

Linear controllers for tracking chained-form systems

Nonlinear Control Network workshop

“Stability and Stabilization”

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Outline

- Model and Problem formulation
- Cascaded systems
- Derivation of linear tracking controller
- Simulations
- Conclusions

Chained form system

Nonholonomic system in chained form:

$$\begin{aligned}\dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_1 \\ &\vdots \\ \dot{x}_n &= x_{n-1} u_1\end{aligned}$$

Examples: Car pulling multiple trailers, rolling Euro, knife-edge (pizza-knife)

Tracking Problem (state-feedback)

System dynamics:

$$\dot{x}_1 = u_1$$

$$\dot{x}_2 = u_2$$

$$\dot{x}_3 = x_2 u_1$$

$$\vdots$$

$$\dot{x}_n = x_{n-1} u_1$$

Reference dynamics:

$$\dot{x}_{1,r} = u_{1,r}$$

$$\dot{x}_{2,r} = u_{2,r}$$

$$\dot{x}_{3,r} = x_{2,r} u_{1,r}$$

$$\vdots$$

$$\dot{x}_{n,r} = x_{n-1,r} u_{1,r}$$

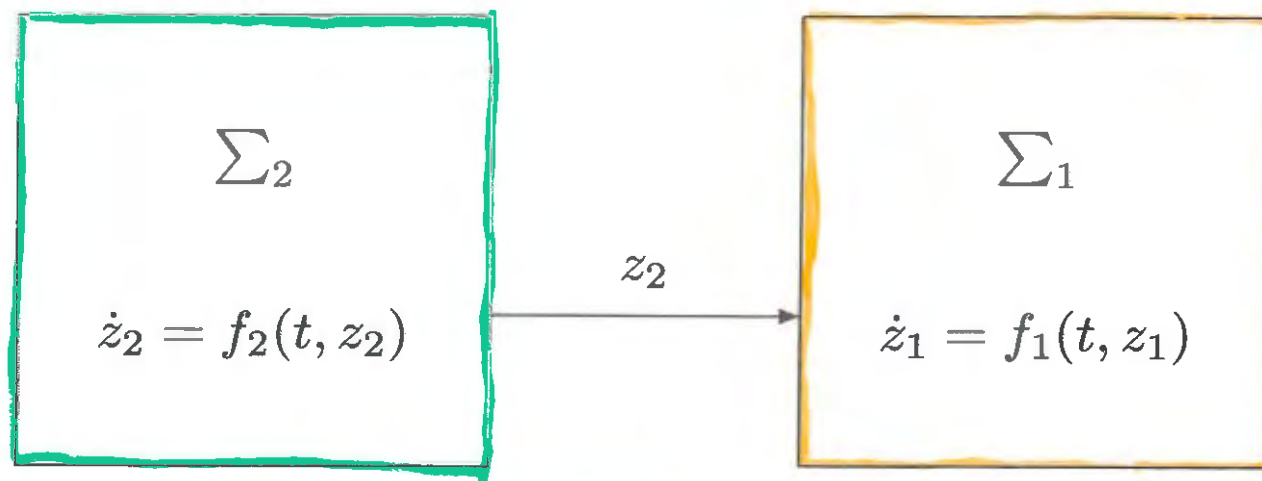
Find control laws

$$u \equiv u(\mathbf{x}, x_r, u_r)$$

that yield

$$\lim_{t \rightarrow \infty} |x(t) - x_r(t)| = 0$$

Cascaded systems



$$\dot{z}_1 = f_1(t, z_1) + g(t, z_1, z_2)z_2$$

$$\dot{z}_2 = f_2(t, z_2)$$

Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_1^2 x_2 \\ \dot{x}_2 &= -\gamma x_2\end{aligned}$$

Solution:

$$\begin{aligned}x_1(t) &= \frac{2x_1(0)}{x_1(0)x_2(0)e^{-\gamma t} + [2 - x_1(0)x_2(0)]e^{\gamma t}} \\ x_2(t) &= x_2(0)e^{-\gamma t}\end{aligned}$$

When $x_1(0)x_2(0) > 2$ we have a finite escape time

$$t_{esc} = \frac{1}{2\gamma} \ln \frac{x_1(0)x_2(0)}{x_1(0)x_2(0) - 2}$$

Conditions

E. Panteley en A. Loría (S&CL 33(2), 1998):

Cascade Globally Uniformly Asymptotically Stable (GUAS) when

- Σ_1 GUAS, polynomial ‘Lyapunov function’
- $g(t, z_1, z_2)$ at most linear in z_1
- Σ_2 GUAS, $z_2(t)$ integrable

Controller Design

Error dynamics ($x_e = x - x_r$):

$$\begin{array}{rclcl} \dot{x}_{2,e} & = & 0 & + & (u_2 - u_{2,r}) & + & 0 \\ \dot{x}_{3,e} & = & u_{1,r}x_{2,e} & + & & + & x_2(u_1 - u_{1,r}) \\ & & \vdots & & & & \vdots \\ \dot{x}_{n,e} & = & u_{1,r}x_{n-1,e} & + & & + & x_{n-1}(u_1 - u_{1,r}) \end{array}$$

$$\dot{x}_{1,e} = (u_1 - u_{1,r})$$

Problem has reduced to (seperately) stabilizing the error-dynamics

$$\begin{bmatrix} \dot{x}_{2,e} \\ \dot{x}_{3,e} \\ \dot{x}_{4,e} \\ \vdots \\ \dot{x}_{n,e} \end{bmatrix} = \begin{bmatrix} 0 & \dots & \dots & \dots & 0 \\ u_{1,r} & \ddots & & & \vdots \\ 0 & u_{1,r} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u_{1,r} & 0 \end{bmatrix} \begin{bmatrix} x_{2,e} \\ x_{3,e} \\ x_{4,e} \\ \vdots \\ x_{n,e} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} (u_2 - u_{2,r})$$

and

$$\dot{x}_{1,e} = (u_1 - u_{1,r})$$

Stabilizing the $x_{1,e}$ dynamics:

$$u_1 = u_{1,r} - c_1 x_{1,e} \quad c_1 > 0$$

For stabilizing the $[x_{2,e}, \dots, x_{n,e}]^T$ dynamics we need *uniform controllability*.

We can show that if $u_{1,r} \not\rightarrow 0$, the controller

$$u_2 = u_{2,r} - c_2 x_{2,e} - c_3 u_{1,r} x_{3,e} - c_4 x_{4,e} - c_5 u_{1,r} x_{5,e} - \dots$$

yields global asymptotic stability of the closed-loop system, provided

$$\lambda^{n-1} + c_2 \lambda^{n-2} + \dots + c_{n-1} \lambda + c_n \text{ is Hurwitz.}$$

To summarize

Consider the error dynamics

$$\dot{x}_{1,e} = u_1 - u_{1,r}$$

$$\dot{x}_{2,e} = u_2 - u_{2,r}$$

$$\dot{x}_{3,e} = x_2 u_1 - x_{2,r} u_{1,r}$$

$$\vdots$$

$$\dot{x}_{n,e} = x_{n-1} u_1 - x_{n-1,r} u_{1,r}$$

in closed loop with the controller

$$u_1 = u_{1,r} - c_1 x_{1,e}$$

$$u_2 = u_{2,r} - c_2 x_{2,e} - c_3 u_{1,r} x_{3,e} - c_4 x_{4,e} - c_5 u_{1,r} x_{5,e} - \dots$$

If $u_{1,r} \not\rightarrow 0$, then the closed loop is globally asymptotically stable.

Tracking Problem (output feedback)

System dynamics:

$$\begin{aligned}\dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_1 \\ &\vdots \\ \dot{x}_n &= x_{n-1} u_1\end{aligned}$$

Reference dynamics:

$$\begin{aligned}\dot{x}_{1,r} &= u_{1,r} \\ \dot{x}_{2,r} &= u_{2,r} \\ \dot{x}_{3,r} &= x_{2,r} u_{1,r} \\ &\vdots \\ \dot{x}_{n,r} &= x_{n-1,r} u_{1,r}\end{aligned}$$

Output: $y = [y_1, y_n]^T$

Find control laws

$$u \equiv u(\hat{x}, x_r, u_r) \quad \dot{\hat{x}} = f(\hat{x}, y, x_r, u_r)$$

that yield $\lim_{t \rightarrow \infty} |x(t) - x_r(t)| = 0$

Using again $u_1 = u_{1,r} - c_1 x_{1,e}$ and the cascaded result, we need to stabilize

$$\begin{bmatrix} \dot{x}_{2,e} \\ \dot{x}_{3,e} \\ \dot{x}_{4,e} \\ \vdots \\ \dot{x}_{n,e} \end{bmatrix} = \begin{bmatrix} 0 & \dots & \dots & \dots & 0 \\ u_{1,r} & \ddots & & & \vdots \\ 0 & u_{1,r} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u_{1,r} & 0 \end{bmatrix} \begin{bmatrix} x_{2,e} \\ x_{3,e} \\ x_{4,e} \\ \vdots \\ x_{n,e} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} (u_2 - u_{2,r})$$

$$y = x_{n,e}$$

by means of output feedback.

Using the *separation principle* and *certainty equivalence* we find:

$$u_2 = u_{2,r} - c_2 \hat{x}_{2,e} - c_3 u_{1,r} \hat{x}_{3,e} - c_4 \hat{x}_{4,e} - c_5 u_{1,r} \hat{x}_{5,e} - \dots$$

$$\begin{bmatrix} \dot{\hat{x}}_{2,e} \\ \dot{\hat{x}}_{3,e} \\ \dot{\hat{x}}_{4,e} \\ \vdots \\ \dot{\hat{x}}_{n,e} \end{bmatrix} = \begin{bmatrix} 0 & \dots & \dots & \dots & 0 \\ u_{1,r} & \ddots & & & \vdots \\ 0 & u_{1,r} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u_{1,r} & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_{2,e} \\ \hat{x}_{3,e} \\ \hat{x}_{4,e} \\ \vdots \\ \hat{x}_{n,e} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} (u_2 - u_{2,r}) +$$

$$+ [\dots, l_5 u_{1,r}, l_4, l_3 u_{1,r}, l_2]^T (x_{n,e} - \hat{x}_{n,e})$$

yields global asymptotic stability, provided

$$\begin{array}{l} \lambda^{n-1} + c_2 \lambda^{n-2} + \dots + c_{n-1} \lambda + c_n \\ \lambda^{n-1} + l_2 \lambda^{n-2} + \dots + l_{n-1} \lambda + l_n \end{array} \quad \text{Hurwitz}$$

To summarize

Error dynamics

$$\dot{x}_{1,e} = u_1 - u_{1,r}$$

$$\dot{x}_{2,e} = u_2 - u_{2,r}$$

$$\dot{x}_{3,e} = x_2 u_1 - x_{2,r} u_{1,r}$$

$$\vdots$$

$$\dot{x}_{n,e} = x_{n-1} u_1 - x_{n-1,r} u_{1,r}$$

$$y = [x_{1,e}, x_{n,e}]^T$$

Observer

$$\dot{\hat{x}}_{2,e} = u_2 - u_{2,r} + l_n [u_{1,r}] \tilde{x}_{n,e}$$

$$\dot{\hat{x}}_{3,e} = u_{1,r} \hat{x}_{2,e} + l_{n-1} [u_{1,r}] \tilde{x}_{n,e}$$

$$\vdots$$

$$\dot{\hat{x}}_{n-1,e} = u_{1,r} \hat{x}_{n-2,e} + l_3 u_{1,r} \tilde{x}_{n,e}$$

$$\dot{\hat{x}}_{n,e} = u_{1,r} \hat{x}_{n-1,e} + l_2 \tilde{x}_{n,e}$$

$$\tilde{x}_{n,e} = x_{n,e} - \hat{x}_{n,e}$$

in closed loop with the controller

$$u_1 = u_{1,r} - c_1 x_{1,e}$$

$$u_2 = u_{2,r} - c_2 \hat{x}_{2,e} - c_3 u_{1,r} \hat{x}_{3,e} - c_4 \hat{x}_{4,e} - c_5 u_{1,r} \hat{x}_{5,e} - \dots$$

If $u_{1,r} \not\rightarrow 0$, then the closed loop is globally asymptotically stable.

Simulations

Car pulling a single trailer:

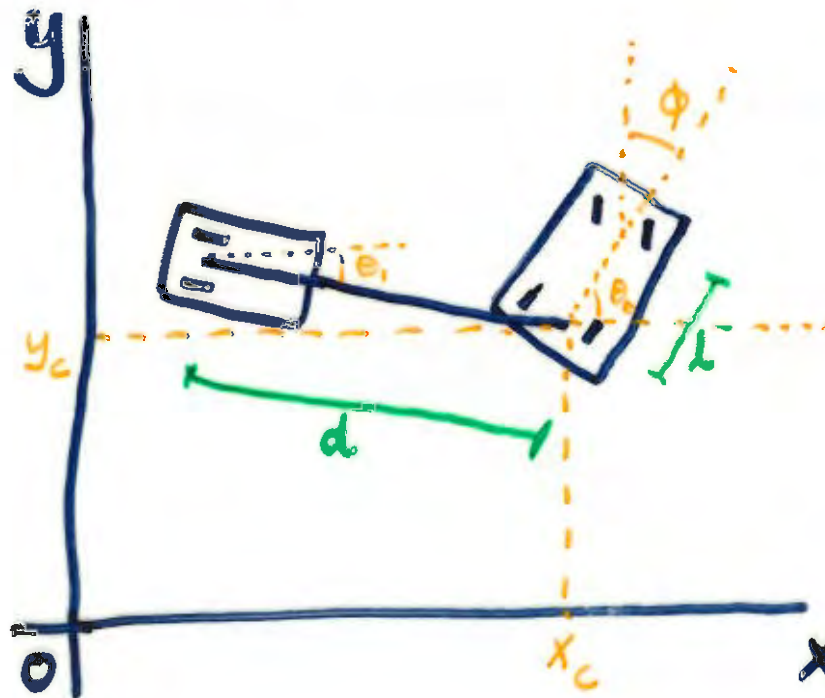
$$\dot{x}_c = v \cos \theta_0$$

$$\dot{y}_c = v \sin \theta_0$$

$$\dot{\phi} = \omega$$

$$\dot{\theta}_0 = \frac{1}{l} v \tan \phi$$

$$\dot{\theta}_1 = \frac{1}{d_1} v \sin(\theta_0 - \theta_1)$$



State-feedback

Initial condition:

$$x_1(0) = x_2(0) = x_3(0) = x_4(0) = x_5(0) = 1$$

Reference:

$$x_{1,r}(0) = x_{2,r}(0) = x_{3,r}(0) = x_{4,r}(0) = x_{5,r}(0) = 0$$

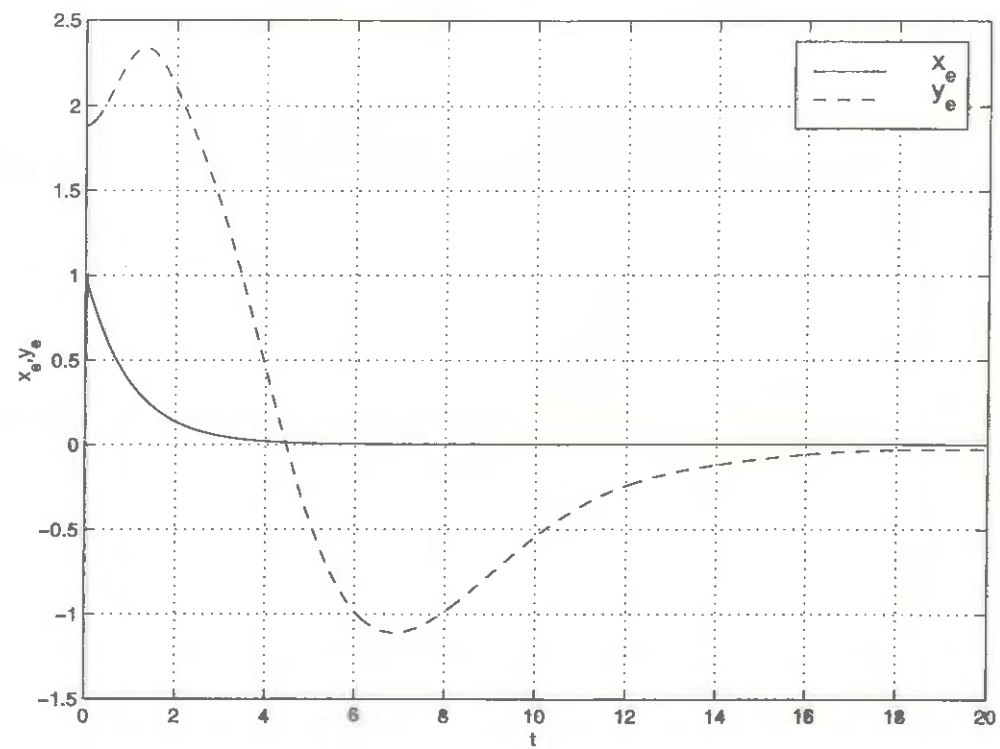
$$u_{1,r} = u_{2,r} = 1$$

Controller-gains

$$c_1 = 1, c_2 = 4, c_3 = 6, c_4 = 4, c_5 = 1$$

We looked at the errors $x_e = x_c - x_{c,r}$ and $y_e = y_c - y_{c,r}$.

State-feedback



Output-feedback

Initial condition:

$$x_1(0) = x_2(0) = x_3(0) = x_4(0) = x_5(0) = 1$$

$$\hat{x}_{1,e}(0) = \hat{x}_{2,e}(0) = \hat{x}_{3,e}(0) = \hat{x}_{4,e}(0) = \hat{x}_{5,e}(0) = -1$$

Reference:

$$x_{1,r}(0) = x_{2,r}(0) = x_{3,r}(0) = x_{4,r}(0) = x_{5,r}(0) = 0$$

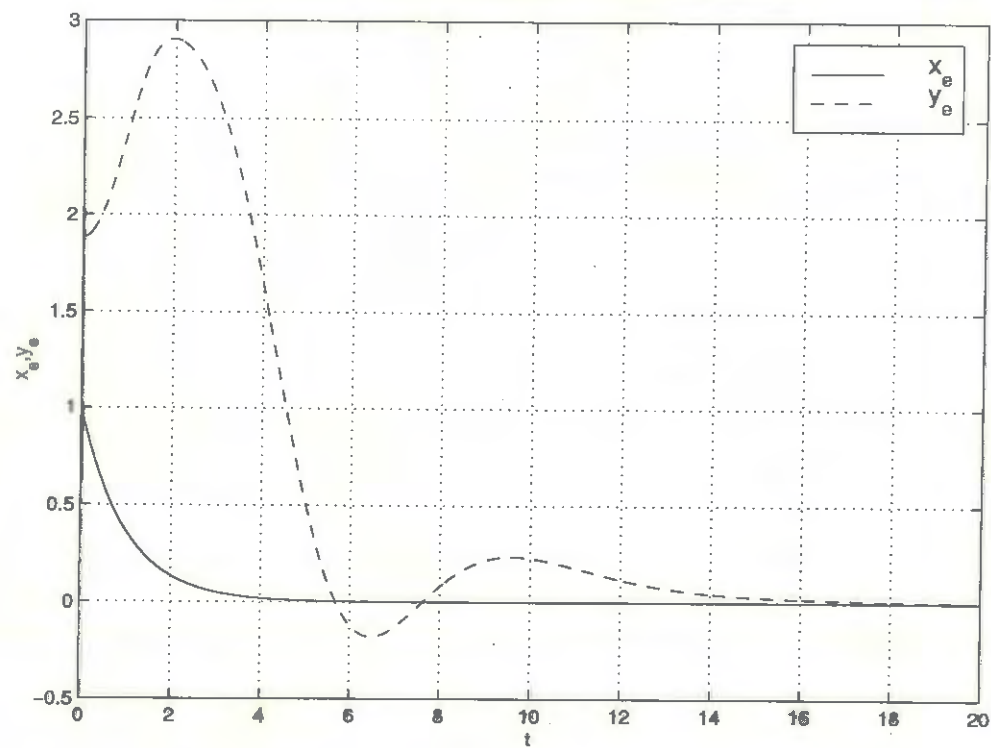
$$u_{1,r} = u_{2,r} = 1$$

Controller and observer gains

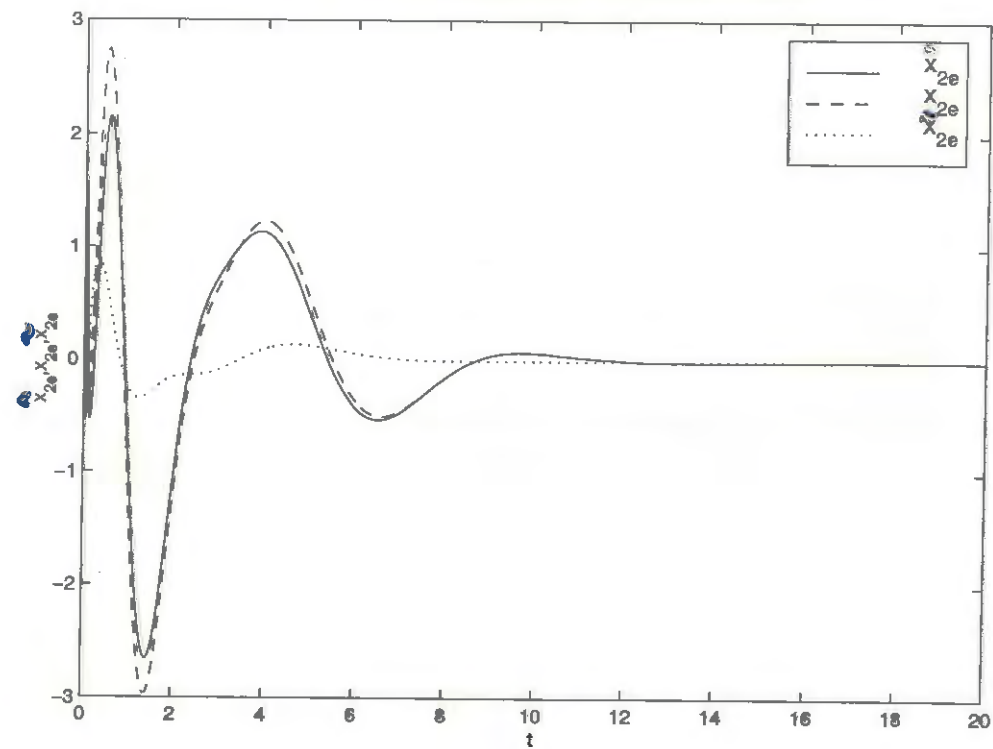
$$c_1 = 1, c_2 = 4, c_3 = 6, c_4 = 4, c_5 = 1, l_2 = 8, l_3 = 24, l_4 = 32, l_5 = 16$$

We looked at the errors $x_e = x_c - x_{c,r}$ and $y_e = y_c - y_{c,r}$.

Output-feedback



Output-feedback



Conclusions

- Linear (time-varying) controllers for nonlinear chained form system.
- Separate controller design by viewing the system as a cascade.
- Both state and output feedback.
- Global results (not based on linearization)
- Similar approach can be used with saturated control inputs.