Bounded Tracking Control of a Wheeled Mobile Robot

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Outline

- Introduction:
  - Model of mobile car
  - Problem formulation

- (Dynamic) Disturbance Decoupling: Local tracking controller

- Integrator Backstepping

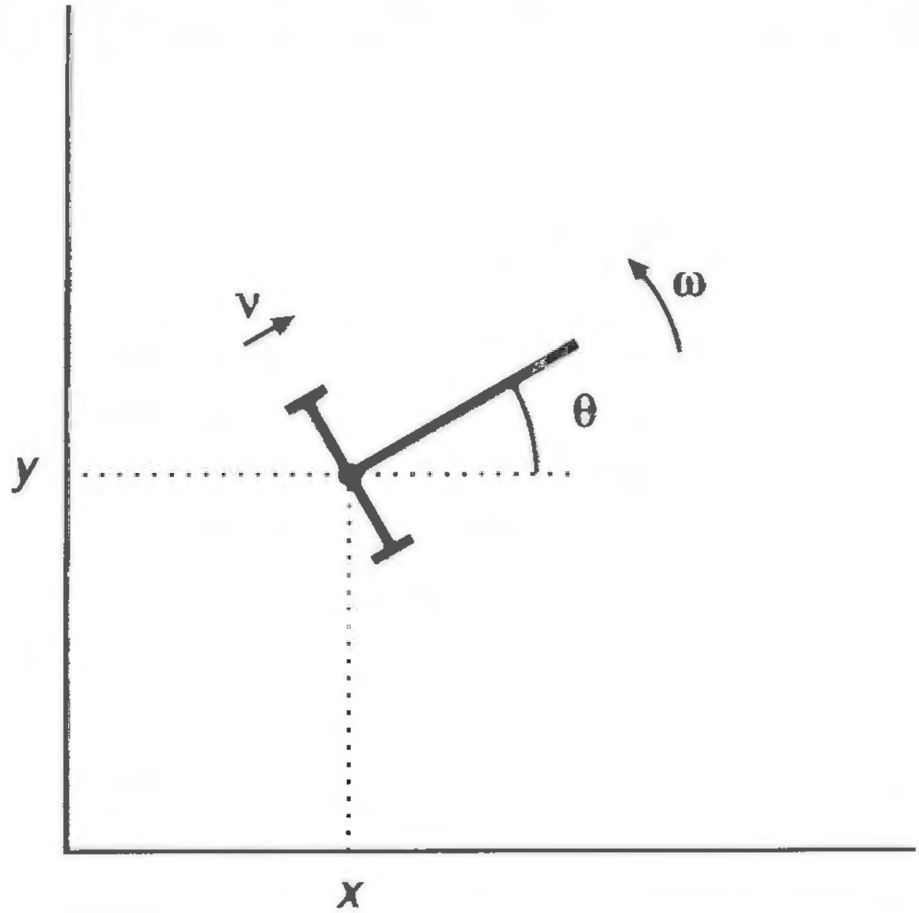
- Global tracking controller of Jiang and Nijmeijer using Integrator Backstepping

- Extension to (semi)globally bounded controller

- Simulations

- Conclusions

A wheeled mobile robot



The dynamics of the car is described by:

$$\begin{aligned}\dot{x} &= \nu \cos\theta \\ \dot{y} &= \nu \sin\theta \\ \dot{\theta} &= \omega\end{aligned}$$

ν: linear velocity of mobile robot

ω: angular velocity of mobile robot

Model:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \nu + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \omega$$

Linearisation $\Rightarrow$ uncontrollable system.
System is however controllable.

Constraint:

$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0$$

can not be integrated to $f(x, y, \theta) = 0$.

Consider the system

$$
\begin{aligned}
\dot{x} &= -y\omega \\
\dot{y} &= x\omega \\
\dot{\omega} &= u
\end{aligned}
$$

Constraint:

$$x\dot{x} + y\dot{y} = 0$$

can be integrated to

$$x^2 + y^2 = \text{constant} = x(0)^2 + y(0)^2$$

Using

$$\theta = \arctan\left(\frac{y}{x}\right) \Leftrightarrow \left\{ \begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \right.$$

we get

$$
\begin{aligned}
\dot{\theta} &= \omega \\
\dot{\omega} &= u
\end{aligned}
$$

Nonholonomic control system $\Rightarrow$ Brockett (83): No continuous state-feedback control law of the kind $u = \mu(x)$ for stabilisation problem.

Problem formulation

Tracking control problem:

Find control laws for ν and ω such that the robot follows a *reference robot*, with position $[x_r, y_r, \theta_r]^T$ and inputs ν_r and ω_r.

Bounded tracking control problem:

Tracking control problem, where the inputs are constrained to

$$|\nu(t)| \leq \nu^{max}, \quad |\omega(t)| \leq \omega^{max} \qquad \forall t \geq 0$$

Assumption:

$$\begin{aligned} \nu^{max} &> \max_{t \geq 0} \; \nu_r(t) \\ \omega^{max} &> \max_{t \geq 0} \; \omega_r(t) \end{aligned}$$

Input-Output decoupling

Model:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \nu + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \omega$$

Output: x and y.

Can we find a regular feedback:

$$\begin{bmatrix} \nu \\ \omega \end{bmatrix} = \alpha(x, y, \theta) + \beta(x, y, \theta) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

such that v_1 only influences x and v_2 only influences y?

Recipe: Differentiate outputs (x and y), until inputs (ν and ω) appear, and form *decoupling matrix*, which has to have full rank:

Differentiating first output (x):

$$\dot{x} = \nu \cos\theta$$

Differentiating second output (y):

$$\dot{y} = \nu \sin\theta$$

Yielding:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{bmatrix}}_{\substack{\text{Decoupling} \\ \text{matrix}}} \begin{bmatrix} \nu \\ \omega \end{bmatrix}$$

Decoupling matrix does not have full rank
$\Rightarrow$ not (locally) input-output decoupable.

Dynamic Input-output decoupling

Model:

$$\begin{bmatrix} x \\ y \\ \theta \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \nu + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \omega$$

Output: x and y.

Can we find a dynamic state feedback:

$$\begin{aligned} \dot{z} &= \gamma(z, x, y, \theta) + \delta(z, x, y, \theta) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ \begin{bmatrix} \nu \\ \omega \end{bmatrix} &= \alpha(z, x, y, \theta) + \beta(z, x, y, \theta) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \end{aligned}$$

where $z \in \mathbb{R}^q$, such that v_1 only influences x and v_2 only influences y?

Useful strategy: Add integrator and repeat previous recipe.

Use dynamic state feedback

$$\begin{aligned} \dot{z} &= u_2 \\ \nu &= z \\ \omega &= u_1 \end{aligned}$$

resulting into:

$$\begin{aligned} \dot{x} &= z\cos\theta \\ \dot{y} &= z\sin\theta \\ \dot{\theta} &= u_1 \\ \dot{z} &= u_2 \end{aligned}$$

Outputs: x and y.

Differentiate outputs (x and y), until inputs (u_1 and u_2) appear, and form decoupling matrix:

Differentiating first output (x):

$$\begin{aligned}\dot{x} &= z\cos\theta \\ \ddot{x} &= u_2\cos\theta - z\sin\theta\, u_1\end{aligned}$$

Differentiating second output (y):

$$\begin{aligned}\dot{y} &= z\sin\theta \\ \ddot{y} &= u_2\sin\theta + z\cos\theta\, u_1\end{aligned}$$

Yielding:

$$\begin{bmatrix}\ddot{x} \\ \ddot{y}\end{bmatrix} = \underbrace{\begin{bmatrix}-z\sin\theta & \cos\theta \\ z\cos\theta & \sin\theta\end{bmatrix}}_{\substack{\text{Decoupling} \\ \text{matrix}}} \begin{bmatrix}u_1 \\ u_2\end{bmatrix}$$

Decoupling matrix full rank, provided $z \neq 0$.

We have:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} -z\sin\theta & \cos\theta \\ z\cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

so choosing

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{z} \begin{bmatrix} -\sin\theta & \cos\theta \\ z\cos\theta & z\sin\theta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

results into

$$\begin{aligned} \ddot{x} &= v_1 \\ \ddot{y} &= v_2 \end{aligned}$$

Now tracking can be easily achieved, use e.g.

$$\begin{aligned} v_1 &= \ddot{x}_r - k_{d,x}\dot{e}_x - k_{p,x}e_x \\ v_2 &= \ddot{y}_r - k_{d,y}\dot{e}_y - k_{p,y}e_y \end{aligned}$$

where $e_x \equiv x - x_r$, $e_y \equiv y - y_r$
$k_{d,x}$, $k_{p,x}$, $k_{d,y}$, and $k_{p,y}$ positive constants.

Concluding

System:

$$\begin{aligned}\dot{x} &= \nu \cos\theta \\ \dot{y} &= \nu \sin\theta \\ \dot{\theta} &= \omega\end{aligned}$$

Dynamic state feedback used:

$$\begin{aligned}\dot{z} &= -\frac{1}{z}\sin\theta\, v_1 + \frac{1}{z}\cos\theta\, v_2 \\ \nu &= z \\ \omega &= \cos\theta\, v_1 + \sin\theta\, v_2\end{aligned}$$

where

$$\begin{aligned}v_1 &= \ddot{x}_r - k_{d,x}\dot{e}_x - k_{p,x}e_x \\ v_2 &= \ddot{y}_r - k_{d,y}\dot{e}_y - k_{p,y}e_y\end{aligned}$$

Disadvantage: $z(t) \neq 0$ for all $t \geq 0$.

Only local result, provided $\liminf_{t\to\infty} \nu_r(t) > 0$.

Where are we now?

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Integrator backstepping

Example:

$$\begin{aligned}\dot{x} &= \cos x - x^3 + \xi \\ \dot{\xi} &= u\end{aligned}$$

Design objective: regulation of $x(t)$.
Only equilibrium $(x, \xi) = (0, -1)$.

Stabilizing function

$$\xi_{des} = -c_1 x - \cos x \triangleq \alpha(x)$$

Results into $\dot{x} = -c_1 x - x^3$. If we use $V = \frac{1}{2}x^2$ we get $\dot{V} = -c_1 x^2 - x^4 \Rightarrow$ glob. as. stable.

Introduce error variable

$$z = \xi - \xi_{des} \qquad \Rightarrow \qquad \xi = \xi_{des} + z$$

Results into:

$$\begin{aligned}\dot{x} &= -c_1 x - x^3 + z \\ \dot{z} &= \dot{\xi} - \dot{\alpha} = u + (c_1 - \sin x)(-c_1 x - x^3 + z)\end{aligned}$$

Consider Lyapunov function candidate

$$V(x,z) = \frac{1}{2}x^2 + \frac{1}{2}z^2.$$

It's time-derivative becomes:

$$\dot{V}(x,z) = -c_1x^2 - x^4 + z[x + u + (c_1 - \sin x)(-c_1x - x^3 + z)$$

Choosing

$$u = -x - (c_1 - \sin x)(-c_1x - x^3 + z) - c_2z$$

results into

$$\dot{V} = -c_1x^2 - x^4 - c_2z^2.$$

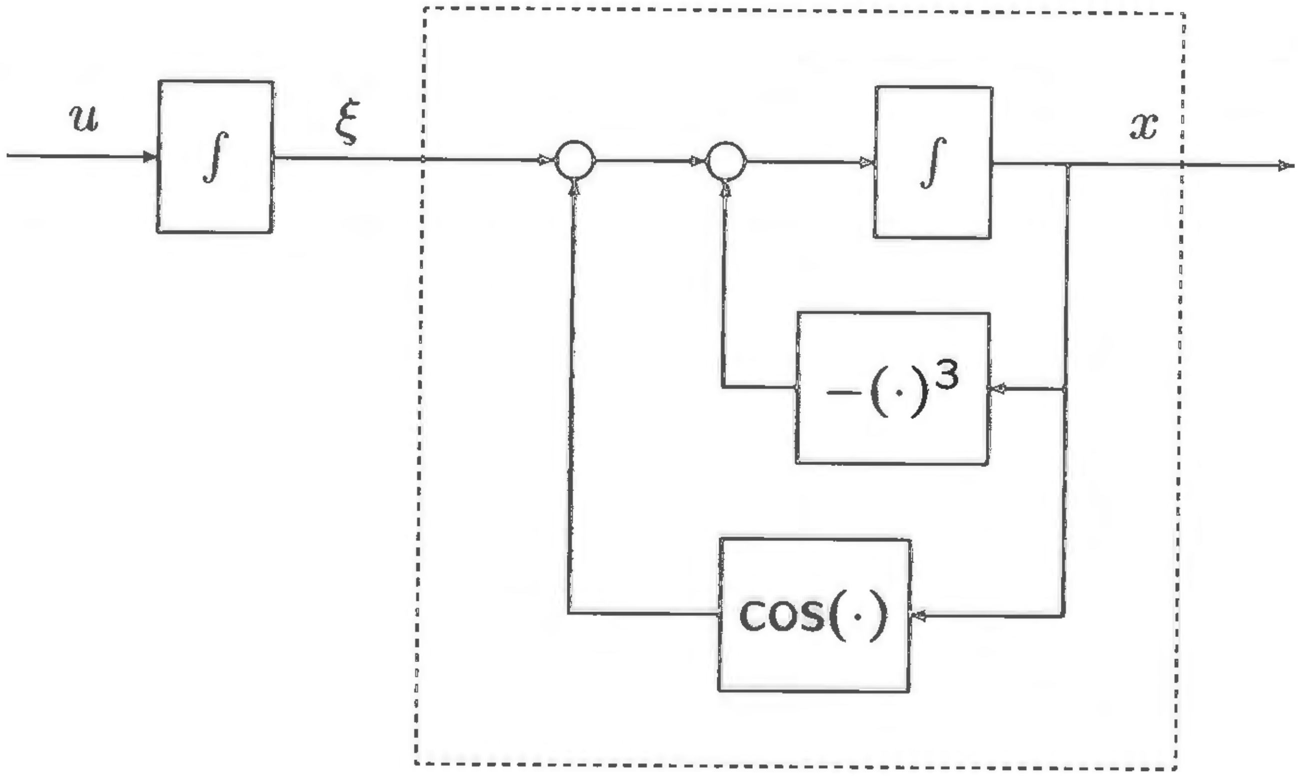
So the controller

$$\begin{aligned} u &= -c_2(\xi + c_1x + \cos x) - x - \\ &\quad -(c_1 - \sin x)(\cos x - x^3 + \xi) \end{aligned}$$

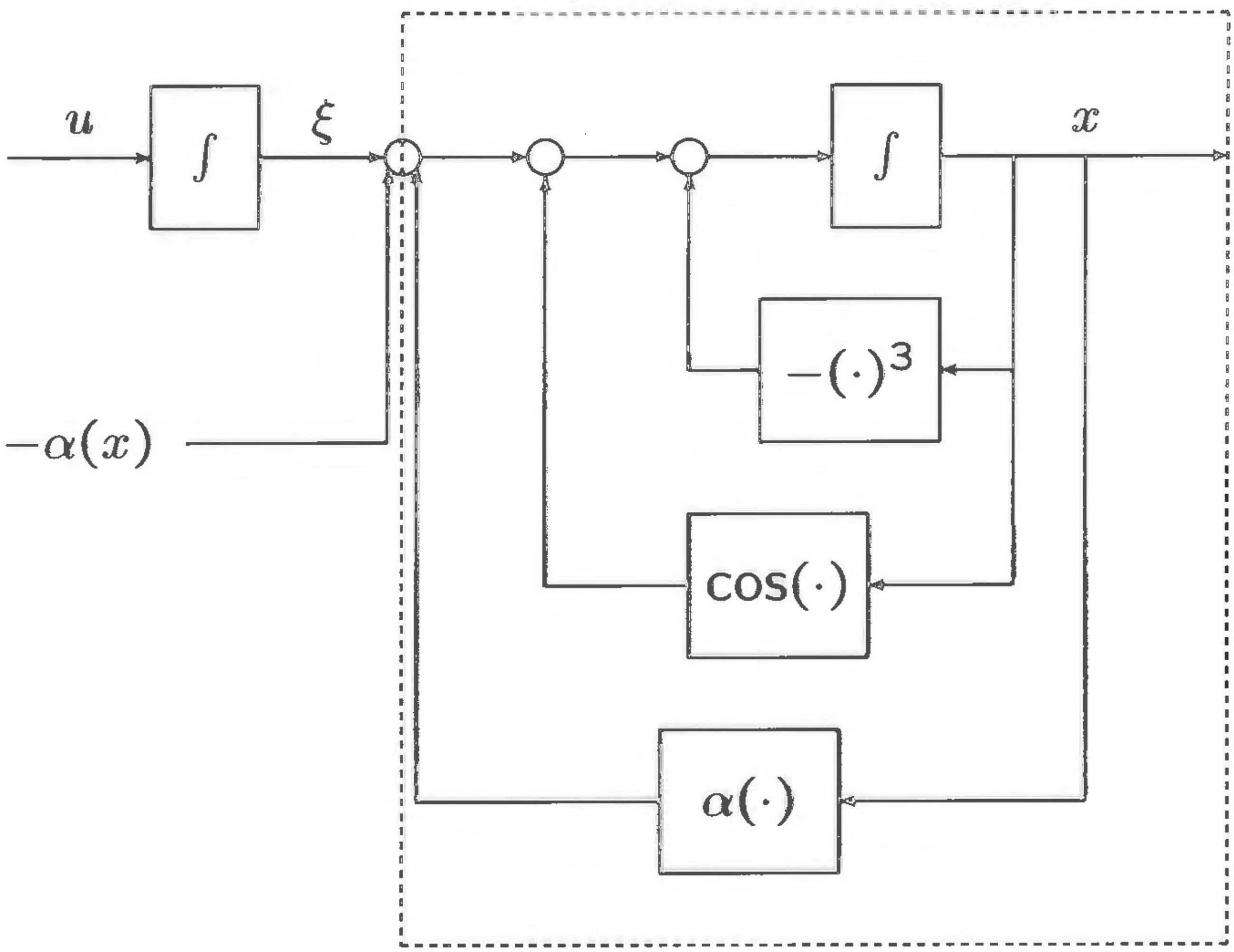
results in global asymptotic stability.

The system

$$\begin{aligned}\dot{x} &= \cos x - x^3 + \xi \\ \dot{\xi} &= u\end{aligned}$$

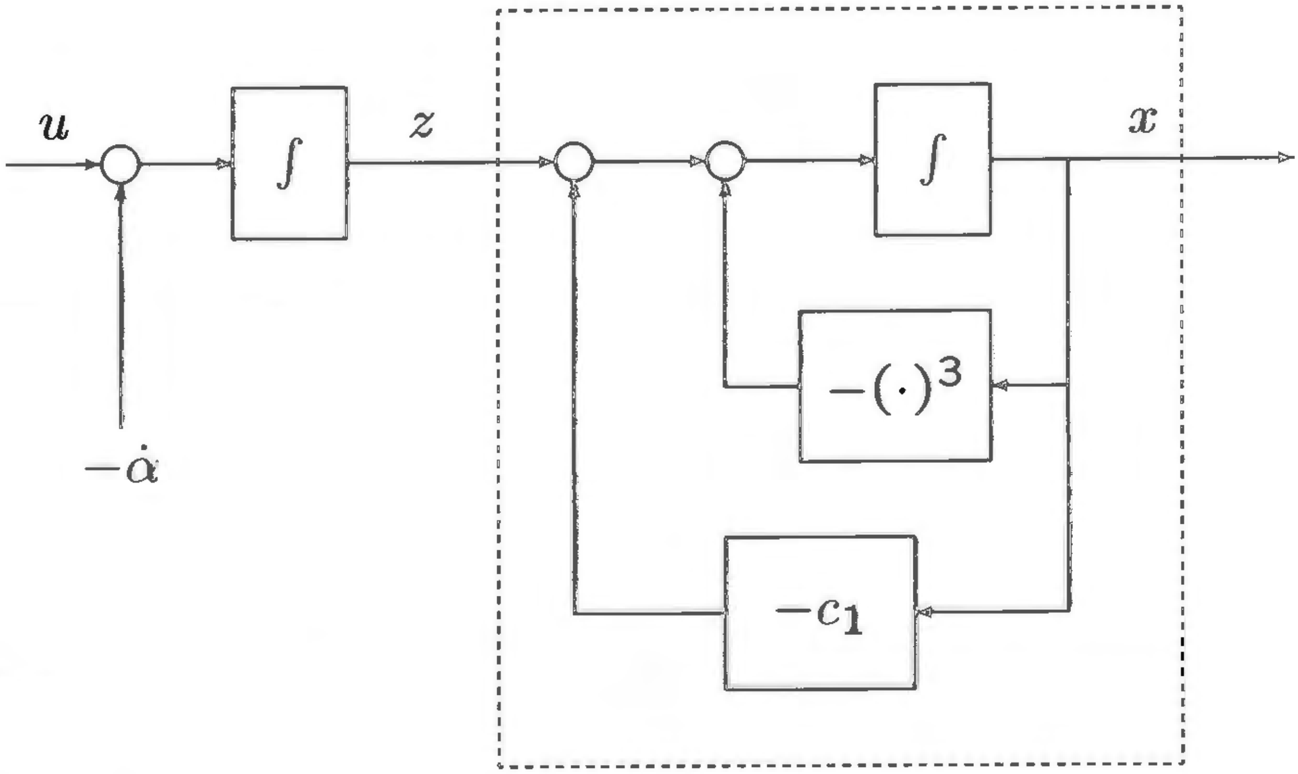


Introduction of $\xi_{des} = \alpha(x)$:



Backstepping $-\alpha$ through the integrator, by definition of $z = \xi - \alpha$, resulting into

$$
\begin{aligned}
\dot{x} &= -c_1 x - x^3 + z \\
\dot{z} &= \dot{\xi} - \dot{\alpha} = u - \dot{\alpha}
\end{aligned}
$$



What did we do?

- Studied the equation without inputs, considering the other state variable as input.
- Tried to stabilize that system
- Defined error variables
- Applied change of coordinates
- Added squared-error to Lyapunov function and derived real control input needed

Also a recursive procedure is possible

Suppose we have the system

$$\begin{aligned} \dot{x}_1 &= \cos x_1 - x_1^3 + x_2 && (1) \\ \dot{x}_2 &= x_3 && (2) \\ \dot{x}_3 &= u && (3) \end{aligned}$$

Consider the subsystem (1,2) assuming x_3 to be the input. Then we now that

$$\begin{aligned} x_3 &= -c_2(x_2 + c_1 x_1 + \cos x_1 - x_1 - \\ &\quad -(c_1 - \sin x_1)(\cos x_1 - x_1^3 + x_2) = \alpha(x) \end{aligned}$$

results in global asymptotic stability. Define $\bar{x}_3 = x_3 - \alpha(x)$ and consider the Lyapunov function candidate

$$V = \frac{1}{2}x_1^2 + \frac{1}{2}[x_2 + c_1 x_1 + \cos x_1]^2 + \frac{1}{2}\bar{x}_3^2$$

Choosing

$$u = -(x_2 + c_1 x - 1 + \cos x_1) + \dot{\alpha} - \bar{x}_3$$

results in

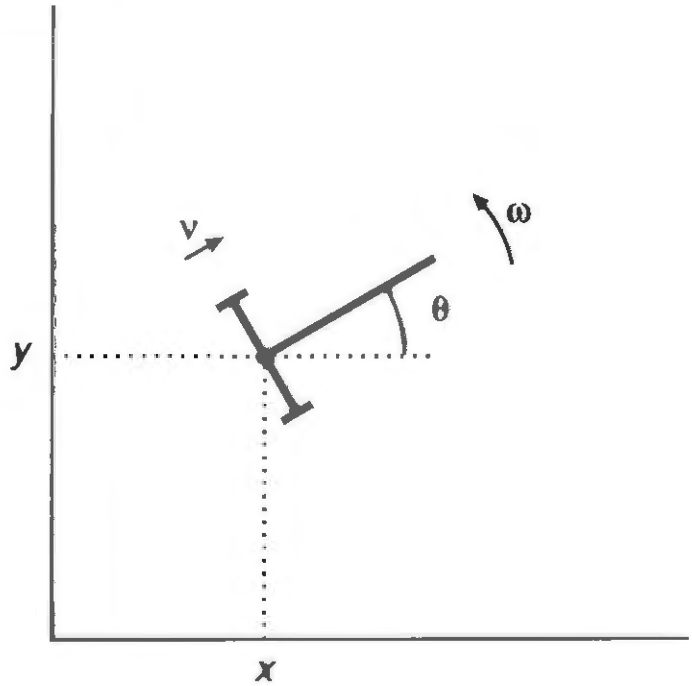
$$\dot{V} = -c_1 x_1^2 - x_1^4 - c_2[x_2 + c_1 x_1 + \cos x_1]^2 - c_3 \bar{x}_3^2$$

and therefore global asymptotic stability.

Result of Jiang and Nijmeijer

Z.-P. Jiang and H. Nijmeijer *Tracking Control of Mobile Robots: A Case Study in Backstepping*, to appear in Automatica.

Recall the model:



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Tracking control problem, where the inputs are constrained to

$$|\nu(t)| \leq \nu^{max}, \quad |\omega(t)| \leq \omega^{max} \qquad \forall t \geq 0$$

Assumption:

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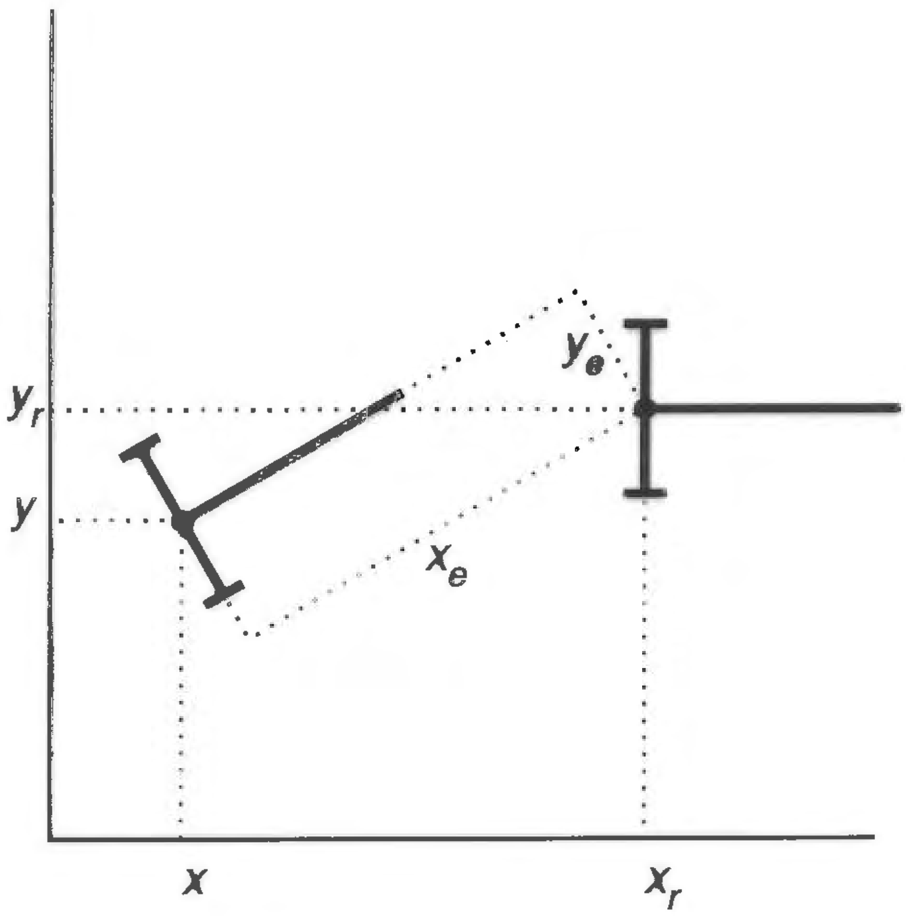


Denote error coordinates as

$$\begin{bmatrix} x_e \\ y_e \\ \theta_e \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_r - x \\ y_r - y \\ \theta_r - \theta \end{bmatrix}$$

resulting in

$$\begin{aligned} \dot{x}_e &= \omega y_e - \nu + \nu_r \cos\theta_e \\ \dot{y}_e &= -\omega x_e + \nu_r \sin\theta_e \\ \dot{\theta}_e &= \omega_r - \omega \end{aligned}$$

The error dynamics:

$$\begin{aligned}\dot{x}_e &= \omega y_e - \nu + \nu_r \cos\theta_e \\ \dot{y}_e &= -\omega x_e + \nu_r \sin\theta_e \\ \dot{\theta}_e &= \omega_r - \omega\end{aligned}$$

We view x_e and θ_e as inputs acting on the y_e system.

Choosing $x_e = c_3 \omega y_e$ and $\theta_e = 0$ results into:

$$\dot{y}_e = -c_3 \omega^2 y_e$$

Using the Lyapunov function candidate

$$V(y_e) = \frac{1}{2} y_e^2$$

results into

$$\dot{V}(t, y_e) = -\omega^2 y_e^2$$

and therefore stability of y_e. We can **not** conclude *asymptotic* stability (yet).

We can use the idea of integrator backstepping and define the new variable

$$\bar{x}_e = x_e - c_3 \omega y_e$$

($\bar{y}_e = y_e - 0$ does not change).

Then we obtain in the new coordinates:

$$\begin{aligned}
\dot{\bar{x}}_e &= \omega y_e - \nu + \nu_r \cos\theta_e - c_3 \dot{\omega} y_e - c_3 \omega \dot{y}_e \\
\dot{y}_e &= -\omega\{\bar{x}_e + c_3 \omega y_e\} + \nu_r \sin\theta_e \\
\dot{\theta}_e &= \omega_r - \omega
\end{aligned}$$

Consider the Lyapunov function candidate

$$V(t, x_e, y_e, \theta_e) = \frac{1}{2}\bar{x}_e^2 + \frac{1}{2}y_e^2 + \frac{1}{2\gamma}\theta_e^2 \quad (\gamma > 0)$$

It's time-derivative becomes:

$$\begin{aligned}
\dot{V} &= \bar{x}_e[\omega y_e - \nu + \nu_r \cos\theta_e - c_3 \dot{\omega} y_e - c_3 \omega \dot{y}_e] + \\
&\quad + y_e[-\omega\{\bar{x}_e + c_3 \omega y_e\} + \nu_r \sin\theta_e] + \frac{1}{\gamma}\theta_e[\omega_r - \omega] \\
&= -\bar{x}_e[\nu - \nu_r \cos\theta_e + c_3 \dot{\omega} y_e + c_3 \omega \dot{y}_e] - \\
&\quad - c_3 \omega^2 y_e^2 - \frac{1}{\gamma}\theta_e[\omega - \omega_r - \gamma y_e \nu_r \frac{\sin\theta_e}{\theta_e}]
\end{aligned}$$

So if we use

$$
\begin{aligned}
\nu &= \nu_r \cos\theta_e - c_3\dot{\omega}y_e - c_3\omega\dot{y}_e + c_4\bar{x}_e \\
\omega &= \omega_r + \gamma y_e \nu_r \frac{\sin\theta_e}{\theta_e} + c_5\theta_e
\end{aligned}
$$

we obtain

$$
\dot{V}(t, x_e, y_e, \theta_e) = -c_4\bar{x}_e^2 - c_3\omega^2 y_e^2 - \frac{1}{\gamma}c_5\theta_e^2
$$

Note that we used, $\dot{\omega}$ and $\dot{y}_e$ which can be expressed in the state variables, using:

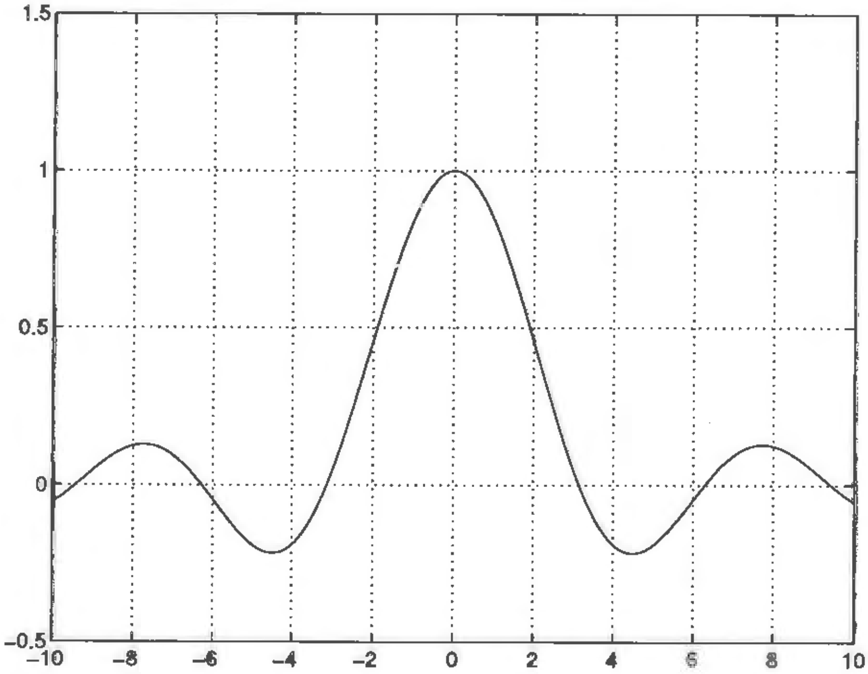
$$
\begin{aligned}
\dot{\omega} &= \dot{\omega}_r + \gamma(\dot{y}_e\nu_r + y_e\dot{\nu}_r)\frac{\sin\theta_e}{\theta_e} + \\
&\quad + \gamma y_e \nu_r \frac{\theta_e\cos\theta_e - \sin\theta_e}{\theta_e^2}\dot{\theta}_e + c_5\dot{\theta}_e \\
\dot{y}_e &= -\omega x_e + \nu_r \sin\theta_e \\
\dot{\theta}_e &= \omega_r - \omega
\end{aligned}
$$

We only need to have $\dot{\nu}_r$ and $\dot{\omega}_r$.

Furthermore, we use

$$\frac{\sin \theta_e}{\theta_e} = \int_0^1 \cos(s\theta_e)ds$$

which looks like



Theorem: Assume that ν_r, $\dot{\nu}_r$, ω_r, and $\dot{\omega}_r$ are bounded on $[0,\infty)$.
Then, all trajectories of the system

$$\begin{aligned}\dot{x}_e &= \omega y_e - \nu + \nu_r \cos\theta_e \\ \dot{y}_e &= -\omega x_e + \nu_r \sin\theta_e \\ \dot{\theta}_e &= \omega_r - \omega\end{aligned}$$

under the feedback

$$\begin{aligned}\nu &= \nu_r \cos\theta_e - c_3\dot{\omega} y_e - c_3\omega\dot{y}_e + c_4\bar{x}_e \\ \omega &= \omega_r + \gamma y_e \nu_r \int_0^1 \cos(s\theta_e)ds + c_5\theta_e\end{aligned}$$

are globally uniformly bounded.

Furthermore, if either $\nu_r \not\to 0$ or $\nu \to 0$ but $\omega_r \not\to 0$ then globally:

$$\lim_{t\to\infty} (|x_e(t)| + |y_e(t)| + |\theta_e(t)|) = 0$$

Additionally, if $\liminf_{t\to\infty} |\omega_r(t)| > 0$:
The zero-equilibrium is exponentially stable for small initial errors, i.e. all closed-loop trajectories go to zero at an exponentially rate after a considerable period of time.

Bounded tracking

The controller

$$
\begin{aligned}
\nu &= \nu_r \cos\theta_e - c_3\dot{\omega} y_e - c_3\omega \dot{y}_e + c_4 \bar{x}_e \\
\omega &= \omega_r + \gamma y_e \nu_r \int_0^1 \cos(s\theta_e) ds + c_5\theta_e
\end{aligned}
$$

does not satisfy

$$
|\nu(t)| \leq \nu^{max}, \quad |\omega(t)| \leq \omega^{max} \qquad \forall t \geq 0
$$

How to extend this result to a bounded result?

Example

Consider the system

$$\dot{x} = u$$

A stabilizing controller is not only

$$u = -c_1 x$$

But also

$$u = -c_1 \tanh(c_2 x)$$

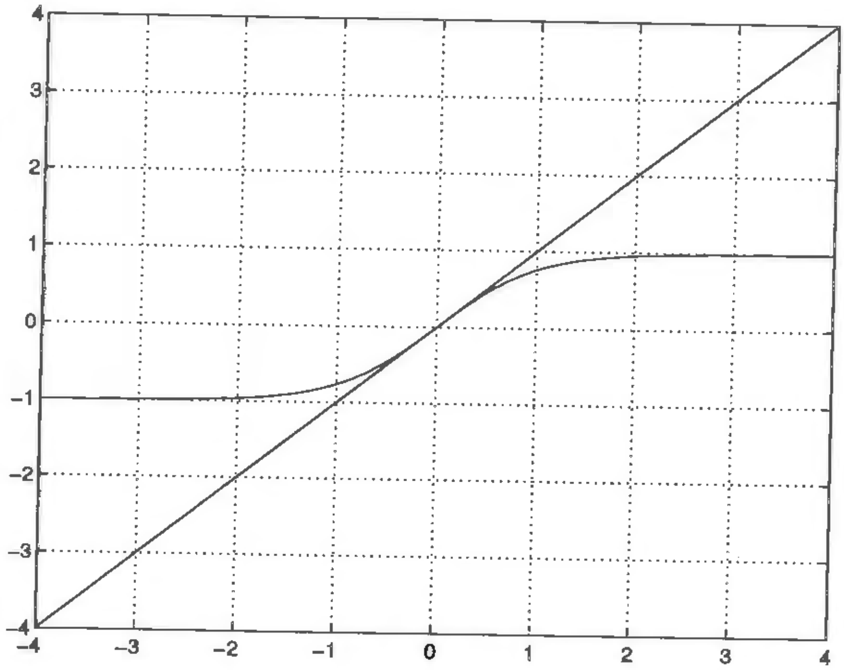
or

$$u = -c_3 \mathrm{sat}(c_4 x)$$

(c_1, c_2, c_3, c_4 positive).

Let $\mathcal{F}$ denote the set of all nondecreasing C^1 functions satisfying $f(0) = 0, f'(0) > 0$.

Examples:



$f(x) = ax$, $f(x) = a\,\mathrm{tanh}(bx)$ $(a, b > 0)$.

Let $\mathcal{B} \triangleq \{f \in \mathcal{F} | f \in L_\infty \wedge f' \in L_\infty\}$.

Other elements of $\mathcal{B}$: $\frac{ax}{b+|x|}$, $a\,\mathrm{arctan}(bx)$,

$$f(x) = \begin{cases} -1 & x < -\pi/2 \\ \sin(x) & -\pi/2 \le x \le \pi/2 \\ 1 & \pi/2 < x \end{cases}$$

Recall the error dynamics

$$\begin{aligned}
\dot{x}_e &= \omega y_e - \nu + \nu_r \cos\theta_e \\
\dot{y}_e &= -\omega x_e + \nu_r \sin\theta_e \\
\dot{\theta}_e &= \omega_r - \omega
\end{aligned}$$

We view x_e and θ_e as inputs acting on the y_e system.

Choosing $x_e = f_1(\omega) f_2(y_e)$ $(f_1, f_2 \in \mathcal{F})$ and $\theta_e = 0$ results into:

$$\dot{y}_e = -\omega f_1(\omega) f_2(y_e).$$

Using the Lyapunov function candidate

$$V(y_e) = \frac{1}{2} y_e^2$$

results into:

$$\dot{V}(t, y_e) = -\omega f_1(\omega) y_e f_2(y_e)$$

and therefore stability of y_e.

Define the new variable

$$\bar{x}_e = x_e - f_1(\omega) f_2(y_e).$$

Then we obtain in the new coordinates:

$$\begin{aligned}
\dot{\bar{x}}_e &= \omega y_e - \nu + \nu_r \cos\theta_e - \dot{\omega} f_1'(\omega) f_2(y_e) - \\
&\quad - f_1(\omega) f_2'(y_e) \dot{y}_e \\
\dot{y}_e &= -\omega\{\bar{x}_e + f_1(\omega) f_2(y_e)\} + \nu_r \sin\theta_e \\
\dot{\theta}_e &= \omega_r - \omega
\end{aligned}$$

Consider the Lyapunov function candidate

$$V(t, x_e, y_e, \theta_e) = \frac{1}{2}\bar{x}_e^2 + \frac{1}{2} y_e^2 + \frac{1}{2\gamma}\theta_e^2 \quad (\gamma > 0)$$

It's timederivative becomes:

$$\begin{aligned}
\dot{V} &= \bar{x}_e[\omega y_e - \nu + \nu_r \cos\theta_e - \dot{\omega} f_1'(\omega) f_2(y_e) - \\
&\quad - f_1(\omega) f_2'(y_e)\dot{y}_e] + y_e[-\omega\{\bar{x}_e + f_1(\omega) f_2(y_e)\} + \\
&\quad + \nu_r \sin\theta_e] + \frac{1}{\gamma}\theta_e[\omega_r - \omega] \\
&= -\bar{x}_e[\nu - \nu_r \cos\theta_e + \dot{\omega} f_1'(\omega) f_2(y_e) + \\
&\quad + f_1(\omega) f_2'(y_e)\dot{y}_e] - \omega f_1(\omega) y_e f_2(y_e) - \\
&\quad - \frac{1}{\gamma}\theta_e[\omega - \omega_r - \gamma y_e \nu_r \frac{\sin\theta_e}{\theta_e}]
\end{aligned}$$

So if we use

$$
\begin{aligned}
\nu &= \nu_r \cos\theta_e - \dot{\omega} f_1'(\omega) f_2(y_e) - \\
&\quad - f_1(\omega) f_2'(y_e)\dot{y}_e + f_3(\bar{x}_e) \\
\omega &= \omega_r + \gamma y_e \nu_r \frac{\sin\theta_e}{\theta_e} + f_4(\theta_e)
\end{aligned}
$$

We obtain:

$$
\dot{V} = -\bar{x}_e f_3(\bar{x}_e) - \omega f_1(\omega) y_e f_2(y_e) - \frac{1}{\gamma}\theta_e f_4(\theta_e)
$$

where ω and y_e are given by

$$
\begin{aligned}
\dot{\omega} &= \dot{\omega}_r + \gamma(\dot{y}_e \nu_r + y_e \dot{\nu}_r)\frac{\sin\theta_e}{\theta_e} + \\
&\quad + \gamma y_e \nu_r \frac{\theta_e \cos\theta_e - \sin\theta_e}{\theta_e^2}\dot{\theta}_e + f_4'(\theta_e)\dot{\theta}_e \\
\dot{y}_e &= -\omega\{\bar{x}_e + f_1(\omega) f_2(y_e)\} + \nu_r \sin\theta_e \\
\dot{\theta}_e &= \omega_r - \omega
\end{aligned}
$$

We only need to have $\dot{\nu}_r$ and $\dot{\omega}_r$.

However, is this bounded?

First choose a γ and $f_1, f_2, f_3, f_4 \in \mathcal{B}$.

We need to satisfy:

$$|\nu(t)| \leq \nu^{max}, \quad |\omega(t)| \leq \omega^{max} \qquad \forall t \geq 0$$

provided that:

$$\begin{aligned} \nu^{max} &> \max_{t\geq 0} \ \nu_r(t) \\ \omega^{max} &> \max_{t\geq 0} \ \omega_r(t) \end{aligned}$$

We know that $V(t) \leq V(0)$. Therefore

$$|\gamma y_e(t)| \leq \sqrt{\gamma^2[x_e(0) + ||f_1||_\infty ||f_2||_\infty]^2 + \gamma^2 y_e(0)^2 + \gamma\theta_e(0)^2}$$

So can premultiply γ and f_4 with some small enough constants (≤ 1) to obtain:

$$|\omega(t)| \leq \omega^{max} \qquad \forall t \geq 0$$

Now we can derive bounds on $\dot{y}_e$, $\dot{\theta}_e$ and $\dot{\omega}$.

Now we can premultiply f_1, f_2 and f_3 with some small enough constants (≤ 1) to obtain:

$$|\nu(t)| \leq \nu^{max} \qquad \forall t \geq 0$$

Note that by this possibly premultiplication of f_1 and f_2 with a small enough contant might decrease the bound on $|\gamma y_e|$, but will never increase it.

Concluding

Provided we know bounds on the initial conditions, we are always able to choose a controller such that

$$|\nu(t)| \leq \nu^{max}, \quad |\omega(t)| \leq \omega^{max} \qquad \forall t \geq 0$$

Theorem: Assume that ν_r, $\dot{\nu}_r$, ω_r, and $\dot{\omega}_r$ are bounded on $[0, \infty)$.
Then, all trajectories of the system

$$\begin{aligned} \dot{x}_e &= \omega y_e - \nu + \nu_r \cos\theta_e \\ \dot{y}_e &= -\omega x_e + \nu_r \sin\theta_e \\ \dot{\theta}_e &= \omega_r - \omega \end{aligned}$$

under the feedback

$$\begin{aligned} \nu &= \nu_r \cos\theta_e - \dot{\omega} f_1'(\omega) f_2(y_e) - \\ &\quad - f_1(\omega) f_2'(y_e)\dot{y}_e + f_3(\bar{x}_e) \\ \omega &= \omega_r + \gamma y_e \nu_r \frac{\sin\theta_e}{\theta_e} + f_4(\theta_e) \end{aligned}$$

are globally uniformly bounded.

If $\nu_r \not\to 0$ or $\nu \to 0$ but $\omega_r \not\to 0$ then globally:

$$\lim_{t\to\infty} (|x_e(t)| + |y_e(t)| + |\theta_e(t)|) = 0$$

Furthermore, for any initial condition $(x_e(0), y_e(0), \theta_e(0))$ there exist $f_1, f_2, f_3, f_4 \in \mathcal{B}$ such that

$$|\nu(t)| \le \nu^{max}, \quad |\omega(t)| \le \omega^{max} \qquad \forall t \ge 0$$

Additionally, if $\liminf_{t\to\infty} |\omega_r(t)| > 0$, the zero equilibrium is exponentially stable for small initial errors.
In other words: all the closed-loop trajectories go to zero at an exponentially rate after a considerable period of time.

Remark By using

$$\begin{aligned} f_1(\omega) &= c_3\omega \\ f_2(y_e) &= y_e \\ f_3(\bar{x}_e) &= c_4\bar{x}_e \\ f_4(\theta_e) &= c_5\theta_e \end{aligned}$$

we obtain the result of Jiang and Nijmeijer

Simulations

Initial conditions:
$(x_e(0), y_e(0), \theta_e(0)) = (16.6, 1.5, -1)$

Reference trajectory:
$\nu_r(t) = 1$, $\omega_r(t) = 0$, i.e. straight line.

Controller Jiang & Nijmeijer (Automatica):
$f_1(\omega) = \omega$, $f_2(y_e) = y_e$, $f_3(\bar{x}_e) = 2\bar{x}_e$, $f_4(\theta_e) = \theta_e$, $\gamma = 1$ (i.e. $c_3 = 1$, $c_4 = 2$, $c_5 = 1$).

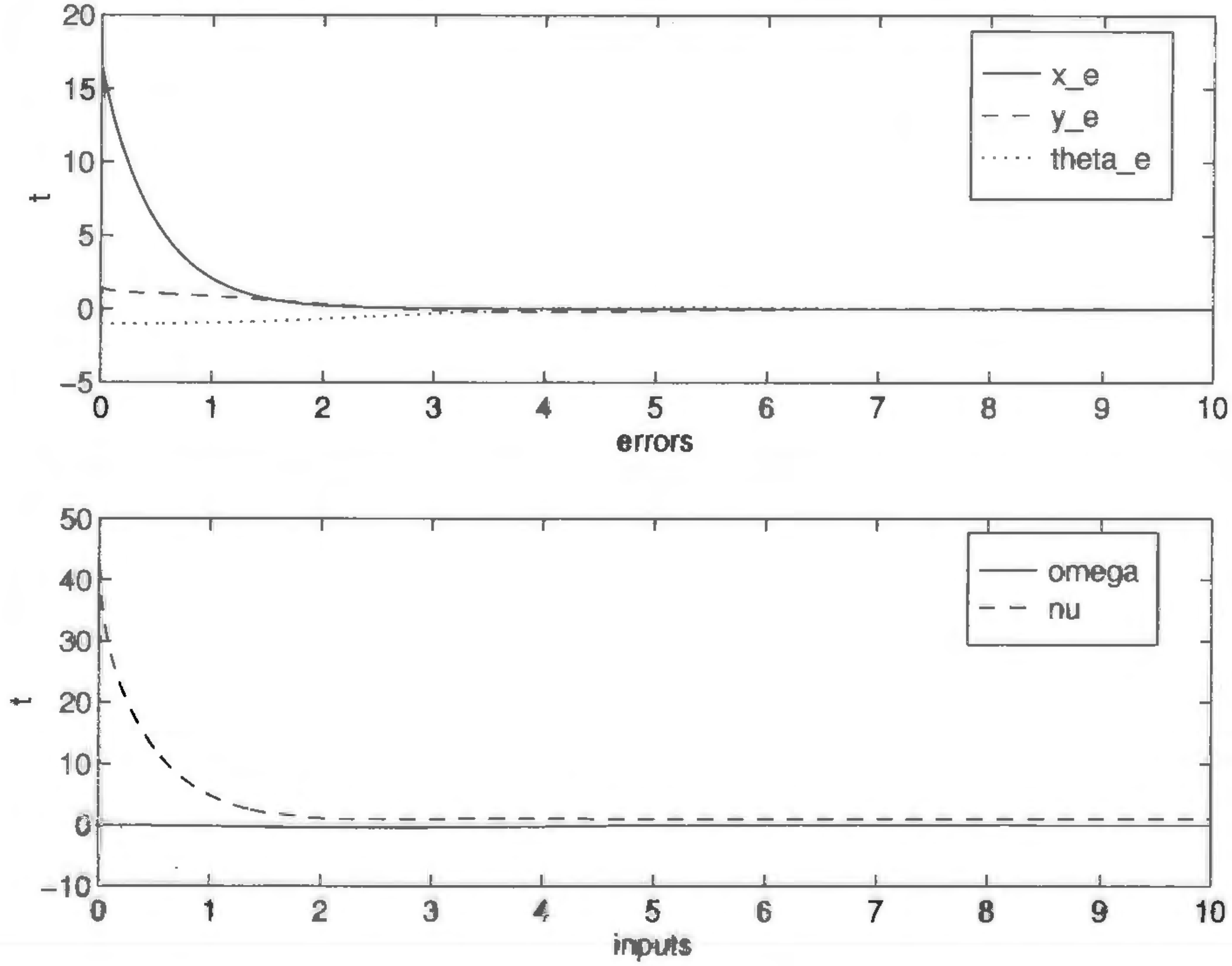
Bounded controller:
$f_1(\omega) = 0.2\tanh(\omega)$, $f_2(y_e) = 0.2\tanh(y_e)$, $f_3(\bar{x}_e) = 0.2\tanh(\bar{x}_e)$, $f_4(\theta_e) = 0.2\tanh(\theta_e)$, $\gamma = 0.045$.

The bounded controller was designed to satisfy

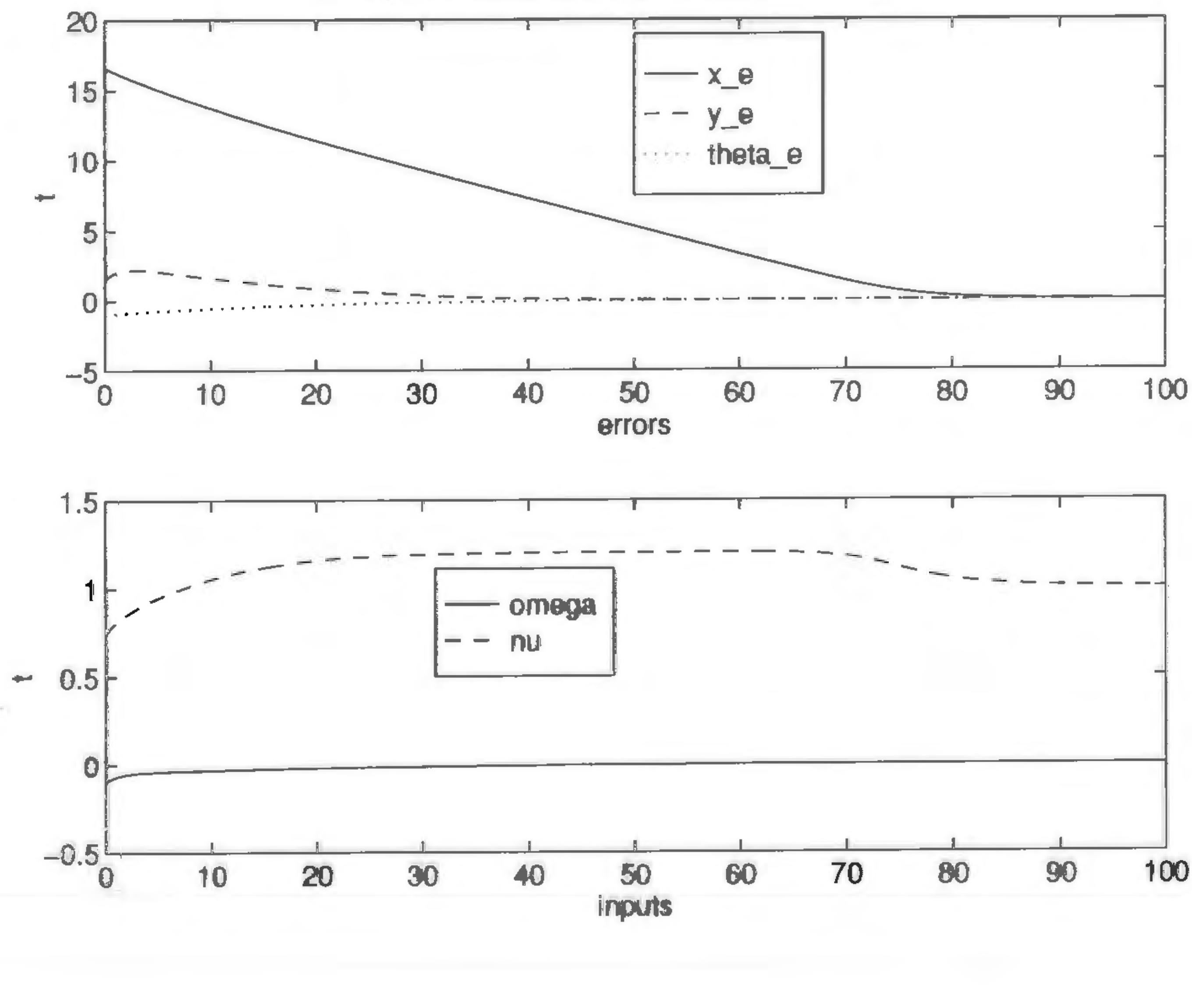
$$|\nu(t)| \leq 2, \quad |\omega(t)| \leq 1 \qquad \forall t \geq 0$$

Controller Jiang & Nijmeijer (II):
$f_1(\omega) = 0.2\omega$, $f_2(y_e) = 0.2y_e$, $f_3(\bar{x}_e) = 0.2\bar{x}_e$, $f_4(\theta_e) = 0.2\theta_e$, $\gamma = 0.045$.

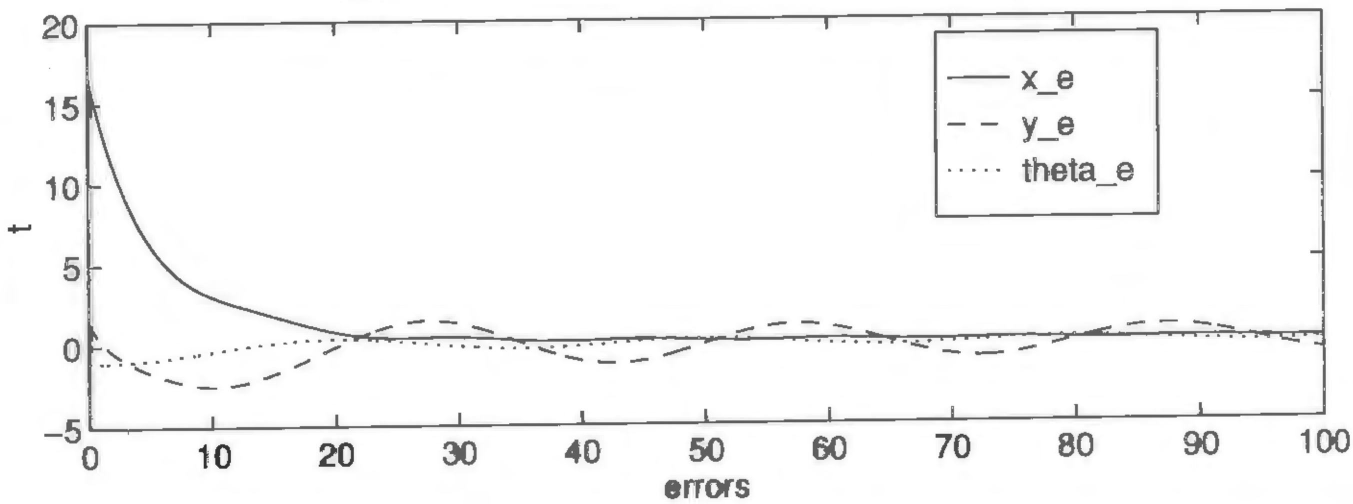
Controller Jiang & Nijmeijer

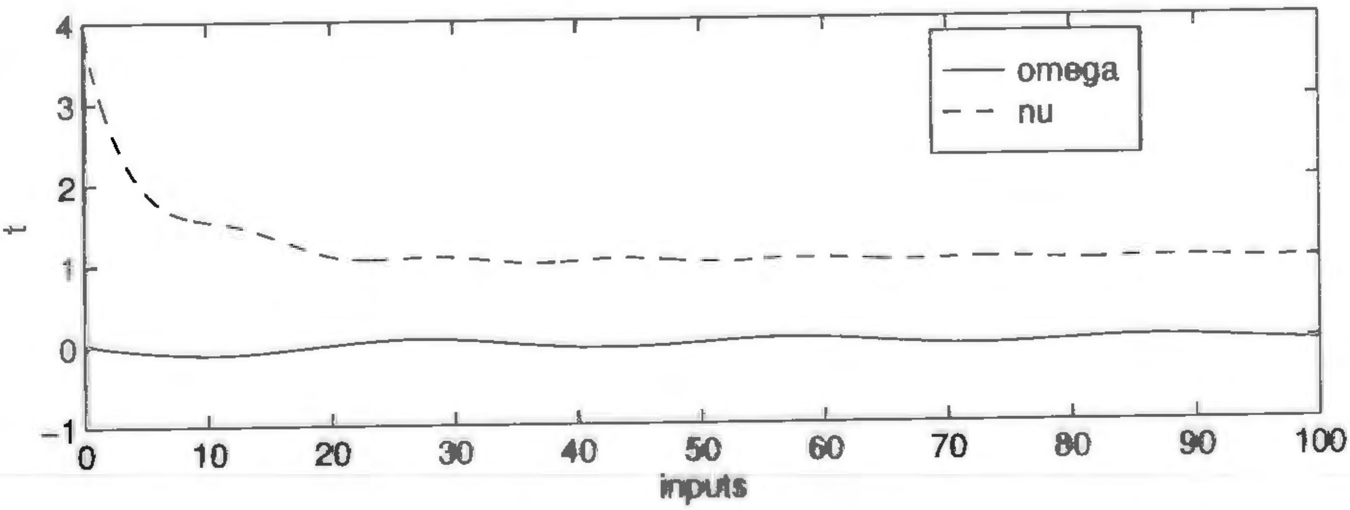


Bounded Controller



Jiang & Nijmeijer (II)





Conclusions

- Using the idea of integrator backstepping resulted in globally tracking controllers for a wheeled mobile robot.

- A richer class of (globally) tracking controllers for a wheeled mobile robot has been derived.

- Under input constraints, (semi)globally tracking controllers have been found.

- There is a trade-off between convergence to the desired trajectory and the satisfaction of input constraints.