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Lyapunov stability: Why uniform results are important, and how to obtain them

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Where innovation starts

- Sphere of radius r: $B_r \triangleq \{x \in \mathbb{R}^n \,|\, \|x\| < r\}$.
- A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be uniformly continuous on a set S if:

$$\forall \epsilon > 0 \; \exists \delta > 0 : \qquad \|x - y\| < \delta \Rightarrow \|f(x) - f(y)\| < \epsilon \qquad \forall x, y \in S.$$

Lemma: Consider a differentiable function $f : \mathbb{R} \to \mathbb{R}$. If a constant $M \in \mathbb{R}$ exists such that

$$\sup_{x \in \mathbb{R}} \left| \frac{\mathrm{d}f}{\mathrm{d}x}(x) \right| \leq M,$$

then f is uniformly continuous on $\mathbb{R}$.

- A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ ($\alpha \in \mathcal{K}$) if:
  - it is strictly increasing, and
  - $\alpha(0) = 0$.
- A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ ($\beta \in \mathcal{KL}$) if
  - for each fixed s the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r, and if
  - for each fixed r the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.

Consider the system

$$\dot{x} = f(t,x) \qquad \text{where} \qquad f(t,0) = 0 \qquad \forall t \geq 0 \tag{1}$$

- The equilibrium point $x = 0$ of (1) is said to be globally asymptotically stable (GAS) if for all $t_0 \in \mathbb{R}_+$ a function $\beta \in \mathcal{KL}$ exists such that for all $x(t_0) \in \mathbb{R}^n$
$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) \qquad \forall t \geq t_0 \geq 0$$
- The equilibrium point $x = 0$ of (1) is said to be uniformly globally asymptotically stable (UGAS) if a function $\beta \in \mathcal{KL}$ exists such that for all $(t_0, x(t_0)) \in \mathbb{R}_+ \times \mathbb{R}^n$
$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) \qquad \forall t \geq t_0 \geq 0$$

Example (Panteley, Loría, Teel, 1999)

Consider the system

$$\dot{x} = \begin{cases} \frac{1}{1+t} & \text{if } x \leq -\frac{1}{1+t} \\ -x & \text{if } |x| \leq \frac{1}{1+t} \\ -\frac{1}{1+t} & \text{if } x \geq \frac{1}{1+t} \end{cases}$$

For each $r > 0$ and $t_0 \geq 0$ there exist $k > 0$ and $\gamma > 0$ such that for all $t \geq t_0$ and $|x(t_0)| \leq r$:

$$|x(t)| \leq k|x(t_0)|e^{-\gamma(t-t_0)} \qquad \forall t \geq t_0 \geq 0$$

However, always a bounded (arbitrarily small) additive perturbation $\delta(t,x)$ and a constant $t_0 \geq 0$ exist such that the trajectories of the perturbed system $\dot{x} = f(t,x) + \delta(t,x)$ are unbounded.
Main reason for this negative result: the constants k and γ are allowed to depend on t_0, i.e., for each value of t_0 different constants k and γ may be chosen.

Lemma (Khalil 1996, Lemma 5.3: Robustness to perturbations for UGAS)

Let $x = 0$ be a uniformly asymptotically stable equilibrium point of the nominal system $\dot{x} = f(t, x)$ where $f : \mathbb{R}_+ \times B_r \to \mathbb{R}^n$ is continuously differentiable, and the Jacobian $\left[\frac{\partial f}{\partial x}\right]$ is bounded on B_r, uniformly in t. Then one can determine constants $\Delta > 0$ and $R > 0$ such that for all perturbations $\delta(t, x)$ that satisfy the uniform bound $\|\delta(t, x)\| \leq \delta < \Delta$ and all initial conditions $\|x(t_0)\| \leq R$, the solution $x(t)$ of the perturbed system $\dot{x} = f(t, x) + \delta(t, x)$ satisfies

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) \qquad \forall t_0 \leq t \leq t_1$$

and

$$\|x(t)\| \leq \rho(\delta) \qquad \forall t \geq t_1$$

for some $\beta \in \mathcal{KL}$ and some finite time t_1, where $\rho(\delta)$ is a class $\mathcal{K}$ function of δ. Furthermore, if $x = 0$ is a uniformly globally exponentially stable equilibrium point, we can allow for arbitrarily large δ by choosing $R > 0$ large enough.

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Lesson learned from example

For robustness we need uniform global asymptotic stability.

Subject of this talk

How to show this when we do not have a proper Lyapunov function, i.e, when $\dot{V}$ is negative semi-definite.

We will see:

- Using Barbălat (+ signal chasing) shows only GAS, whereas we want UGAS.
- How to show UGAS using different tools

We need one more slide with preliminaries before we move to an illustrative example. . .

Lemma (Barbălat, 1959)

Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be a uniformly continuous function. Suppose that $\lim_{t\to\infty} \int_0^t \phi(\tau)\mathrm{d}\tau$ exists and is finite. Then $\lim\limits_{t\to\infty} \phi(t) = 0$.

Corollary

If $f \in \mathcal{L}_\infty$, $\dot{f} \in \mathcal{L}_\infty$, and $f \in \mathcal{L}_p$ for some $p \in [1,\infty)$, then $\lim\limits_{t\to\infty} f(t)^p = 0$, so $\lim\limits_{t\to\infty} f(t) = 0$.

Lemma (Micaelli, Samson, 1993)

Let $f : \mathbb{R}_+ \to \mathbb{R}$ be any differentiable function. If $\lim_{t\to\infty} f(t) = 0$ and

$$\dot{f}(t) = f_0(t) + \eta(t) \qquad\qquad t \geq 0$$

where f_0 is a uniformly continuous function (e.g., $\dot{f}_0$ is bounded) and $\lim_{t\to\infty} \eta(t) = 0$, then $\lim_{t\to\infty} \dot{f}(t) = \lim_{t\to\infty} f_0(t) = 0$.

Consider tracking error dynamics for kinematic model of mobile robot tracking a reference, expressed in its body fixed frame:

$$\begin{aligned}
\dot{x}_e &= \omega y_e - v + v_r \cos\theta_e \\
\dot{y}_e &= -\omega x_e + v_r \sin\theta_e \\
\dot{\theta}_e &= \omega_r - \omega
\end{aligned}$$

where ω_r and v_r are given functions of time, and $0 < v_r^{\min} \leq v_r(t) \leq v_r^{\max}$, $|\dot{v}_r| \leq a^{\max}$, $|\omega_r| \leq \omega^{\max}$. Using

$$\begin{aligned}
v &= v_r \cos\theta_e + k_1 x_e \\
\omega &= \omega_r + k_2 y_e v_r \frac{\sin\theta_e}{\theta_e} + k_3\theta_e
\end{aligned}$$

NB: More correct: $\frac{\sin\theta_e}{\theta_e} \Rightarrow \int_0^1 \cos(\theta_e s)\mathrm{d}s$

with $k_1, k_2, k_3 > 0$, results in the closed-loop system

$$\begin{aligned}
\dot{x}_e &= \omega y_e - k_1 x_e \\
\dot{y}_e &= -\omega x_e + v_r \sin\theta_e \\
\dot{\theta}_e &= -k_2 y_e v_r \frac{\sin\theta_e}{\theta_e} - k_3\theta_e
\end{aligned}$$

Closed-loop system:

$$\dot{x}_e = \omega y_e - k_1 x_e \qquad \dot{y}_e = -\omega x_e + v_r \sin\theta_e \qquad \dot{\theta}_e = -k_2 y_e v_r \frac{\sin\theta_e}{\theta_e} - k_3\theta_e$$

Lyapunov function candidate: $V = \frac{1}{2}x_e^2 + \frac{1}{2}y_e^2 + \frac{1}{2k_2}\theta_e^2$
Differentiating along solutions:

$$\dot{V} = x_e(\omega y_e - k_1 x_e) + y_e(-\omega x_e + v_r \sin\theta_e) + \frac{1}{k_2}\theta_e(-k_2 y_e v_r \frac{\sin\theta_e}{\theta_e} - k_3\theta_e) = -k_1 x_e^2 - \frac{k_3}{k_2}\theta_e^2$$

Barbălat (or Corollary): $V \in \mathcal{L}_\infty$, $\dot{V} \in \mathcal{L}_\infty$, $\dot{V} \in \mathcal{L}_1$, so $\lim_{t\to\infty} \dot{V}(t) = 0$, i.e., $\lim_{t\to\infty} x_e(t) = \lim_{t\to\infty} \theta_e(t) = 0$.

Lemma of Micaelli and Samson: $\dot{\theta}_e = -\underbrace{k_2 y_e v_r}_{f_0(t)} + \underbrace{k_2 y_e v_r \left(1 - \frac{\sin\theta_e}{\theta_e}\right) - k_3\theta_e}_{\eta(t)}$

f_0 uniformly continuous, $\lim_{t\to\infty} \eta(t) = 0$, so $\lim_{t\to\infty} y_e(t)v_r(t) = 0$ and therefore $\lim_{t\to\infty} y_e(t) = 0$.
From the above we can conclude global asymptotic stability of the closed-loop system.

Previous example is standard proof. More general case: $\dot{x}_1 = f_1(x_1, x_2, t)$, $\dot{x}_2 = f_2(x_1, x_2, t)$

- Lyapunov function: $V(x_1, x_2, t)$ positive definite.
- Derivative along dynamics: $\dot{V}(x_1, t)$ negative semi-definite.
- Using Barbălat: $x_1 \to 0$.
- Using Micaelli, Samson: $f_1(0, x_2, t) \to 0$, which implies $x_2 \to 0$.

Or even more general: $\dot{x}_1 = f_1(x_1, x_2, x_3, t)$, $\dot{x}_2 = f_2(x_1, x_2, x_3, t)$, $\dot{x}_3 = f_3(x_1, x_2, x_3, t)$

- Lyapunov function: $V(x_1, x_2, x_3, t)$ positive definite.
- Derivative along dynamics: $\dot{V}(x_1, t)$ negative semi-definite.
- Using Barbălat: $x_1 \to 0$.
- Using Micaelli, Samson: $f_1(0, x_2, x_3, t) \to 0$, which implies $x_2 \to 0$.
- Using Micaelli, Samson: $f_2(0, 0, x_3, t) \to 0$, which implies $x_3 \to 0$.

Or even more general...

Using this approach we can show global asymptotic stability. However, we look for uniform result!

Theorem (Corollary of Matrosov like theorem by Loría, Panteley, Popović, Teel, 2005)

Consider the dynamical system

$$\dot{x} = f(t,x) \qquad x(t_0) = x_0 \qquad f(t,0) = 0 \tag{2}$$

$f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ *locally bounded, continuous almost everywhere, and locally uniformly continuous in* t*.*
If there exist

- j *differentiable functions* $V_i : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$*, bounded in* t*, and*
- *continuous functions* $Y_i : \mathbb{R}^n \to \mathbb{R}$ *for* $i \in \{1, 2, \dots j\}$ *such that*
- V_1 *is positive definite and radially unbounded,*
- $\dot{V}_i(t,x) \leq Y_i(x)$*, for all* $i \in \{1, 2, \dots, j\}$*,*
- $Y_i(x) = 0$ *for* $i \in \{1, 2, \dots, k-1\}$ *implies* $Y_k(x) \leq 0$*, for all* $k \in \{1, 2, \dots, j\}$*,*
- $Y_i(x) = 0$ *for all* $i \in \{1, 2, \dots, j\}$ *implies* $x = 0$*,*

then the origin $x = 0$ *of (2) is uniformly globally asymptotically stable.*

Question: how to determine suitable functions V_i and Y_i (for $i > 1$)?

Closed-loop system:

$$\dot{x}_e = \omega y_e - k_1 x_e \qquad \dot{y}_e = -\omega x_e + v_r \sin\theta_e \qquad \dot{\theta}_e = -k_2 y_e v_r \frac{\sin\theta_e}{\theta_e} - k_3\theta_e$$

Lyapunov function candidate: $V_1 = \frac{1}{2}x_e^2 + \frac{1}{2}y_e^2 + \frac{1}{2k_2}\theta_e^2$
Differentiating along solutions:

$$\dot{V}_1 = -k_1 x_e^2 - \frac{k_3}{k_2}\theta_e^2 = Y_1$$

Consider $V_2 = -\theta_e\dot{\theta}_e$. Then

$$\dot{V}_2 = -\dot{\theta}_e^2 - \theta_e\ddot{\theta}_e = -\left(-k_2 y_e v_r + k_2 y_e v_r\left(1 - \frac{\sin\theta_e}{\theta_e}\right) - k_3\theta_e\right)^2 - \theta_e\ddot{\theta}_e$$

$$= -(k_2 y_e v_r)^2 + 2k_2 y_e v_r\left[k_2 y_e v_r\left(1 - \frac{\sin\theta_e}{\theta_e}\right) - k_3\theta_e\right] - \left[k_2 y_e v_r\left(1 - \frac{\sin\theta_e}{\theta_e}\right) - k_3\theta_e\right]^2 - \theta_e\ddot{\theta}_e = Y_2$$

Note that $Y_1 = 0$ implies $Y_2 \leq 0$. Furthermore, $Y_1 = Y_2 = 0$ implies $x_e = y_e = \theta_e = 0$.
Therefore: uniform global asymptotic stability.

More general case: $\dot{x}_1 = f_1(x_1, x_2, t)$, $\dot{x}_2 = f_2(x_1, x_2, t)$

- Lyapunov function: $V_1(x_1, x_2, t)$ positive definite.
- Derivative along dynamics: $\dot{V}_1(x_1, t) \leq Y_1(x_1)$ negative semi-definite.
- Use $V_2 = -x_1^T \dot{x}_1$. Then $\dot{V}_2 = Y_2$.
- Note that $Y_1 = 0$ implies $Y_2 = -f_1(0, x_2, t)^2 \leq 0$. Furthermore $Y_1 = Y_2 = 0$ implies $x_1 = x_2 = 0$.
- Conclusion: uniform global asymptotic stability.

Or even more general: $\dot{x}_1 = f_1(x_1, x_2, x_3, t)$, $\dot{x}_2 = f_2(x_1, x_2, x_3, t)$, $\dot{x}_3 = f_3(x_1, x_2, x_3, t)$

- Lyapunov function: $V_1(x_1, x_2, x_3, t)$ positive definite.
- Derivative along dynamics: $\dot{V}_1(x_1, t) \leq Y_1(x_1)$ negative semi-definite.
- Use $V_2 = -x_1^T \dot{x}_1$. Then $\dot{V}_2 = Y_2$.
- Note that $Y_1 = 0$ implies $Y_2 = -f_1(0, x_2, x_3, t)^2 \leq 0$. Furthermore $Y_1 = Y_2 = 0$ implies $x_1 = x_2 = 0$.
- Use $V_3 = -x_2^T \dot{x}_2$. Then $\dot{V}_3 = Y_3$.
- $Y_1 = Y_2 = 0$ implies $Y_3 = -f_2(0, 0, x_3, t)^2 \leq 0$. Also, $Y_1 = Y_2 = Y_3 = 0$ implies $x_1 = x_2 = x_3 = 0$.
- Conclusion: uniform global asymptotic stability.

Conclusions

- We showed the need for uniform asymptotic stability
- We provided a way how to modify commonly used techniques for showing GAS to prove UGAS instead.