Lyapunov stability: Why uniform results are important, and how to obtain them

Erjen Lefeber. D&C Lunch colloquium, November 14, 2019, Eindhoven Slides available via `https://dc.wtb.tue.nl/lefeber/`

Department of Mechanical Engineering

Personal background

- Some historical facts
- Personal information
- Research topics

Some historical facts

- July 1990: member of Dutch team International Mathematical Olympiad (Beijing)
- March 1995: First journal paper (IEEE TAC)
  On the possible divergence of the projection algorithm
- April 1995: First experience with Mechanical Engineering A mathematical approach to come to an optimal velocity profile for the endurance stage of the Tour de Sol.
- June 1996: MSc in Applied Mathematics at University of Twente
  (Adaptive) control of chaotic and robot systems via bounded feedback control
- October 1999: First experiment (wearing waders) Tracking control of an underactuated ship
- April 2000: PhD in Applied Mathematics at University of Twente
  Tracking Control of Nonlinear Mechanical Systems
- Since January 2000: Assistant Professor at TU/e: Mechanical Engineering
  2000–2014 Systems Engineering Group (since 2011: Manufacturing Networks)
  2015–now Dynamics and Control Group

Personal information

- Married to Wieke, since 2000
- Four children: Jiska (15), Nathan (13), Tobias (11), Mikal (10)
- Hobby: scuba diving (Dive Master)
- Goal: Half marathon of Eindhoven 2020

Research topics

- Control of drones
- Cooperative Adaptive Cruise Control
- Intersection Control
- Network control/synchronization

Passion

Finding a Lyapunov-based stability proof

Lyapunov stability: Why uniform results are important, and how to obtain them

- Standard approach of using Barbălat + signal chasing
- Need for **uniform** asymptotic stability
- Modified approach for showing **UGAS**.

Example (Jiang, Nijmeijer, 1997)

Consider tracking error dynamics for kinematic model of mobile robot tracking a reference, expressed in its body fixed frame:

$$\dot{x}_e = \omega y_e - v + v_r \cos\theta_e \qquad \dot{y}_e = -\omega x_e + v_r \sin\theta_e \qquad \dot{\theta}_e = \omega_r - \omega$$

where ω_r and v_r are given functions of time, and $0 < v_r^{\min} \leq v_r(t) \leq v_r^{\max}$, $|\dot{v}_r| \leq a^{\max}$, $|\omega_r| \leq \omega^{\max}$. Using

$$v = v_r \cos\theta_e + k_1 x_e$$

$$\omega = \omega_r + k_2 y_e v_r \tfrac{\sin\theta_e}{\theta_e} + k_3 \theta_e \qquad \text{NB: } \tfrac{\sin\theta_e}{\theta_e} = \int_0^1 \cos(\theta_e s)\,\mathrm{d}\,s$$

with $k_1, k_2, k_3 > 0$, results in the closed-loop system

$$\dot{x}_e = \omega y_e - k_1 x_e \qquad \dot{y}_e = -\omega x_e + v_r \sin\theta_e \qquad \dot{\theta}_e = -k_2 y_e v_r \tfrac{\sin\theta_e}{\theta_e} - k_3 \theta_e$$

Example (Jiang, Nijmeijer, 1997)

Closed-loop system:

$$\dot{x}_e = \omega y_e - k_1 x_e \qquad \dot{y}_e = -\omega x_e + v_r \sin\theta_e \qquad \dot{\theta}_e = -k_2 y_e v_r \tfrac{\sin\theta_e}{\theta_e} - k_3\theta_e$$

Lyapunov function candidate: $V = \frac{1}{2}x_e^2 + \frac{1}{2}y_e^2 + \frac{1}{2k_2}\theta_e^2 > 0$

Differentiating along solutions:

$$\dot{V} = x_e(\omega y_e - k_1 x_e) + y_e(-\omega x_e + v_r \sin\theta_e) + \frac{1}{k_2}\theta_e(-k_2 y_e v_r \tfrac{\sin\theta_e}{\theta_e} - k_3\theta_e) = -k_1 x_e^2 - \frac{k_3}{k_2}\theta_e^2 \leq 0$$

How to complete the proof?

- We can not use LaSalle (1959), since closed-loop dynamics is not autonomous.
- We might use LaSalle (1976)...

Questions

Assume that $\lim_{t\to\infty} x(t) = 0$. Do we have $\lim_{t\to\infty} \dot{x}(t) = 0$?

No: Consider $x(t) = e^{-t} \sin e^{2t}$ for which $\dot{x}(t) = -e^{-t} \sin e^{2t} + 2e^{t} \cos e^{2t}$

Assume that $x(t)$ is bounded and $\lim_{t\to\infty} \dot{x}(t) = 0$. Do we have $\lim_{t\to\infty} x(t) = C$ for some constant C?

No: Consider $\dot{x}(t) = \frac{\cos(\ln(t+1))}{t+1}$ for which $x(t) = \sin(\ln(1+t))$

We need some results to complete the proof

Commonly used tools for completing the proof

Lemma (Barbălat, 1959)

Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be a uniformly continuous function (e.g., $\dot{\phi}$ bounded). Suppose that $\lim_{t\to\infty} \int_0^t \phi(\tau)\,\mathrm{d}\tau$ exists and is finite. Then $\lim\limits_{t\to\infty} \phi(t) = 0$.

Idea: For $\phi(t)$ use $\dot{V}(t)$.

Lemma (Micaelli, Samson, 1993)

Let $f : \mathbb{R}_+ \to \mathbb{R}$ be any differentiable function. If $\lim_{t\to\infty} f(t) = 0$ and

$$\dot{f}(t) = f_0(t) + \eta(t) \qquad\qquad t \geq 0$$

where f_0 is a uniformly continuous function (e.g., $\dot{f}_0$ is bounded) and $\lim_{t\to\infty} \eta(t) = 0$, then $\lim_{t\to\infty} \dot{f}(t) = \lim_{t\to\infty} f_0(t) = 0$.

Idea: Signal chasing by (repeatedly) applying to signals that converge to zero

Example (Jiang, Nijmeijer, 1997)

Since $\dot{V} \leq 0$ we have: x_e, y_e, θ_e bounded.

Barbălat: $\ddot{V}$ bounded, $\lim_{t\to\infty} \int_0^t \dot{V}\, \mathrm{d}\, t = \lim_{t\to\infty} V(t) - V(0)$ exists and finite, so $\lim_{t\to\infty} \dot{V}(t) = 0$, i.e., $\lim_{t\to\infty} x_e(t) = \lim_{t\to\infty} \theta_e(t) = 0$.

Lemma of Micaelli and Samson: $\dot{\theta}_e = -\underbrace{k_2 y_e v_r}_{f_0(t)} + \underbrace{k_2 y_e v_r \left(1 - \frac{\sin\theta_e}{\theta_e}\right) - k_3\theta_e}_{\eta(t)}$

f_0 uniformly continuous, $\lim_{t\to\infty} \eta(t) = 0$, so $\lim_{t\to\infty} y_e(t) v_r(t) = 0$ and therefore $\lim_{t\to\infty} y_e(t) = 0$.

From the above we can conclude global asymptotic stability of the closed-loop system.

Standard form

Previous example is standard proof.

More general: $\dot{x}_1 = f_1(x_1, x_2, x_3, t)$, $\dot{x}_2 = f_2(x_1, x_2, x_3, t)$, $\dot{x}_3 = f_3(x_1, x_2, x_3, t)$

- Lyapunov function: $V(x_1, x_2, x_3, t)$ positive definite.
- Derivative along dynamics: $\dot{V}(x_1, t)$ negative semi-definite.
- Using Barbălat: $\dot{V}(x_1, t) \to 0$, which implies $x_1 \to 0$.
- Using Micaelli, Samson: $f_1(0, x_2, x_3, t) \to 0$, which implies $x_2 \to 0$.
- Using Micaelli, Samson: $f_2(0, 0, x_3, t) \to 0$, which implies $x_3 \to 0$.

Or even more general. . .

Using this approach we can show global asymptotic stability. However, is that what we want?

Example (Panteley, Loría, Teel, 1999)

Consider the system

$$\dot{x} = \begin{cases} \frac{1}{1+t} & \text{if } x \leq -\frac{1}{1+t} \\ -x & \text{if } |x| \leq \frac{1}{1+t} \\ -\frac{1}{1+t} & \text{if } x \geq \frac{1}{1+t} \end{cases}$$

For each $r > 0$ and $t_0 \geq 0$ there exist $k > 0$ and $\gamma > 0$ such that for all $t \geq t_0$ and $|x(t_0)| \leq r$:

$$|x(t)| \leq k|x(t_0)|e^{-\gamma(t-t_0)} \qquad \forall t \geq t_0 \geq 0$$

However, always a bounded (arbitrarily small) additive perturbation $\delta(t,x)$ and a constant $t_0 \geq 0$ exist such that the trajectories of the perturbed system $\dot{x} = f(t,x) + \delta(t,x)$ are unbounded.

Main reason for this negative result: the constants k and γ are allowed to depend on t_0, i.e., for each value of t_0 different constants k and γ may be chosen.

Robustness to perturbations for UGAS

Lemma (Khalil 1996 (2nd ed), Lemma 5.3; Khalil 2002 (3rd ed), Lemma 9.3)

Let $x = 0$ be a uniformly asymptotically stable equilibrium point of the nominal system $\dot{x} = f(t,x)$ where $f : \mathbb{R}_+ \times B_r \to \mathbb{R}^n$ is continuously differentiable, and the Jacobian $\left[\frac{\partial f}{\partial x}\right]$ is bounded on B_r, uniformly in t. Then one can determine constants $\Delta > 0$ and $R > 0$ such that for all perturbations $\delta(t,x)$ that satisfy the uniform bound $\|\delta(t,x)\| \leq \delta < \Delta$ and all initial conditions $\|x(t_0)\| \leq R$, the solution $x(t)$ of the perturbed system $\dot{x} = f(t,x) + \delta(t,x)$ satisfies

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) \qquad \forall t_0 \leq t \leq t_1 \qquad \text{and} \qquad \|x(t)\| \leq \rho(\delta) \qquad \forall t \geq t_1$$

for some $\beta \in \mathcal{KL}$ and some finite time t_1, where $\rho(\delta)$ is a class $\mathcal{K}$ function of δ. Furthermore, if $x = 0$ is a uniformly globally exponentially stable equilibrium point, we can allow for arbitrarily large δ by choosing $R > 0$ large enough.

Problem

Lesson learned from example

For robustness we need uniform global asymptotic stability.

Subject of remainder of this talk (10 minutes)

How to show UGAS when we do not have a proper Lyapunov function, i.e, when $\dot{V}$ is negative semi-definite, but are able to complete the proof using Barbălat + signal chasing

After this talk, you (hopefully) know:

- How to complete a proof using Barbălat + signal chasing
- Using Barbălat + signal chasing shows only GAS, whereas we want UGAS.
- How to show UGAS using different tools

Matrosov like theorem (Loría et.al., 2005)

Consider the dynamical system

$$\dot{x} = f(t, x) \qquad x(t_0) = x_0 \qquad f(t, 0) = 0 \tag{1}$$

$f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ loc. bounded, continuous a.e., loc. unif. continuous in t. If there exist

- j differentiable functions $V_i : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$, bounded in t, and
- continuous functions $Y_i : \mathbb{R}^n \to \mathbb{R}$ for $i \in \{1, 2, \ldots j\}$ such that

- V_1 is positive definite and radially unbounded,
- $\dot{V}_i(t, x) \leq Y_i(x)$, for all $i \in \{1, 2, \ldots, j\}$,
- $Y_i(x) = 0$ for $i \in \{1, 2, \ldots, k-1\}$ implies $Y_k(x) \leq 0$, for all $k \in \{1, 2, \ldots, j\}$,
- $Y_i(x) = 0$ for all $i \in \{1, 2, \ldots, j\}$ implies $x = 0$,

then the origin $x = 0$ of (1) is uniformly globally asymptotically stable.

Question: how to determine suitable functions V_i and Y_i (for $i > 1$)?

Example (revisited)

Closed-loop system: $\dot{x}_e = \omega y_e - k_1 x_e$, $\dot{y}_e = -\omega x_e + v_r \sin\theta_e$, $\dot{\theta}_e = -k_2 y_e v_r \frac{\sin\theta_e}{\theta_e} - k_3\theta_e$.

Lyapunov function candidate: $V_1 = \frac{1}{2}x_e^2 + \frac{1}{2}y_e^2 + \frac{1}{2k_2}\theta_e^2$.

Differentiating along solutions: $\dot{V}_1 = -k_1 x_e^2 - \frac{k_3}{k_2}\theta_e^2 = Y_1$.

Consider $V_2 = -\theta_e\dot{\theta}_e$. Then

$$\begin{aligned}\dot{V}_2 &= -\dot{\theta}_e^2 - \theta_e\ddot{\theta}_e = -[-k_2 y_e v_r + \eta(t)]^2 - \theta_e\ddot{\theta}_e = -(k_2 y_e v_r)^2 + 2k_2 y_e v_r \eta(t) - \eta(t)^2 - \theta_e\ddot{\theta}_e \\ &\leq -k_2^2 (v_r^{\min})^2 y_e^2 \ + \ M_1\|\eta\| + \|\eta\|^2 + M_2\|\theta_e\| = Y_2.\end{aligned}$$

Note that $Y_1 = 0$ implies $Y_2 \leq 0$. Furthermore, $Y_1 = Y_2 = 0$ implies $x_e = y_e = \theta_e = 0$.

Therefore: uniform global asymptotic stability.

New standard approach for uniform results

More general case: $\dot{x}_1 = f_1(x_1, x_2, x_3, t)$, $\dot{x}_2 = f_2(x_1, x_2, x_3, t)$, $\dot{x}_3 = f_3(x_1, x_2, x_3, t)$

- Lyapunov function: $V_1(x_1, x_2, x_3, t)$ positive definite.
- Derivative along dynamics: $\dot{V}_1(x_1, t) = \cdots \leq Y_1(x_1)$ negative semi-definite.
- Use $V_2 = -x_1^T \dot{x}_1$. Then $\dot{V}_2 \leq -[f_1(0, x_2, x_3, t)]^2 + F_2(\|x_1\|) \leq Y_2(x)$.
- $Y_1 = 0$ implies $Y_2 \leq 0$. Furthermore $Y_1 = Y_2 = 0$ implies $x_1 = x_2 = 0$.
- Use $V_3 = -x_2^T \dot{x}_2$. Then $\dot{V}_3 \leq -[f_2(0, 0, x_3, t)]^2 + F_3(\|x_1\|, \|x_2\|) \leq Y_3(x)$.
- $Y_1 = Y_2 = 0$ implies $Y_3 =\leq 0$. Also, $Y_1 = Y_2 = Y_3 = 0$ implies $x_1 = x_2 = x_3 = 0$.
- Conclusion: uniform global asymptotic stability.

NB: Often simpler functions can be found for V_i, e.g., $V_2 = -f_1(0, x_2, x_3, t)^T \dot{x}_1$, etc.

Conclusions

- Got to know Erjen Lefeber slightly better

- We recalled the standard approach of using Barbălat + signal chasing
- We illustrated the need for uniform asymptotic stability
- We showed how to modify the standard approach for showing GAS to prove UGAS instead.