An Example of a Bounded Tracking Controller

A prototype example for designing a globally bounded tracking controller is presented.

The solution obtained by using composite or switching controllers.

First, the system is steered 'close' to the desired trajectory, after which a local tracking controller is applied.

For the considered system, both controllers are bounded and therefore boundedness of the composite controller follows.

A simulation illustrates the idea.

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1 Introduction

In this paper we study the problem of finding global tracking controllers that meet input constraints. This study is motivated by the existence of actuator saturation or constraints on actuators as commonly encountered in control engineering.
Very often controllers are designed without taking into account the actuator constraints and simply saturating the designed controller in case the actuator limits are encountered.
For one or two-dimensional linear systems this strategy may work, but in [3] it has been shown that for a triple integrator such saturated linear feedback is not sufficient.

The fact that even linear feedback laws when saturated can even lead to instability, known as anti-windup (AWR), has motivated a large amount of research.
Solutions to the bounded feedback stabilisation of linear systems can be found in [12] and [13].

On the other hand, for nonlinear systems relatively few results on bounded feedback stabilisation are available. Most of today's results in nonlinear saturated control deal with rigid robot systems, see e.g. [1,5], or mobile robots, see [4]. Also extensions with bounded output feedback and tracking can be found in [9] and references therein.
Another recent contribution on nonlinear saturated feedback is [2] where a backstepping procedure for the design of globally stabilising state feedback control laws that meet input constraints is presented.

In this paper we treat the problem of finding global tracking controllers that meet input constraints by introducing the idea of composite controllers.
The control scheme essentially combines a bounded stabilising controller with a local asymptotically stable tracking controller and is illustrated on the driven van der Pol system.
However, the idea is applicable for general second order nonlinear systems for state feedback, output feedback and adaptive state feedback, as is shown in [6], see also [7,8].

The organisation of this paper is as follows. Section 2 contains the problem formulation and preliminaries. In section 3 the key idea for the construction of a globally bounded controller is explained. In section 4 the controller design for the driven van der Pol system in case of state feedback is considered in detail.
The resulting performance is shown in section 5 by means of numerical simulations. Section 6 contains concluding remarks.

2 Preliminaries and problem formulation

2.1 The van der Pol system

The example on which we illustrate the main ideas is the van der Pol system.

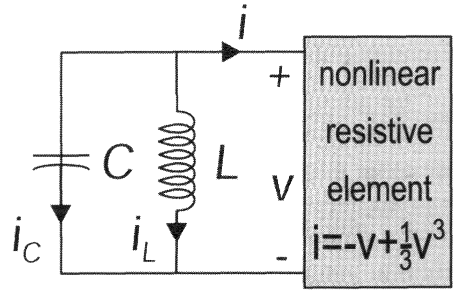

Fig. 2.1. The van der Pol circuit

The van der Pol system can be realised as the electrical circuit of Figure 2.1.
We assume the inductor and capacitor to be linear, time-invariant and passive, i.e. $L > 0$ and $C > 0$.
The resistive element is an active circuit characterised by the voltage-controlled i - v characteristic $h(v) = \frac{1}{3}v^3 - v$.

Using Kirchhoff's current law we obtain

$$C\frac{dv}{dt} + \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau + \frac{1}{3}v^3 - v = 0$$

Differentiating with respect to t and rescaling the time-variable results in

$$\ddot{v} + \sqrt{L/C}(v^2 - 1)\dot{v} + v = 0$$

In case we add a source of alternating voltage to the circuit, we obtain

$$\ddot{v} + \mu(v^2 - 1)\dot{v} + v = q\cos\omega t \tag{1}$$

which is known as the driven van der Pol oscillator, with $\mu = \sqrt{L/C}$.
Van der Pol used this equation to model an electrical circuit with a triode valve.

It is interesting to note that this system exhibits chaotic behaviour, for various parameter values as for instance $\mu = 5$, and $q = 5$, and $\omega = 2.463$, cf. [11], showing that the dynamics is very rich and complex.

Throughout, we consider a controlled version of the driven van der Pol equation :

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = u + q\cos\omega t \tag{2}$$

where $\mu > 0$, q and ω are known constants and u is the input.

2.2 Problem formulation

Consider the (complex) system (2). Suppose that measurements of x and $\dot{x}$ are available. Let $x_d(t)$ be a desired trajectory for the system (2), and assume that $x_d(t)$ is at least two times continuously differentiable in t and satisfies

$$|x_d(t)| \le B_0, \quad |\dot{x}_d(t)| \le B_1, \tag{3}$$

for given positive constants B_0, B_1 and B_2. Then the tracking control problem under actuator constraints consists of designing, if possible, a state feedback law $u(t) = \alpha(x, \dot{x}, x_d, \dot{x}_d, t)$ for the control $u(t)$ such that $x(t)$ and $\dot{x}(t)$ approach their desired values $x_d(t)$ and $\dot{x}_d(t)$ while keeping the applied input $u(t)$ within in advance specified bounds, i.e. there exists a constant u_{max} such that

$$|u(t)| \le u_{\max} \text{ for all } t \ge 0. \tag{4}$$

In other words : design a controller for $u(t)$ such that

$$\lim_{t \to \infty} |x(t) - x_d(t)| = \lim_{t \to \infty} |\dot{x}(t) - \dot{x}_d(t)| = 0 \tag{5}$$

while satisfying (4) for all initial conditions $(x(0),\dot{x}(0))$.

3 The main idea: using composite controllers

The problem under consideration is that of finding a controller $u(t) = \alpha(x, \dot{x}, x_d, \dot{x}_d, t)$ that guarantees the tracking condition (5) under the constraint (4). No matter how large the initial errors are, the control effort has to remain within the bounds (4).

To solve this problem we introduce the idea of composite controllers. The problem of finding an a priori bounded tracking controller can be divided into two subproblems. First of all we need to find a local tracking controller. In general this controller is not a priori bounded : the larger the errors are, the larger the control action becomes. How do we meet the input constraints globally ?

Notice that even though this local tracking controller might not be a priori bounded, we can determine a bound on the input when we restrict our selves to a (small) region of attraction around the origin of the tracking-error dynamics. That is, we can find a subset of points in the error-dynamics state-space that is such that if we start in this set, we will remain in that set for all future time. Furthermore for any point in the set the tracking control meets the input constraints. There is no need to modify the controller on this set. We only need to redefine the controller due to the input constraints on the other points in state space.

If we are able to find a controller that globally brings our system into the region of attraction of the local tracking controller, while meeting the input constraints, our problem has been solved. Since, then we first use this controller until we are in the region of attraction of the tracking controller and then switch to the tracking controller to achieve tracking. This composite controller yields global tracking and meets the input constraints

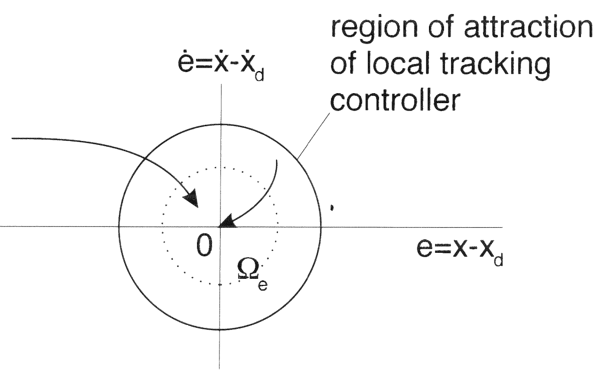

Fig. 3.1. A graphical representation of the main idea

The idea can be depicted as follows (see Figure 3.1). Assume we have a local tracking controller with the solid circle as its region of attraction. On this region the tracking controller is bounded. Then we look for a second controller that is bounded and brings our system into this region of attraction. Using these two controllers we can compose an a priori bounded global tracking controller. The composition consists of first applying the bounded global controller that brings us into the region of attraction (solid circle) and when we are in that region of attraction, e.g. in the dashed circle Ω_ϵ, we switch to the tracking controller.

The key-idea is that when the tracking error is large we mainly have to concentrate on reducing the tracking error rather than on tracking itself. Only when the tracking errors have become small enough we start tracking. In this way we find a global controller that guarantees tracking while meeting the input constraints.

Therefore the problem of finding a global tracking controller that meets input constraints can be divided into two subproblems, namely that of

1. Finding a tracking controller.
2. Finding a bounded controller that brings us into the region of attraction of the controller found in 1.

Since the desired trajectory is assumed to be a priori bounded (c.f. (3)), the bounded controller that brings us into the region of attraction of the tracking controller is for simplicity chosen to be a controller that steers the system towards a fixed point for the system (2). Of course several other, more sophisticated, choices are possible.

4 Bounded tracking control of the van der Pol system

In this section we consider the problem of finding a global tracking controller for the driven van der Pol system (2), that remains a priori bounded. We assume that the reference trajectory to be tracked is bounded, i.e. that it satisfies (3). To arrive at a bounded tracking controller we use the composite controller idea as presented in the previous section.

First we need to have a (not necessarily a priori bounded) tracking controller

Proposition 4.1. *Consider the system (2) together with the feedback-linearising control law*

$$u = \ddot{x}_d - \mu(1-x^2)\dot{x} + x - K_d\dot{e} - K_p e - q\cos\omega t \tag{6}$$

where $K_p>0$ and $K_d>0$ are constant and $e = x - x_d$, $\dot{e} = \dot{x} - \dot{x}_d$. *The resulting closed-loop system is globally asymptotically stable.*

The proof of this proposition in given in [6] and merely follows from the fact that the error dynamics is given by

$$\ddot{e} + K_d\,\dot{e} + K_p\,e = 0, \tag{7}$$

which is asymptotically stable for K_p and K_d positive.

Although the controller (6) is a global tracking controller, it is **not** a bounded tracking controller. As proposed in section 3 we now look for a globally bounded controller that steers the system towards a fixed point for the system (2). In this case the point $(x,\dot{x})=(2,0)$ in order to arrive at a simple controller. As mentioned in section 3, more clever choices for bounded controllers that bring us into the region of attraction of the tracking controller can be made.

Proposition 4.2 *Consider the system (2) together with the control law*

$$u = 2 - q\cos\omega t - K_p\tanh(\lambda(x-2)) \tag{8}$$

where $K_p \geq 0$ and $\lambda > 0$ are constants and tanh(*y*) = $\dfrac{e^y - e^{-y}}{e^y + e^{-y}}$, *see Figure 4.1. For the resulting closed-loop system we have* $\lim_{t\to\infty} x(t) = 2$.

A proof of this proposition can be found in [6].

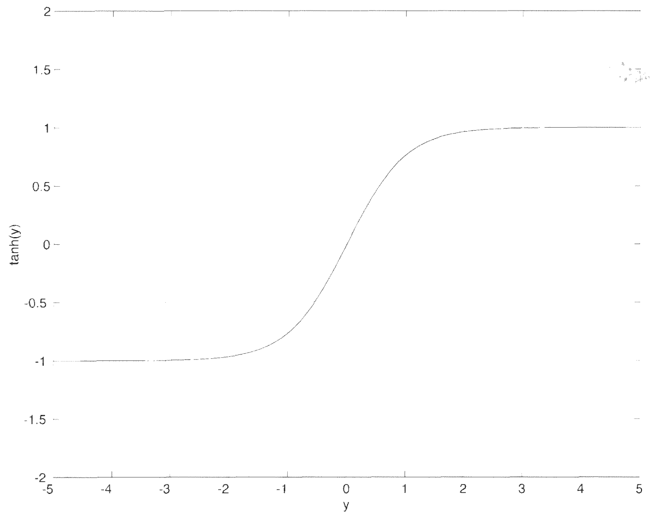

Fig. 4.1. The function tanh(*y*)

Following section 3, we combine the two control laws (6) and (8) into a globally tracking controller that is bounded. We first use the controller (8) to reduce the tracking error within reasonable bounds. Since the desired trajectory to track satisfies (3) we know that, no matter what initial conditions we have, eventually our tracking errors will be within in advance known bounds. It can be shown that we are guaranteed to enter the set

$$\Omega_\varepsilon \triangleq \{(e,\dot{e}) \mid \tfrac{1}{2}\dot{e}^2 + \tfrac{1}{2}K_p\,e^2 \leq \tfrac{1}{2}B_1^2 + \tfrac{1}{2}K_p\,(B_0+2)^2 + \varepsilon\} \tag{9}$$

for all $\epsilon > 0$. That is, for all $\epsilon > 0$ we have that for all initial conditions, $x(0), \dot{x}(0)$, no matter how large, there exists a time $t_s \geq 0$ (depending on the initial conditions) such that when applying (8) to (2) we eventually have $(e(t_s), \dot{e}(t_s)) \in \Omega_\varepsilon$.

Once we are in Ω_ϵ we can switch to the controller (6). From the proof of Proposition 4.1 we know that when we are in Ω_ϵ and apply the controller (6) we remain in Ω_ϵ.

Furthermore we have asymptotic stability. Since we only use (6) once we are in Ω_ϵ we can determine a bound on $u(t)$ for $t \geq t_s$. It is clear that (8) also results in a bounded control. Therefore, the composite controller yields global asymptotic stability of the tracking error dynamics, while meeting the input constraints.

The foregoing can be summarised as follows.

Proposition 4.3 *Consider the system (2). Then there exists a switching time $t_s \geq 0$ such that the composite controller*

$$u = \begin{cases} 2 - q\cos\omega t - K_{p,1}\tanh(\lambda(x-2)) & \text{for } t < t_s \\ \ddot{x}_d - \mu(1-x^2)\dot{x} + x - K_d\,\dot{e} - K_p\,e - q\cos\omega t & \text{for } t \geq t_s \end{cases} \tag{10}$$

yields global asymptotic stability of the tracking error dynamics. Furthermore, we can determine u_{max} such that $\|u(t)\| \leq u_{max}$ for all $t \geq 0$.

5 Simulations

To illustrate the results we simulated the system (2) with $\mu = 5$, $q = 5$, and $\omega = 2.463$. We consider the problem of tracking the desired trajectory $x_d(t) = \sin t$. We start our simulations from the initial conditions $x(0) = 3$, $\dot{x}(0) = -2$.

For our first simulation we use feedback linearising controller

$$u = \ddot{x}_d - 5(1 - x^2)\dot{x} + x - 4\dot{e} - 4e - 5\cos 2.463t$$

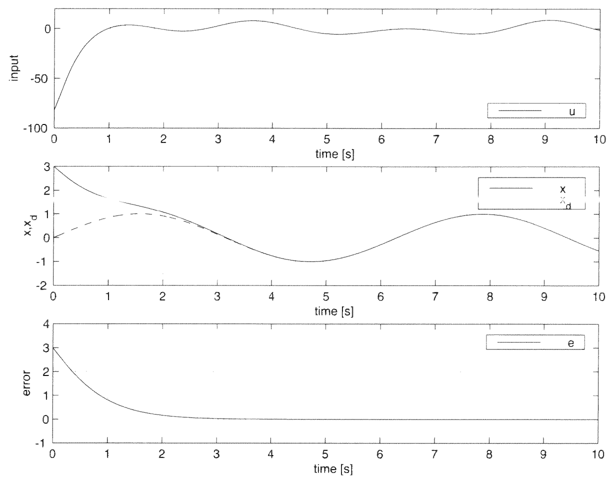

Fig. 5.1. Feedback linearising controller

The results are depicted in Figure 5.1. We see that a quick convergence to the desired trajectory. The price we have to pay is that we need an initial control effort of $u(0) = -82$, almost ten times as much as will be needed for tracking.

In our second simulation, we use the same local tracking controller, but in combination with the controller (8), as proposed in the previous section. This to reduce the control effort needed. To be precise: we use the control law (10) with $\lambda = 5$, $K_p = 4$, $K_d = 4$, where t_s is defined as the first time-instant that $8(x-2)^2 + \dot{x}^2 \leq 0.01$. The resulting performance is depicted in Figure 5.2.

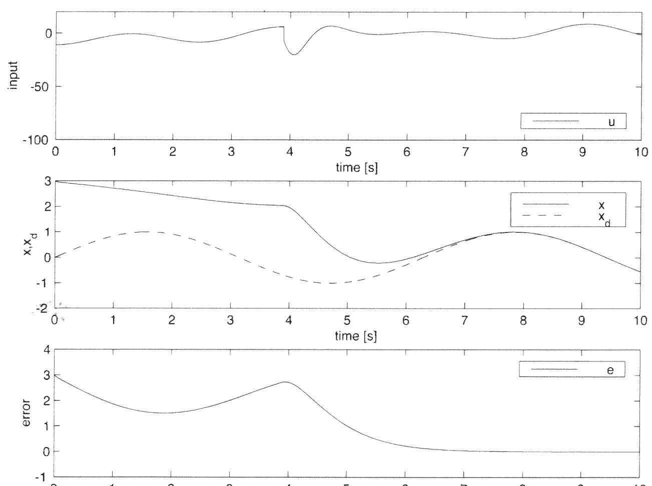

Fig. 5.2. Composite controller

We see that in the beginning the main emphasis is put on decreasing x. At time $t = 3.88$ s the controller (8) has brought our system into the region where the tracking controller is guaranteed to remain 'small' we switch to the tracking controller resulting in tracking of the desired trajectory.
The control effort needed for the composite controller to achieve tracking is only two times the control effort needed to track the desired trajectory, which forms a clear reduction.
The price we obviously have to pay is a slower convergence, which is not surprising since we use less control effort.

6 Concluding remarks

We introduced the idea of using composite controllers as a solution to the tracking control problem under input constraints. The idea has been illustrated by means of the driven van der Pol system. A solution to the tracking control problem using state feedback has been presented and illustrated by numerical simulations.

The idea of composite controllers is also applicable for the problem of output feedback and for the adaptive tracking control problem. It is not restricted to the driven van der Pol system but applicable to any second order nonlinear system. Using this idea bounded tracking controllers (both state feedback, output feedback and adaptive state feedback) are presented in [6] for the driven Duffing system, the van der Pol system, and rigid robot manipulators.

The idea of using composite controllers turned out to be useful for showing global asymptotic stability of (delayed) PID control for rigid robot manipulators [10]. The controller presented in [10] is the composition of a local PID controller and a global PD controller that brings the system into the region of attraction of the PID controller. The switch simply consists of turning on the integrator when the errors have become small enough.

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