On the tracking of nonholonomic systems

SSC-seminar

June 10, 1999

Outline

- Chained form system + problem formulation
- Cascaded approach
- Controller design
- Elegant(!?) proof
- Simulation
- Underactuated ship tracking
- Conclusions

Tracking Problem (state-feedback)

System dynamics:

$$
\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= x_2 u_1 \\
&\vdots \\
\dot{x}_n &= x_{n-1} u_1
\end{aligned}
$$

Reference dynamics:

$$
\begin{aligned}
\dot{x}_{1,r} &= u_{1,r} \\
\dot{x}_{2,r} &= u_{2,r} \\
\dot{x}_{3,r} &= x_{2,r} u_{1,r} \\
&\vdots \\
\dot{x}_{n,r} &= x_{n-1,r} u_{1,r}
\end{aligned}
$$

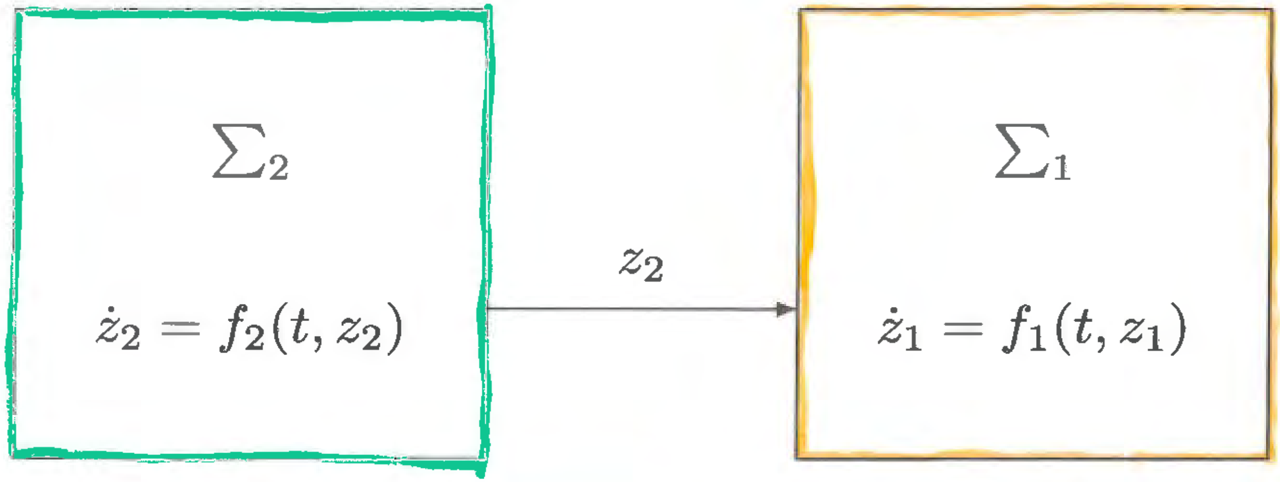
Find control laws

$$u \equiv u(x, x_r, u_r)$$

that yield

$$\lim_{t \to \infty} |x(t) - x_r(t)| = 0$$

Cascaded systems



$$
\begin{aligned}
\dot{z}_1 &= f_1(t, z_1) + g(t, z_1, z_2)\boxed{z_2} \\
\dot{z}_2 &= f_2(t, z_2)
\end{aligned}
$$

Consider the system

$$
\begin{aligned}
\boxed{\dot{x}_1 = -x_1} \quad &+ \quad x_1^2 \boxed{x_2} \\
\boxed{\dot{x}_2 = -\gamma x_2} &
\end{aligned}
$$

Solution:

$$
\begin{aligned}
x_1(t) &= \frac{2x_1(0)}{x_1(0)x_2(0)e^{-t\gamma} + [2 - x_1(0)x_2(0)]e^{t\gamma}} \\
x_2(t) &= x_2(0)e^{-\gamma t}
\end{aligned}
$$

When $x_1(0)x_2(0) > 2$ we have a finite escape time

$$
t_{esc} = \frac{1}{2\gamma} \ln \frac{x_1(0)x_2(0)}{x_1(0)x_2(0) - 2}
$$

Conditions

E. Panteley en A. Loría (S&CL 33(2), 1998):

Cascade Globally Uniformly Asymptotically Stable (GUAS) when

- Σ_1 GUAS, polynomial 'Lyapunov function'
- $g(t, z_1, z_2)$ at most linear in z_1
- Σ_2 GUAS, $z_2(t)$ integrable

Controller Design

Error dynamics ($x_e = x - x_r$):

$$
\begin{aligned}
x_{1,e} &= u_1 - u_{1,r} \\
x_{2,e} &= u_2 - u_{2,r} \\
x_{3,e} &= x_2 u_1 - x_{2,r} u_{1,r} \qquad\qquad +x_2 u_{1,r} - x_2 u_{1,r} \\
&= u_{1,r} x_{2,e} + x_2 (u_1 - u_{1,r}) \\
&\vdots \\
x_{n,e} &= x_{n-1} u_1 - x_{n-1,r} u_{1,r} \qquad\qquad +x_{n-1} u_{1,r} - x_{n-1} u_{1,r} \\
&= u_{1,r} x_{n-1,e} + x_{n-1} (u_1 - u_{1,r})
\end{aligned}
$$

Controller Design

Error dynamics ($x_e = x - x_r$):

$$
\begin{array}{rclcccc}
\dot{x}_{2,e} & = & 0 & + & (u_2 - u_{2,r}) & + & 0 \\
\dot{x}_{3,e} & = & u_{1,r}x_{2,e} & & & + & x_2(u_1 - u_{1,r}) \\
 & \vdots & & & & \vdots & \\
\dot{x}_{n,e} & = & u_{1,r}x_{n-1,e} & & & + & x_{n-1}(u_1 - u_{1,r})
\end{array}
$$

$$\boxed{\dot{x}_{1,e} = (u_1 - u_{1,r})}$$

Problem has reduced to (seperately) stabilizing the error-dynamics

$$\begin{bmatrix} \dot{x}_{2,e} \\ \dot{x}_{3,e} \\ \dot{x}_{4,e} \\ \vdots \\ \dot{x}_{n,e} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ u_{1,r} & \ddots & & & \vdots \\ 0 & u_{1,r} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & u_{1,r} & 0 \end{bmatrix} \begin{bmatrix} x_{2,e} \\ x_{3,e} \\ x_{4,e} \\ \vdots \\ x_{n,e} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} (u_2 - u_{2,r})$$

and

$$\dot{x}_{1,e} = (u_1 - u_{1,r})$$

Stabilizing the $x_{1,e}$ dynamics:

$$u_1 = u_{1,r} - c_1 x_{1,e} \qquad c_1 > 0$$

For stabilizing the $[x_{2,e}, \ldots, x_{n,e}]^T$ dynamics we need *uniform controllability.*

This turns out to be the case if and only if the vector

$$\begin{bmatrix} 1 \\ \int_{t_0}^{t} u_{1,r}(\sigma)d\sigma \\ \vdots \\ \left(\int_{t_0}^{t} u_{1,r}(\sigma)d\sigma\right)^{n-1} \end{bmatrix}$$

or simpler:
$\exists \varepsilon > 0, \exists \delta > 0$ s.t.
$\forall t \geq 0 \;\; \exists s \in [t, t+\delta]$
s.t. $u_{1,r}(s) \geq \varepsilon$

is persistently exciting.

We can use several controllers $u_2 = u_{2,r} - K(t)[x_{2,e}, \ldots, x_{n,e}]^T$ for globally exponentially stabilizing

$$\begin{bmatrix} \dot{x}_{2,e} \\ \dot{x}_{3,e} \\ \dot{x}_{4,e} \\ \vdots \\ \dot{x}_{n,e} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ u_{1,r} & \ddots & & & \vdots \\ 0 & u_{1,r} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & u_{1,r} & 0 \end{bmatrix} \begin{bmatrix} x_{2,e} \\ x_{3,e} \\ x_{4,e} \\ \vdots \\ x_{n,e} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} (u_2 - u_{2,r})$$

- $K(t) = -B^T \left[\int_t^{t+\delta} 2e^{4\alpha(t-\sigma)} \Phi(t,\sigma) B B^T \Phi^T(t,\sigma) d\sigma \right]^{-1}$
- Pole-placement (cf. Valasek & Olgac, needs $\dot{u}_{1,r}, \ldots, \dot{u}_{1,r}^{(n-2)}$)
- $K(t) = \gamma B^T \Phi^{-T}(t,t_0) \Phi^{-1}(t,t_0)$ (M.-S. Chen)

Note that we need $\Phi(t, t_0)$:

$$\Phi(t,t_0) = \begin{bmatrix} f_0(t,t_0) & 0 & \cdots & 0 \\ f_1(t,t_0) & f_0(t,t_0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ f_{n-1}(t,t_0) & \cdots & f_1(t,t_0) & f_0(t,t_0) \end{bmatrix}$$

where

$$f_k(t,t_0) = \frac{1}{k!}\left[\int_{t_0}^{t} u_{1,r}(\sigma)d\sigma\right]^k = \frac{1}{k!}\left[x_{1,r}(t) - x_{1,r}(t_0)\right]^k$$

However, a simpler controller is given by

- $K(t) = [c_2,\ c_3 u_{1,r}(t),\ c_4,\ c_5 u_{1,r}(t),\ \ldots]$ where c_i are such that $\lambda^{n-1} + c_2\lambda^{n-2} + \ldots + c_{n-1}\lambda + c_n$ is Hurwitz.

Proof

Consider the stability of the differential equation

$$\frac{d^m}{dt^m}y(t) + a_1 \frac{d^{m-1}}{dt^{m-1}}y(t) + \ldots + a_{m-1}\frac{d}{dt}y(t) + a_m y(t) = 0 \quad (*)$$

Define Hurwitz-determinants

$$\Delta_i = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & a_{2i-1} \\ 1 & a_2 & a_4 & \cdots & a_{2i-2} \\ 0 & a_1 & a_3 & \cdots & a_{2i-3} \\ 0 & 1 & a_2 & \cdots & a_{2i-4} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_i \end{vmatrix} \qquad (i = 1, \ldots, m)$$

The system $(*)$ is GES if and only if $\Delta_i > 0$ for $i = 1, \ldots, m$.

If we define

$$b_1 = \Delta_1, \quad b_2 = \frac{\Delta_2}{\Delta_1}, \quad b_3 = \frac{\Delta_3}{\Delta_1 \Delta_2}, \quad b_i = \frac{\Delta_{i-3}\Delta_i}{\Delta_{i-2}\Delta_{i-1}} \quad (i = 4, \ldots, m)$$

the system $(*)$ can also be expressed as

$$\dot{w} = \begin{bmatrix} -b_1 & -b_2 & 0 & \ldots & 0 \\ 1 & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -b_m \\ 0 & \ldots & 0 & 1 & 0 \end{bmatrix} w$$

Define

$$V = b_1 w_1^2 + b_1 b_2 w_2^2 + \ldots + b_1 b_2 \cdots b_{m-1} w_{m-1}^2 + b_1 b_2 \cdots b_m w_m^2$$

Then

$$\dot{V} = -b_1^2 w_1^2$$

Does there exists a transformation $z = Sw$ that transforms the system

$$\dot{w} = \begin{bmatrix} -b_1 & -b_2 & 0 & \dots & 0 \\ 1 & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -b_m \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix} w$$

into

$$\dot{z} = \begin{bmatrix} -a_1 & -a_2 & \dots & \dots & -a_m \\ 1 & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix} z$$

Using the same transformation $z = Sw$ we can transform

$$\dot{z} = \begin{bmatrix} -a_1 & -a_2 u_{1,r}(t) & -a_3 & -a_4 u_{1,r}(t) & \cdots \\ u_{1,r}(t) & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & u_{1,r}(t) & 0 \end{bmatrix} z$$

into

$$\dot{w} = \begin{bmatrix} -b_1 & -b_2 u_{1,r}(t) & 0 & \cdots & 0 \\ u_{1,r}(t) & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -b_m u_{1,r}(t) \\ 0 & \cdots & 0 & u_{1,r}(t) & 0 \end{bmatrix} w$$

Again

$$V = b_1 w_1^2 + b_1 b_2 w_2^2 + \ldots + b_1 b_2 \cdots b_{m-1} w_{m-1}^2 + b_1 b_2 \cdots b_m w_m^2$$

results into

$$\dot{V} = -b_1^2 w_1^2$$

We can conclude GES if the pair

$$\left(\begin{bmatrix} -b_1 & -b_2 u_{1,r}(t) & 0 & \ldots & 0 \\ u_{1,r}(t) & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -b_m u_{1,r}(t) \\ 0 & \ldots & 0 & u_{1,r}(t) & 0 \end{bmatrix}, [b_1, 0, \ldots, 0] \right)$$

is uniformly completely observable (UCO).

Corollary from a result by G. Kern (1982)

Assume that $A(t)$ satisfies a Lipschitz condition, i.e.

$$|A(t) - A(t')| \leq L|t - t'| \qquad \forall t, t' \leq 0$$

The pair $(A(t), C(t))$ is UCO if there exists $\delta > 0$ and s with $t - \delta \leq s \leq t$ such that $W(t - \delta, t)$ where

$$W(t - \delta, t) \int_{t}^{t-\delta} e^{A(s)(t-\tau)} C^T(\tau) C(\tau) e^{A^T(s)(t-\tau)} d\tau$$

satisfies

$$0 < \alpha_1(\delta) I \leq W(t - \delta, t) \qquad \forall t \leq 0$$

To summarize

The error dynamics

$$\begin{aligned}
\dot{x}_{1,e} &= u_1 - u_{1,r} \\
\dot{x}_{2,e} &= u_2 - u_{2,r} \\
\dot{x}_{3,e} &= x_2 u_1 - x_{2,r} u_{1,r} \\
&\vdots \\
\dot{x}_{n,e} &= x_{n-1} u_1 - x_{n-1,r} u_{1,r}
\end{aligned}$$

in closed loop with the controller

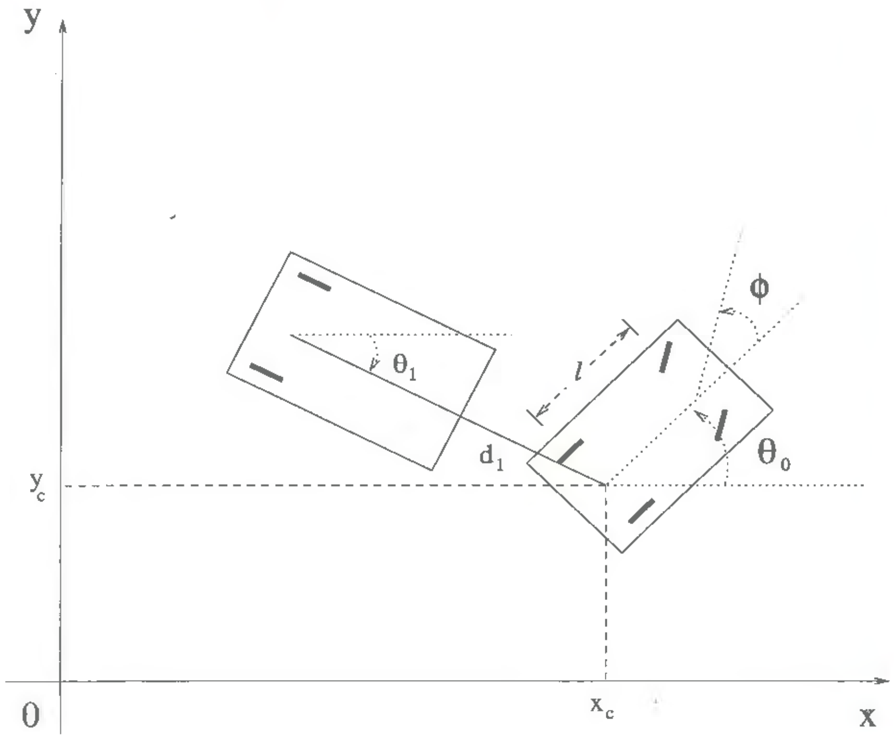
$$\begin{aligned}
u_1 &= u_{1,r} - c_1 x_{1,e} \\
u_2 &= u_{2,r} - K(t)[x_{2,e}, \ldots, x_{n,e}]^T
\end{aligned}$$

is globally (K-exponentially) asymptotically stable.

Simulations

Car pulling a single trailer:

$$
\begin{aligned}
\dot{x}_c &= v\cos\theta_0 \\
\dot{y}_c &= v\sin\theta_0 \\
\dot{\phi} &= \omega \\
\dot{\theta}_0 &= \tfrac{1}{l} v\tan\phi \\
\dot{\theta}_1 &= \tfrac{1}{d_1} v\sin(\theta_0-\theta_1)
\end{aligned}
$$



Moving along a straight line:

$$u_{1,r} = 1 \quad u_{2,r} = 0$$

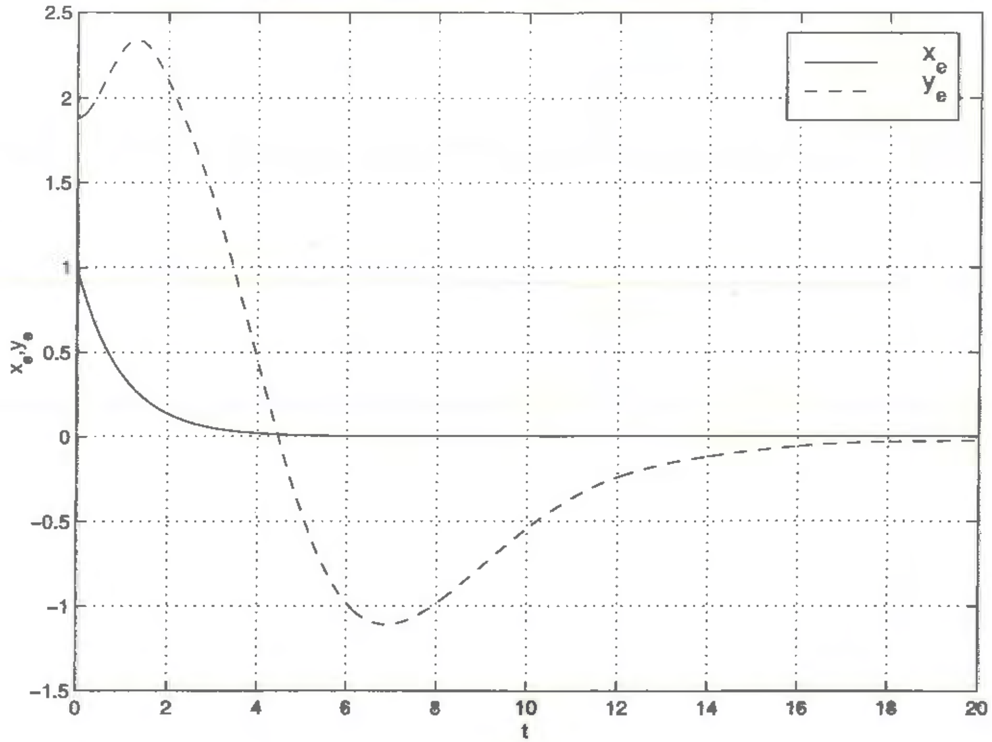
with initial conditions

$$x_{ir}(0) = 0 \;\; (i = 1, \ldots, 5) \qquad x_1(0) = 1, \; x_j(0) = 0.5 \;\; (j = 2, \ldots, 5)$$

controller polynomial: poles in -2

observer polynomial: poles in -3.

State-feedback



Saturated control

Problem has reduced to stabilizing the error-dynamics

$$\begin{bmatrix} \dot{x}_{2,e} \\ \dot{x}_{3,e} \\ \dot{x}_{4,e} \\ \vdots \\ \dot{x}_{n,e} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ u_{1,r} & \ddots & & & \vdots \\ 0 & u_{1,r} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & u_{1,r} & 0 \end{bmatrix} \begin{bmatrix} x_{2,e} \\ x_{3,e} \\ x_{4,e} \\ \vdots \\ x_{n,e} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} (u_2 - u_{2,r})$$

and

$$\dot{x}_{1,e} = (u_1 - u_{1,r})$$

We can also use $u_1 = u_{1,r} - \text{sat}(x_{1,e})$.

Saturated controller:

$$\begin{aligned} u_{1,sat} &= u_{1,r} - \sigma(x_{1,e}) \\ u_{2,sat} &= u_{2,r} - \sum_{k=1}^{4} \epsilon^i \sigma(y_i) \end{aligned}$$

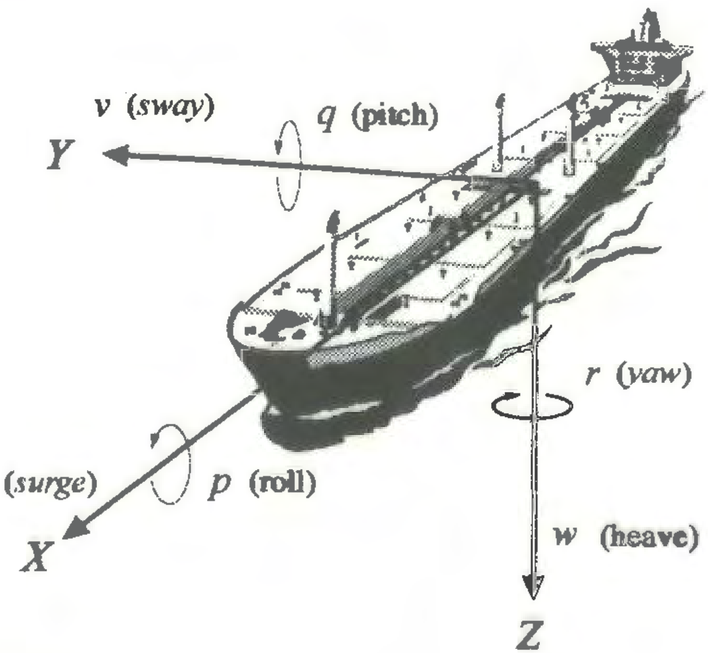
where

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \epsilon & 0 & 0 \\ 1 & \epsilon^2 + \epsilon & \epsilon & 0 \\ 1 & \epsilon^3 + \epsilon^2 + \epsilon & \epsilon^5 + \epsilon^4 + \epsilon^3 & \epsilon^6 \end{bmatrix} \begin{bmatrix} x_{2,e} \\ x_{3,e} \\ x_{4,e} \\ x_{5,e} \end{bmatrix}$$

Ship-dynamics

$$
\begin{aligned}
\dot{x} &= u\cos\psi - v\sin\psi \\
\dot{y} &= u\sin\psi + v\cos\psi \\
\dot{\psi} &= r \\
\dot{u} &= \tau_u \\
\dot{v} &= -cur - dv \\
\dot{r} &= \tau_r
\end{aligned}
$$

where $c, d > 0$



Using

$$
\begin{aligned}
x_1 &= u & x_4 &= x\sin\psi + y\cos\psi \\
x_2 &= v & x_5 &= z_r \\
x_3 &= x\cos\psi + y\sin\psi & x_6 &= \psi
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\dot{x}_1 &= \tau_u \\
\dot{x}_2 &= -cx_1x_5 - dx_2 \\
\dot{x}_3 &= x_1 + x_4x_5 \\
\dot{x}_4 &= x_2 - x_3x_5 \\
\dot{x}_5 &= \tau_r \\
\dot{x}_6 &= x_5
\end{aligned}
$$

Tracking Problem

System dynamics:

$$\begin{aligned}\dot{x}_1 &= \tau_u\\ \dot{x}_2 &= -cx_1x_5 - dx_2\\ \dot{x}_3 &= x_1 + x_4x_5\\ \dot{x}_4 &= x_2 - x_3x_5\\ \dot{x}_5 &= \tau_r\\ \dot{x}_6 &= x_5\end{aligned}$$

Reference dynamics:

$$\begin{aligned}\dot{x}_{1,d} &= \tau_{u,d}\\ \dot{x}_{2,d} &= -cx_{1,d}x_{5,d} - dx_{2,d}\\ \dot{x}_{3,d} &= x_{1,d} + x_{4,d}x_{5,d}\\ \dot{x}_{4,d} &= x_{2,d} - x_{3,d}x_{5,d}\\ \dot{x}_{5,d} &= \tau_{r,d}\\ \dot{x}_{6,d} &= x_{5,d}\end{aligned}$$

Find control laws

$$\tau \equiv \tau(x, x_d, \tau_d)$$

that yield

$$\lim_{t\to\infty} |x(t) - x_d(t)| = 0$$

Tracking error dynamics:

$$\begin{bmatrix} \dot{x}_{1,e} \\ \dot{x}_{2,e} \\ \dot{x}_{3,e} \\ \dot{x}_{4,e} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -cx_{5,d}(t) & -d & 0 & 0 \\ 1 & 0 & 0 & x_{5,d}(t) \\ 0 & 1 & -x_{5,d} & 0 \end{bmatrix} \begin{bmatrix} x_{1,e} \\ x_{2,e} \\ x_{3,e} \\ x_{4,e} \end{bmatrix} +$$

$$+ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} (\tau_u - \tau_{u,d}) + \begin{bmatrix} 0 & 0 \\ -cx_1 & 0 \\ x_4 & 0 \\ -x_3 & 0 \end{bmatrix} \begin{bmatrix} x_{5,e} \\ x_{6,e} \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_{5,e} \\ \dot{x}_{6,e} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{5,e} \\ x_{6,e} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} (\tau_r - \tau_{r,d})$$

Assume that $x_{1,d}$, $x_{3,d}$ and $x_{4,d}$ are bounded and that $x_{5,d}(t) \not\to 0$.

Then for the state-feedback problem we can use the controller

$$\begin{aligned}\tau_u &= \tau_{u,d} - k_1 x_{1,e} + k_2 x_{5,d}(t) x_{2,e} - k_3 x_{3,e} + k_4 x_{5,d}(t) x_{4,e} \\ \tau_r &= \tau_{r,d} - k_5 x_{5,e} - k_6 x_{6,e}\end{aligned}$$

If additionally $x_{5,d}(t)$ is bounded

Then for the output-feedback problem we can use the controller

$$\begin{aligned}\tau_u &= \tau_{u,d} - k_1 \hat{x}_{1,e} + k_2 x_{5,d}(t) \hat{x}_{2,e} - k_3 \hat{x}_{3,e} + k_4 x_{5,d}(t) \hat{x}_{4,e} \\ \tau_r &= \tau_{r,d} - k_5 \hat{x}_{5,e} - k_6 \hat{x}_{6,e}\end{aligned}$$

$\hat{x}$ is generated from the observer

$$\begin{aligned}
\dot{\hat{x}}_{1e-4e} &= A(t)\hat{x}_{1e-4e} + B(\tau_u - \tau_{u,d}) + L(t)(y_{1-2} - \hat{y}_{1-2}) \\
\hat{y}_{1-2} &= C\hat{x}_{1e-4e}
\end{aligned}$$

$$\begin{aligned}
\dot{\hat{x}}_{5e-6e} &= \bar{A}\hat{x}_{5e-6e} + \bar{B}(\tau_r - \tau_{r,d}) + \bar{L}(y_3 - \hat{y}_3) \\
\hat{y}_3 &= \bar{C}\hat{x}_{5e-6e}
\end{aligned}$$

where

$$L(t) = \begin{bmatrix} l_{11} & 0 \\ 0 & l_{22} \\ l_{31} & x_{5,d}(t) \\ -x_{5,d}(t) & l_{42} \end{bmatrix} \qquad \bar{L} = [l_5 \;\; l_6]$$

Conclusions

- Linear (time-varying) controllers for nonlinear chained form system.
- Separate controller design by viewing the system as a cascade.
- Both state and output feedback.
- Global results (not based on linearization)
- Similar approach can be used with saturated control inputs.
- Cascaded approach works for ship too!