the observer into the complete closed-loop system:

Proposition 1. *Consider the tracking error dynamics (8) with output (14) in the closed loop with the control law*

$$\omega = \omega_r + c_1\theta_e, \qquad c_1 > 0, \tag{17a}$$

$$v = v_r + c_2\hat{x}_e - c_3\omega_r y_e, \quad c_2 > 0, \quad c_3 > -1, \tag{17b}$$

where $\hat{x}_e$ is generated by the reduced-order observer

$$\dot{\hat{z}} = -l\omega_r^2\hat{z} + (l^2\omega_r^3 + \omega_r + l\dot{\omega}_r)y_e + (v_r - v), \quad \text{(18a)}$$

$$\hat{x}_e = \hat{z} - l\omega_r y_e, \qquad l > 0. \quad \text{(18b)}$$

If v_r is bounded and ω_r is persistently exciting (PE), then the closed-loop system (8), (17) and (18) is globally $\mathcal{K}$-exponentially stable.

Proof. We can view the closed-loop system (8), (17) and (18) as a cascaded system, i.e. the system of the form (6), where

$$z_1 = \begin{bmatrix} x_e & y_e & x_e - \hat{x}_e \end{bmatrix}^T, \quad z_2 = \theta_e,$$

$$f_1(t, z_1) = \begin{bmatrix} -c_2 & (c_3+1)\omega_r & c_2 \\ -\omega_r & 0 & 0 \\ 0 & 0 & -l\omega_r^2 \end{bmatrix} z_1,$$

$$f_2(t, z_2) = -c_1 z_2,$$

$$g(t, z_1, z_2) = \begin{bmatrix} c_1 y_e + v_r \dfrac{\cos\theta_e - 1}{\theta_e} \\ -c_1 x_e + v_r \dfrac{\sin\theta_e}{\theta_e} \\ c_1 y_e + v_r \dfrac{\cos\theta_e - 1}{\theta_e} \\ \quad + l\omega_r\Big(-c_1 x_e + v_r \dfrac{\sin\theta_e}{\theta_e}\Big) \end{bmatrix}.$$

To be able to apply Corollary 1, we need to verify the global exponential stability (GES) of the subsystem $\dot{z}_1 = f_1(t, z_1)$. To do so, we rewrite it in the cascaded form as

$$\bar{z}_1 = \begin{bmatrix} x_e & y_e \end{bmatrix}^T, \quad \bar{z}_2 = x_e - \hat{x}_e,$$

$$\begin{bmatrix} \dot{x}_e \\ \dot{y}_e \end{bmatrix} = \underbrace{\begin{bmatrix} -c_2 & (c_3+1)\omega_r \\ -\omega_r & 0 \end{bmatrix} \begin{bmatrix} x_e \\ y_e \end{bmatrix}}_{\bar{f}_1(t,\bar{z}_1)} + \underbrace{\begin{bmatrix} c_2 \\ 0 \end{bmatrix} \bar{z}_2}_{\bar{g}(t,\bar{z}_1,\bar{z}_2)}, \quad \text{(19a)}$$

$$\dot{\bar{z}}_2 = -l\omega_r^2 \bar{z}_2. \quad \text{(19b)}$$

Solutions of the subsystem (19b) are given by

$$\bar{z}_2(t) = \bar{z}_2(t_0) e^{-l\int_{t_0}^{t} \omega_r^2(\tau)\mathrm{d}\tau}.$$

Since ω_r is PE, the subsystem (19b) is GES. Furthermore, the term $\bar{g}(t, \bar{z}_1, \bar{z}_2)$ is bounded and the system $\dot{\bar{z}}_1 = \bar{f}_1(t, \bar{z}_1)$ is GES. From Corollary 1 we can conclude that the system $\dot{z}_1 = f_1(t, z_1)$ is GUAS. Since it is a linear time-varying system, Theorem 2 allows us to conclude that $\dot{z}_1 = f_1(t, z_1)$ is GES. Since also the system $\dot{z}_2 = f_2(t, z_2)$ is GES and the boundedness of both v_r and ω_r (cf. Definition 5) guarantees that the condition on $g(t, z_1, z_2)$ is met, Corollary 1 yields the desired result. ■

4. Orientation-Error Observer

In this section we assume that the available output is

$$\mathbf{y} = [\, y_1 \; y_2 \,]^T = [\, x_e \; y_e \,]^T. \quad \text{(20)}$$

We notice that the unknown orientation error θ_e appears in the system equation (8) only as an argument of the sine and cosine. Hence we expect that it is not possible to retrieve the exact value of θ_e from the available output, but only $\sin\theta_e$ and $\cos\theta_e$, i.e. the value of θ_e limited to one full period ($-\pi < \theta_e \le \pi$).

Therefore we modify the controller (12) to include $\sin\theta_e$ instead of θ_e:

$$\begin{aligned} \omega &= \omega_r + c_1 \sin\theta_e, \\ v &= v_r + c_2 x_e - c_3\omega_r y_e. \end{aligned} \quad \text{(21)}$$

The controller (21) ensures the local exponential stability of the closed-loop control system (8) and (21) if $|\theta_e| \le \theta_0 < \pi$.

The θ_e dynamics in the closed loop are given by

$$\dot{\theta}_e = -c_1 \sin\theta_e. \quad \text{(22)}$$

Define the Lyapunov function

$$V(\theta_e) = 1 - \cos\theta_e, \quad \text{(23)}$$

and differentiate it along the dynamics (22):

$$\dot{V} = -c_1 \sin^2\theta_e \le 0. \quad \text{(24)}$$

If c_1 is a positive constant and $|\theta_e| \le \theta_0 < \pi$, the system (22) is asymptotically stable.

We can also find $\delta(\theta_0) > 0$ such that

$$\sin^2\theta_e \ge \delta(\theta_0)(1 - \cos\theta_e).$$

Then (24) satisfies

$$\dot{V} \le -c_1\delta(\theta_0)(1 - \cos\theta_e) = -c_1\delta(\theta_0)V,$$

and the system (22) is LES.

In order to modify the state-feedback controller (21) to an output-feedback controller for the system (8), (20), we shall apply an observer estimating $\sin\theta_e$.

To this end, we define the new variable z

$$z = \sin\theta_e - av_r y_e.$$

Its derivative along the dynamics (8) is given by

$$\dot{z} = (\omega_r - \omega)\cos\theta_e - a\dot{v}_r y_e + av_r\omega x_e - av_r^2\sin\theta_e.$$

Set $\psi = \sin\theta_e$ and its estimate $\hat{\psi} = \widehat{\sin\theta_e}$. Hence we define the observer

$$\dot{\hat{z}} = -a\dot{v}_r y_e + av_r\omega x_e - av_r^2\hat{z} - a^2v_r^3 y_e, \quad (25a)$$

$$\hat{\psi} = \hat{z} + av_r y_e. \quad (25b)$$

With the observer error $\tilde{\psi} = \psi - \hat{\psi}$, we obtain the observer error dynamics

$$\dot{\tilde{\psi}} = (\omega_r - \omega)\cos\theta_e - av_r^2\tilde{\psi}. \quad (26)$$

Before we define the complete control law for the system (8), we examine the stability of the combined observer (25) with the control of angular velocity ω

$$\omega = \omega_r + c_1(t)\hat{\psi}, \quad (27)$$

where $c_1(t)$ is a non-negative function of time. The system consisting of θ_e and the observer error $\tilde{\psi}$ with the control (27) yields

$$\dot{\theta}_e = -c_1(t)\hat{\psi},$$

$$\dot{\tilde{\psi}} = -c_1(t)\hat{\psi}\cos\theta_e - av_r^2\tilde{\psi}.$$

Then, for $\hat{\psi} = \sin\theta_e - \tilde{\psi}$, we obtain

$$\begin{aligned}\dot{\theta}_e &= -c_1(t)\big(\sin\theta_e - \tilde{\psi}\big),\\ \dot{\tilde{\psi}} &= -c_1(t)\frac{1}{2}\sin 2\theta_e + c_1(t)\tilde{\psi}\cos\theta_e - av_r^2\tilde{\psi}.\end{aligned} \quad (28)$$

Define a Lyapunov function V for the system (28):

$$V = (1 - \cos\theta_e) + \frac{1}{2}\tilde{\psi}^2. \quad (29)$$

The derivative of V along trajectories (28) is equal to

$$\begin{aligned}\dot{V} &= -c_1(t)\sin^2\theta_e + c_1(t)\tilde{\psi}\sin\theta_e\\ &\quad - \big(-c_1(t)\cos\theta_e + av_r^2\big)\tilde{\psi}^2 - c_1(t)\frac{1}{2}\sin 2\theta_e\tilde{\psi}\\ &\le -c_1(t)\sin^2\theta_e - \big(-c_1(t)\cos\theta_e + av_r^2\big)\tilde{\psi}^2\\ &\quad + c_1(t)\Big(|\sin\theta_e| + \Big|\frac{1}{2}\sin 2\theta_e\Big|\Big)|\tilde{\psi}|.\end{aligned}$$

Since $\frac{1}{2}|\sin 2\theta_e| \le |\sin\theta_e|$ and $c_1(t)\cos\theta_e \le c_1(t)$,

$$\begin{aligned}\dot{V} \le &-c_1(t)\sin^2\theta_e\\ &- \big(-c_1(t) + av_r^2\big)\tilde{\psi}^2 + 2c_1(t)|\sin\theta_e|\,|\tilde{\psi}|.\end{aligned}$$

Assume that $c_1(t) = \frac{1}{2}\gamma av_r^2$, where $0 < \gamma < 1$. Then

$$\dot{V} \le -av_r^2\left(\frac{\gamma}{2}\big(|\sin\theta_e| - |\tilde{\psi}|\big)^2 + (1-\gamma)\tilde{\psi}^2\right) \le 0. \quad (30)$$

We also assume that θ_e is inside the interval $(-\pi, \pi)$, and we choose a very small constant δ such that $\cos\theta_e > -1 + \delta$ and $\sin^2\theta_e \ge \delta(1 - \cos\theta_e)$ hold and (30) can be transformed into the following form:

$$\begin{aligned}\dot{V} &\le -av_r^2\Big(\alpha^2\sin^2\theta_e + \beta^2\tilde{\psi}^2 - 2\alpha\beta|\sin\theta_e|\,|\tilde{\psi}|\\ &\qquad + \frac{\eta}{\delta}\sin^2\theta_e + \Big(\frac{\eta}{2} + \kappa\Big)\tilde{\psi}^2\Big)\\ &\le -av_r^2\Big(\alpha|\sin\theta_e| - \beta|\tilde{\psi}|\Big)^2\\ &\quad - \eta av_r^2\left(1 - \cos\theta_e + \frac{1}{2}\tilde{\psi}^2\right)\\ &\le -\eta av_r^2 V,\end{aligned}$$

so the system (28) is locally exponentially stable. For given $0 < \gamma < 1$ and small $\delta > 0$ we find constants α, β, η and κ by solving the set of equations

$$\alpha\beta = \frac{\gamma}{2},$$

$$\alpha^2 + \frac{\eta}{\delta} = \frac{\gamma}{2},$$

$$\beta^2 + \frac{\eta}{2} + \kappa = 1 - \frac{\gamma}{2}.$$

Finally, we shall extend our deliberations to the entire closed-loop controller.

Proposition 2. *Consider the system (8) with the control law*

$$\begin{aligned}v &= v_r + c_2x_e - c_3\omega_r y_e,\\ \omega &= \omega_r + \frac{1}{2}\gamma av_r^2\hat{\psi},\end{aligned} \quad (31)$$

and the observer given by

$$\begin{aligned}\dot{\hat{z}} &= -a\dot{v}_r y_e + av_r\omega x_e - av_r^2\hat{z} - a^2v_r^3 y_e,\\ \hat{\psi} &= \hat{z} + av_r y_e,\end{aligned} \quad (32)$$

where $c_3 > -1$, c_2, and a are positive constants, $0 < \gamma < 1$. If v_r, ω_r are bounded and persistently exciting and $\dot{\omega}_r$, $\dot{v}_r$ are bounded, the closed-loop system (8), (31) and (32) is locally exponentially stable.

Proof. The closed-loop dynamics, defined by (8), (31) and (32):

$$\begin{aligned} \dot{x}_e &= \Big((1+c_3)\omega_r + \frac{\gamma}{2} a v_r^2 \hat{\psi}\Big) y_e \\ &\quad + v_r(\cos\theta_e - 1) - c_2 x_e, \\ \dot{y}_e &= -\Big(\omega_r + \frac{\gamma}{2} a v_r^2 \hat{\psi}\Big) x_e + v_r \sin\theta_e, \\ \dot{\theta}_e &= -\frac{\gamma}{2} a v_r^2 \hat{\psi} \\ \dot{\tilde{\psi}} &= -\frac{\gamma}{2} a v_r^2 \Big(\frac{1}{2}\sin 2\theta_e - \tilde{\psi}\cos\theta_e\Big) - a v_r^2 \tilde{\psi}, \end{aligned} \tag{33}$$

can be transformed to the cascaded form

$$\underbrace{\begin{bmatrix}\dot{x}_e \\ \dot{y}_e\end{bmatrix} = \begin{bmatrix} -c_2 & (1+c_3)\omega_r \\ -\omega_r & 0 \end{bmatrix} \begin{bmatrix} x_e \\ y_e \end{bmatrix}}_{\dot{z}_1 = f_1(t,z_1)}$$

$$+ g(t, \underbrace{[x_e\ y_e]^T}_{z_1}, \underbrace{[\theta_e\ \tilde{\psi}]^T}_{z_2}) \begin{bmatrix}\theta_e \\ \tilde{\psi}\end{bmatrix}, \tag{34a}$$

$$\underbrace{\begin{bmatrix}\dot{\theta}_e \\ \dot{\tilde{\psi}}\end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\gamma a v_r^2 (\sin\theta_e - \tilde{\psi}) \\ -\frac{\gamma}{2} a v_r^2 \Big(\frac{1}{2}\sin 2\theta_e - \tilde{\psi}\cos\theta_e\Big) - a v_r^2 \tilde{\psi} \end{bmatrix}}_{\dot{z}_2 = f_2(t,z_2)} \tag{34b}$$

where the interconnection term $g(t, [x_e\ y_e]^T, [\theta_e\ \tilde{\psi}]^T)$ is in the form

$$g\big(t, [x_e\ y_e]^T, [\theta_e\ \tilde{\psi}]^T\big)$$

$$= \left[\begin{array}{c|c} \frac{\gamma}{2} a v_r^2 y_e \int_0^1 \cos s\theta_e \,\mathrm{d}s + v_r \int_0^1 \sin s\theta_e \,\mathrm{d}s & -\frac{\gamma}{2} a v_r^2 y_e \\ -\frac{\gamma}{2} a v_r^2 x_e \int_0^1 \cos s\theta_e \,\mathrm{d}s + v_r \int_0^1 \cos s\theta_e \,\mathrm{d}s & -\frac{\gamma}{2} a v_r^2 x_e \end{array}\right].$$

If v_r is persistently exciting and bounded, the subsystem (34b) is locally exponentially stable and the interconnection term satisfies Assumption 1. Furthermore, if ω_r is PE, we obtain that the subsystem $\dot{z}_1 = f_1(t, z_1)$ is GES. From Corollary 1 we conclude that the system (33) is locally exponentially stable. ■

Remark 1. We notice that both forward and angular velocities need to be persistently exciting. The assumption on v_r is needed to ensure the convergence of the observer, while the condition on ω_r results from the controller used.

5. Simulations

In order to illustrate the behaviour of the output-feedback state-tracking controllers derived in this paper, a number of simulations have been done. The simulations were carried out using MATHEMATICA. We considered the problem of tracking a circle with a constant velocity, i.e. a reference trajectory that is given by $v_r = 1$, $\omega_r = 1$, where, as in (Jiang and Nijmeijer, 1997), we took for the initial error $(x_e(0),\ y_e(0),\ \theta_e(0)) = (-0.5,\ 0.5,\ 1)$. For comparison, we first simulated the state-feedback controller (12) using the gains

$$c_1 = 5.9460, \quad c_2 = 1.3522, \quad c_3 = -0.4142, \tag{35}$$

which arise by minimizing the cost

$$\int_0^\infty x_e^2(\tau) + y_e^2(\tau) + (v_r(\tau) - v(\tau))^2 \,\mathrm{d}\tau$$

for the system (15) with an arbitrarily chosen convergence of θ_e. The resulting performance is depicted in Fig. 3.

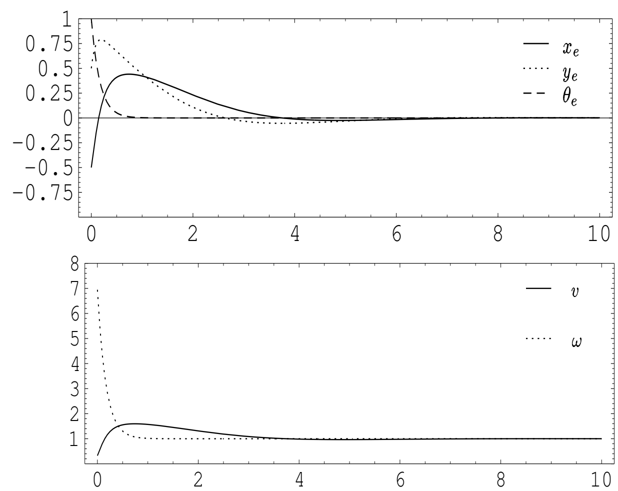

Fig. 3. Tracking errors and inputs for the state-feedback controller (12) with the controller gains (35).

For comparison, a simulation was performed for the controller (21) with the use of the same initial values and gains. The results are shown in Fig. 4.

For studying the behaviour of the position-error observer, we simulated the output-feedback controller (17) and (18) with the controller gains (35) and the observer gain

$$l = 24.7461, \tag{36}$$

which guarantees that the error dynamics for the convergence of the controller (17) and (18) are comparable to the state-feedback controller (12). The results are depicted in Fig. 5.

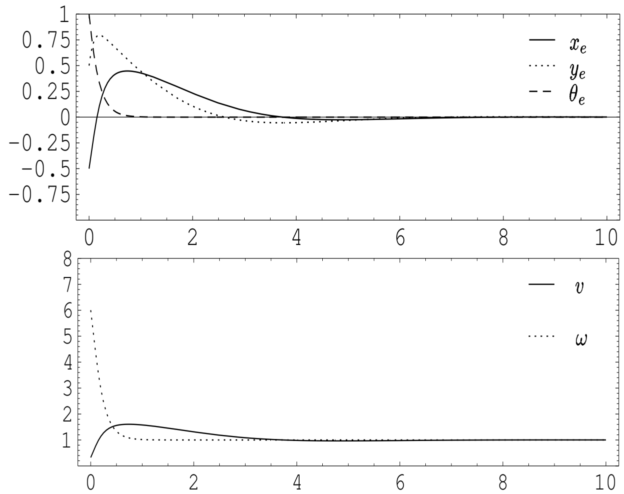

Fig. 4. Tracking errors and inputs for the state-feedback controller (21).

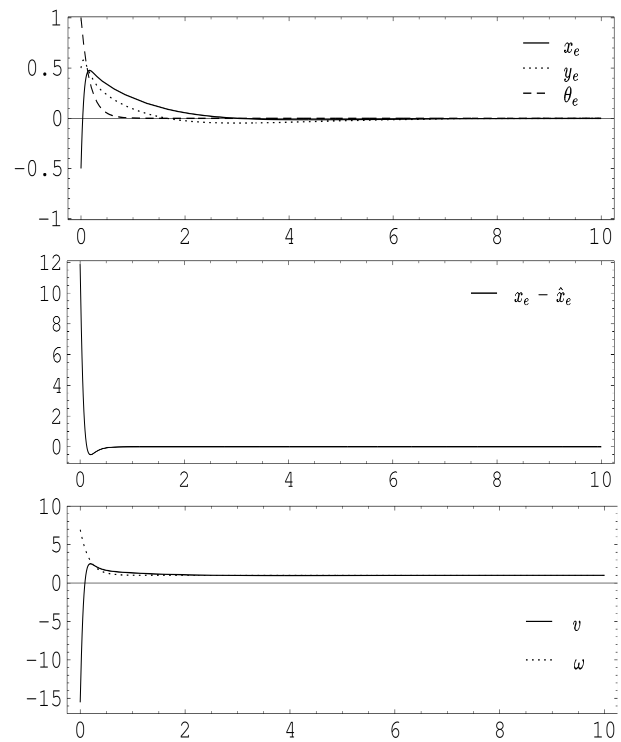

Fig. 5. Tracking errors and inputs for the output-feedback controller (17) and (18) with the controller and observer gains (35) and (36).

In Fig. 6 the results for the orientation-angle observer (32), combined with the controller (31), are presented. To draw a comparison of this controller with the previous ones, we used the same controller gains (35) and the observer gains

$$a = 10, \quad \gamma = 0.5,$$

which ensure the fast convergence of the observer error.

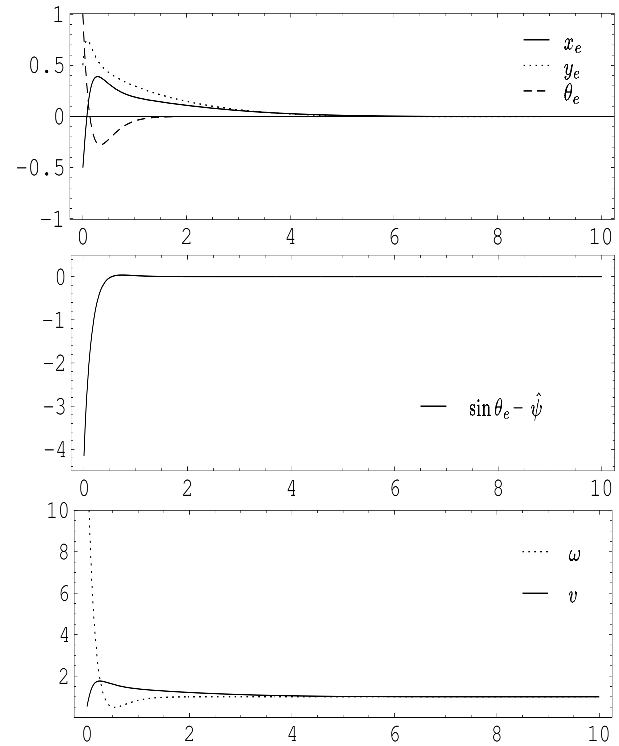

Fig. 6. Tracking errors and inputs for the output-feedback controller (31) and (32).

6. Concluding Remarks

In this paper we have designed two output-feedback tracking controllers for the unicycle-type mobile robot assuming that only the measurements of two out of three state variables are available. It corresponds to two situations encountered in some pursuit navigation problems: the position-error observer can be used when one of the distances between escaper and pursuer robots is outside the range of pursuer robot sensors, or measured with high disturbance error, or for any other reason unreliable. The second observer, estimating the orientation error, replaces the requirement of practically difficult measurements of the orientation angle with much simpler measurements of distances. Both observers can be used either to replace the real sensors or to stand as a parallel system to provide data for the controller in the case when the measurements are temporarily unavailable. In our solution we took advantage of the fact that modified observers for linear systems might be in some cases applied to nonlinear systems. We considered the tracking problem when one of the trajectory tracking error coordinates was unmeasurable. When the position error coordinate is unavailable, we are able to achieve global $\mathcal{K}$-exponential stability. In the case of the unmeasured orientation angle, only local exponential stability was shown.

It is worth noticing that the stability of both controllers assumed persistent excitation of the angular reference velocity. As a result, the output-feedback controllers are not capable of tracking, e.g., straight line trajectories. The additional requirement of the persistent excitation of v_r, appearing in the case of an unmeasured orientation angle error, means that the turning of the steering wheel is not a sufficient movement to estimate the orientation angle of the vehicle. A way of overcoming the PE-problem with the use of the concept of uδ-PE was presented in (Loría *et al.*, 1999). We believe that it is worth investigating if it also applies to the output-feedback case.

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