Tracking of Chained Form Systems via Output Feedback

18th Benelux Meeting on Systems and Control
3–5 March 1999, Houthalen-Helchteren

E. Lefeber*, A. Robertsson**, and H. Nijmeijer*,***

* University of Twente, Enschede, The Netherlands

** Lund Institute of Technology, Lund, Sweden

*** Eindhoven University of Technology, Eindhoven, The Netherlands

Outline

- Model and Problem formulation
- Cascaded systems
- Derivation of linear tracking controller
- Simulations
- Conclusions

Chained form system

Nonholonomic system in chained form:

$$
\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= x_2 u_1 \\
&\vdots \\
\dot{x}_n &= x_{n-1} u_1
\end{aligned}
$$

Examples: Car pulling multiple trailers, rolling Euro, knife-edge (pizza-knife)

Tracking Problem (state-feedback)

System dynamics:

$$
\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= x_2 u_1 \\
&\vdots \\
\dot{x}_n &= x_{n-1} u_1
\end{aligned}
$$

Reference dynamics:

$$
\begin{aligned}
\dot{x}_{1,r} &= u_{1,r} \\
\dot{x}_{2,r} &= u_{2,r} \\
\dot{x}_{3,r} &= x_{2,r} u_{1,r} \\
&\vdots \\
\dot{x}_{n,r} &= x_{n-1,r} u_{1,r}
\end{aligned}
$$

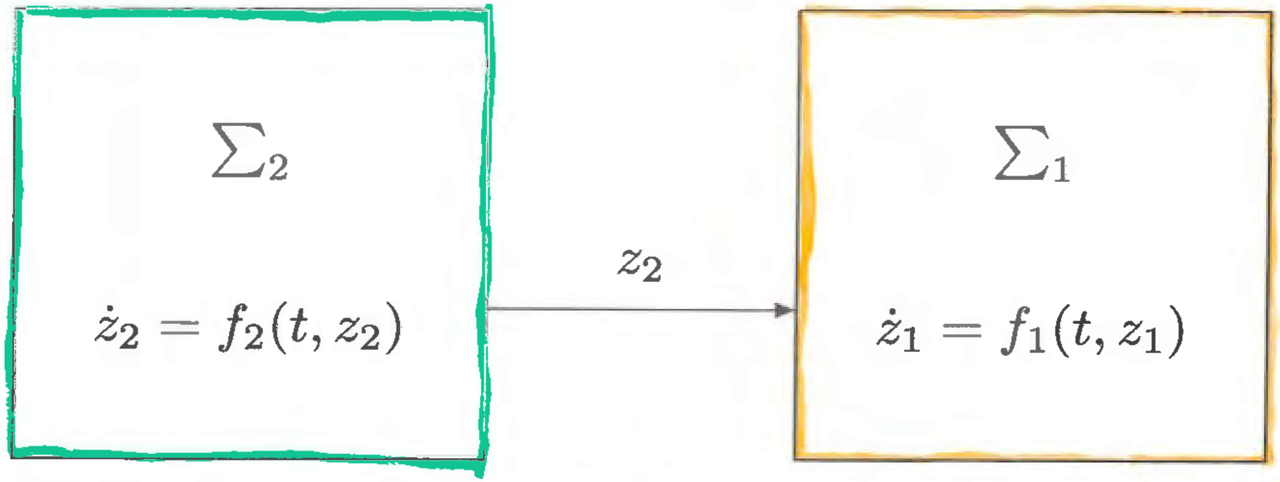
Find control laws

$$u \equiv u(x, x_r, u_r)$$

that yield

$$\lim_{t \to \infty} |x(t) - x_r(t)| = 0$$

Cascaded systems



$$
\begin{aligned}
\dot{z}_1 &= f_1(t, z_1) + g(t, z_1, z_2)\boxed{z_2} \\
\dot{z}_2 &= f_2(t, z_2)
\end{aligned}
$$

Consider the system

$$\begin{aligned} \boxed{\dot{x}_1 = -x_1} &+ x_1^2 \boxed{x_2} \\ \boxed{\dot{x}_2 = -\gamma x_2} & \end{aligned}$$

Solution:

$$\begin{aligned} x_1(t) &= \frac{2x_1(0)}{x_1(0)x_2(0)e^{-t\gamma} + [2 - x_1(0)x_2(0)]e^{t\gamma}} \\ x_2(t) &= x_2(0)e^{-\gamma t} \end{aligned}$$

When $x_1(0)x_2(0) > 2$ we have a finite escape time

$$t_{esc} = \frac{1}{2\gamma} \ln \frac{x_1(0)x_2(0)}{x_1(0)x_2(0) - 2}$$

Conditions

E. Panteley en A. Loría (S&CL 33(2), 1998):

Cascade Globally Uniformly Asymptotically Stable (GUAS) when

- Σ_1 GUAS, polynomial 'Lyapunov function'
- $g(t, z_1, z_2)$ at most linear in z_1
- Σ_2 GUAS, $z_2(t)$ integrable

Controller Design

Error dynamics ($x_e = x - x_r$):

$$
\begin{array}{rclcccc}
\dot{x}_{2,e} & = & 0 & + & (u_2 - u_{2,r}) & + & 0 \\
\dot{x}_{3,e} & = & u_{1,r} x_{2,e} & & & + & x_2 (u_1 - u_{1,r}) \\
& \vdots & & & & \vdots & \\
\dot{x}_{n,e} & = & u_{1,r} x_{n-1,e} & & & + & x_{n-1} (u_1 - u_{1,r})
\end{array}
$$

$$
\dot{x}_{1,e} = (u_1 - u_{1,r})
$$

Problem has reduced to (seperately) stabilizing the error-dynamics

$$\begin{bmatrix} \dot{x}_{2,e} \\ \dot{x}_{3,e} \\ \dot{x}_{4,e} \\ \vdots \\ \dot{x}_{n,e} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ u_{1,r} & \ddots & & & \vdots \\ 0 & u_{1,r} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & u_{1,r} & 0 \end{bmatrix} \begin{bmatrix} x_{2,e} \\ x_{3,e} \\ x_{4,e} \\ \vdots \\ x_{n,e} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} (u_2 - u_{2,r})$$

and

$$\dot{x}_{1,e} = (u_1 - u_{1,r})$$

Stabilizing the $x_{1,e}$ dynamics:

$$u_1 = u_{1,r} - c_1 x_{1,e} \qquad\qquad c_1 > 0$$

For stabilizing the $[x_{2,e}, \ldots, x_{n,e}]^T$ dynamics we need *uniform controllability.*

We can show that if $u_{1,r} \not\to 0$, the controller

$$u_2 = u_{2,r} - c_2 x_{2,e} - c_3 u_{1,r} x_{3,e} - c_4 x_{4,e} - c_5 u_{1,r} x_{5,e} - \ldots$$

yields global asymptotic stability of the closed-loop system, provided

$$\lambda^{n-1} + c_2 \lambda^{n-2} + \ldots + c_{n-1}\lambda + c_n \ \text{ is Hurwitz.}$$

To summarize

Consider the error dynamics

$$
\begin{aligned}
\dot{x}_{1,e} &= u_1 - u_{1,r} \\
\dot{x}_{2,e} &= u_2 - u_{2,r} \\
\dot{x}_{3,e} &= x_2 u_1 - x_{2,r} u_{1,r} \\
&\vdots \\
\dot{x}_{n,e} &= x_{n-1} u_1 - x_{n-1,r} u_{1,r}
\end{aligned}
$$

in closed loop with the controller

$$
\begin{aligned}
u_1 &= u_{1,r} - c_1 x_{1,e} \\
u_2 &= u_{2,r} - c_2 x_{2,e} - c_3 u_{1,r} x_{3,e} - c_4 x_{4,e} - c_5 u_{1,r} x_{5,e} - \cdots
\end{aligned}
$$

If $u_{1,r} \not\to 0$, then the closed loop is globally asymptotically stable.

Tracking Problem (output feedback)

System dynamics:

$$\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= x_2 u_1 \\
&\vdots \\
\dot{x}_n &= x_{n-1} u_1
\end{aligned}$$

Reference dynamics:

$$\begin{aligned}
\dot{x}_{1,r} &= u_{1,r} \\
\dot{x}_{2,r} &= u_{2,r} \\
\dot{x}_{3,r} &= x_{2,r} u_{1,r} \\
&\vdots \\
\dot{x}_{n,r} &= x_{n-1,r} u_{1,r}
\end{aligned}$$

Output: $\quad y = [y_1, y_n]^T$

Find control laws

$$u \equiv u(\hat{x}, x_r, u_r) \qquad \dot{\hat{x}} = f(\hat{x}, y, x_r, u_r)$$

that yield $\lim\limits_{t \to \infty} |x(t) - x_r(t)| = 0$

Using again $u_1 = u_{1,r} - c_1 x_{1,e}$ and the cascaded result, we need to stabilize

$$
\begin{aligned}
\begin{bmatrix} \dot{x}_{2,e} \\ \dot{x}_{3,e} \\ \dot{x}_{4,e} \\ \vdots \\ \dot{x}_{n,e} \end{bmatrix} &= \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ u_{1,r} & \ddots & & & \vdots \\ 0 & u_{1,r} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & u_{1,r} & 0 \end{bmatrix} \begin{bmatrix} x_{2,e} \\ x_{3,e} \\ x_{4,e} \\ \vdots \\ x_{n,e} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} (u_2 - u_{2,r}) \\
y &= x_{n,e}
\end{aligned}
$$

by means of output feedback.

Using the *separation principle* and *certainty equivalence* we find:

$$u_2 = u_{2,r} - c_2\hat{x}_{2,e} - c_3 u_{1,r}\hat{x}_{3,e} - c_4\hat{x}_{4,e} - c_5 u_{1,r}\hat{x}_{5,e} - \cdots$$

$$\begin{bmatrix} \dot{\hat{x}}_{2,e} \\ \dot{\hat{x}}_{3,e} \\ \dot{\hat{x}}_{4,e} \\ \vdots \\ \dot{\hat{x}}_{n,e} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ u_{1,r} & \ddots & & & \vdots \\ 0 & u_{1,r} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & u_{1,r} & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_{2,e} \\ \hat{x}_{3,e} \\ \hat{x}_{4,e} \\ \vdots \\ \hat{x}_{n,e} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} (u_2 - u_{2,r}) +$$

$$+ [\ldots, \; l_5 u_{1,r}, \; l_4, \; l_3 u_{1,r}, \; l_2]^T (x_{n,e} - \hat{x}_{n,e})$$

yields global asymptotic stability, provided

$$\left.\begin{array}{l} \lambda^{n-1} + c_2\lambda^{n-2} + \ldots + c_{n-1}\lambda + c_n \\ \lambda^{n-1} + l_2\lambda^{n-2} + \ldots + l_{n-1}\lambda + l_n \end{array}\right. \quad \text{Hurwitz}$$

To summarize

Error dynamics

$$
\begin{aligned}
\dot{x}_{1,e} &= u_1 - u_{1,r} \\
\dot{x}_{2,e} &= u_2 - u_{2,r} \\
\dot{x}_{3,e} &= x_2 u_1 - x_{2,r} u_{1,r} \\
&\vdots \\
\dot{x}_{n,e} &= x_{n-1} u_1 - x_{n-1,r} u_{1,r} \\
y &= [x_{1,e}, \;\; x_{n,e}]^T
\end{aligned}
$$

Observer

$$
\begin{aligned}
\dot{\hat{x}}_{2,e} &= u_2 - u_{2,r} + l_n[u_{1,r}]\tilde{x}_{n,e} \\
\dot{\hat{x}}_{3,e} &= u_{1,r}\hat{x}_{2,e} + l_{n-1}[u_{1,r}]\tilde{x}_{n,e} \\
&\vdots \\
\dot{\hat{x}}_{n-1,e} &= u_{1,r}\hat{x}_{n-2,e} + l_3 u_{1,r}\tilde{x}_{n,e} \\
\dot{\hat{x}}_{n,e} &= u_{1,r}\hat{x}_{n-1,e} + l_2\tilde{x}_{n,e} \\
\tilde{x}_{n,e} &= x_{n,e} - \hat{x}_{n,e}
\end{aligned}
$$

in closed loop with the controller

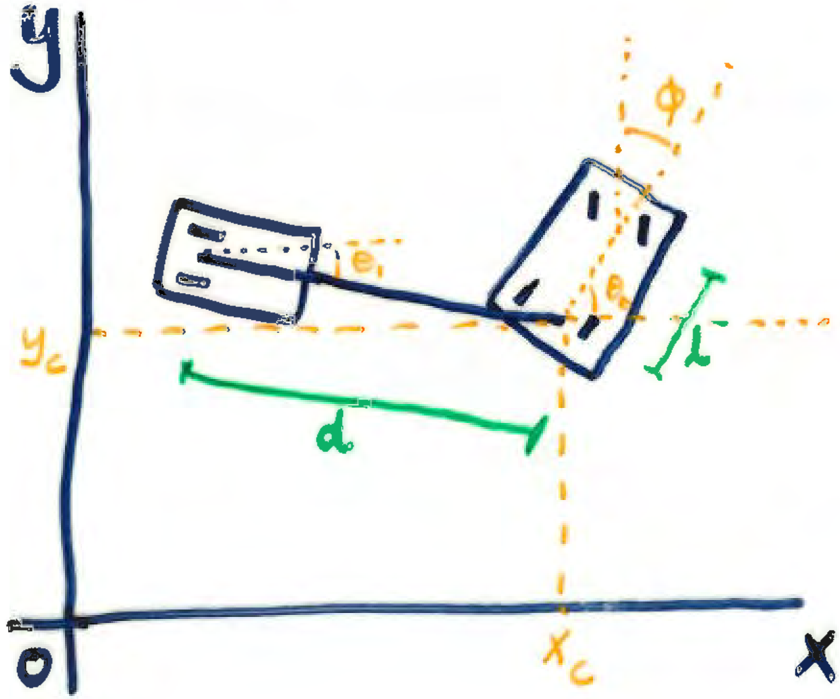
$$
\begin{aligned}
u_1 &= u_{1,r} - c_1 x_{1,e} \\
u_2 &= u_{2,r} - c_2\hat{x}_{2,e} - c_3 u_{1,r}\hat{x}_{3,e} - c_4\hat{x}_{4,e} - c_5 u_{1,r}\hat{x}_{5,e} - \cdots
\end{aligned}
$$

If $u_{1,r} \not\to 0$, then the closed loop is globally asymptotically stable.

Simulations

Car pulling a single trailer:

$$\begin{aligned}
\dot{x}_c &= v\cos\theta_0 \\
\dot{y}_c &= v\sin\theta_0 \\
\dot{\phi} &= \omega \\
\dot{\theta}_0 &= \tfrac{1}{l}v\tan\phi \\
\dot{\theta}_1 &= \tfrac{1}{d_1}v\sin(\theta_0-\theta_1)
\end{aligned}$$



State-feedback

Initial condition:

$$x_1(0) = x_2(0) = x_3(0) = x_4(0) = x_5(0) = 1$$

Reference:

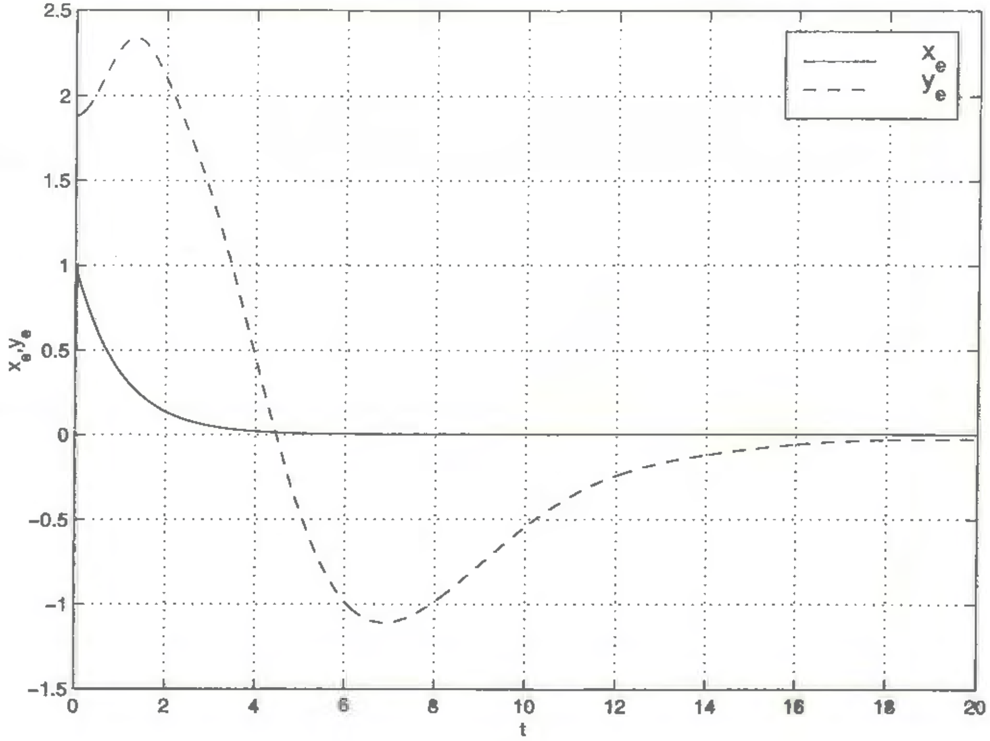
$$x_{1,r}(0) = x_{2,r}(0) = x_{3,r}(0) = x_{4,r}(0) = x_{5,r}(0) = 0$$
$$u_{1,r} = u_{2,r} = 1$$

Controller-gains

$$c_1 = 1, c_2 = 4, c_3 = 6, c_4 = 4, c_5 = 1$$

We looked at the errors $x_e = x_c - x_{c,r}$ and $y_e = y_c - y_{c,r}$.

State-feedback



Output-feedback

Initial condition:

$$x_1(0) = x_2(0) = x_3(0) = x_4(0) = x_5(0) = 1$$
$$\hat{x}_{1,e}(0) = \hat{x}_{2,e}(0) = \hat{x}_{3,e}(0) = \hat{x}_{4,e}(0) = \hat{x}_{5,e}(0) = -1$$

Reference:

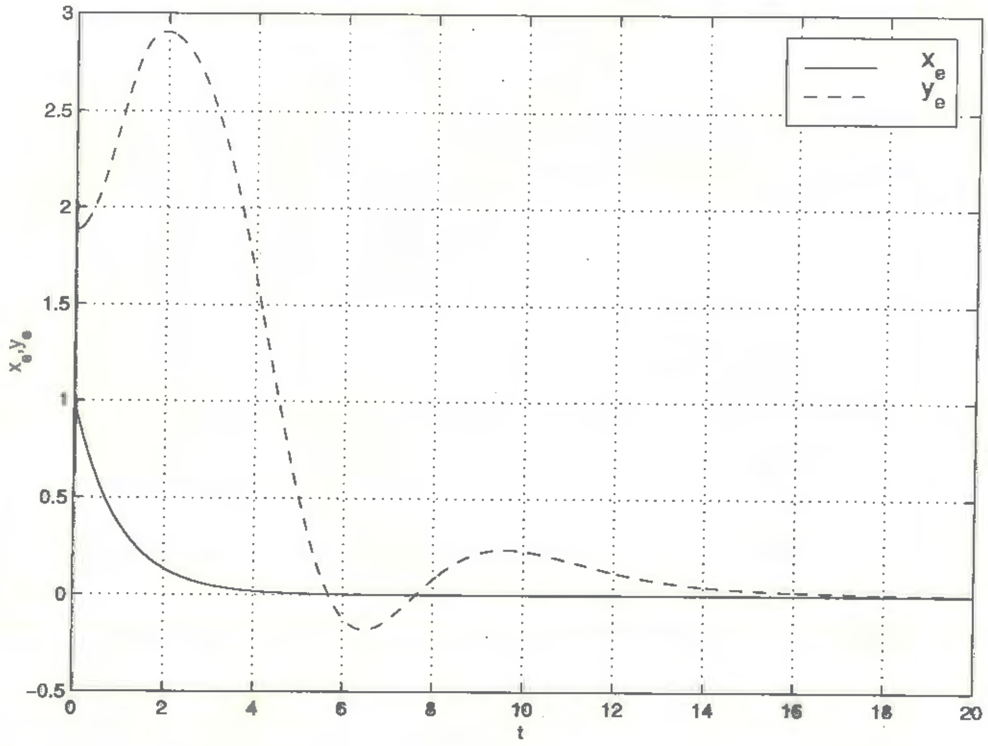
$$x_{1,r}(0) = x_{2,r}(0) = x_{3,r}(0) = x_{4,r}(0) = x_{5,r}(0) = 0$$
$$u_{1,r} = u_{2,r} = 1$$

Controller and observer gains

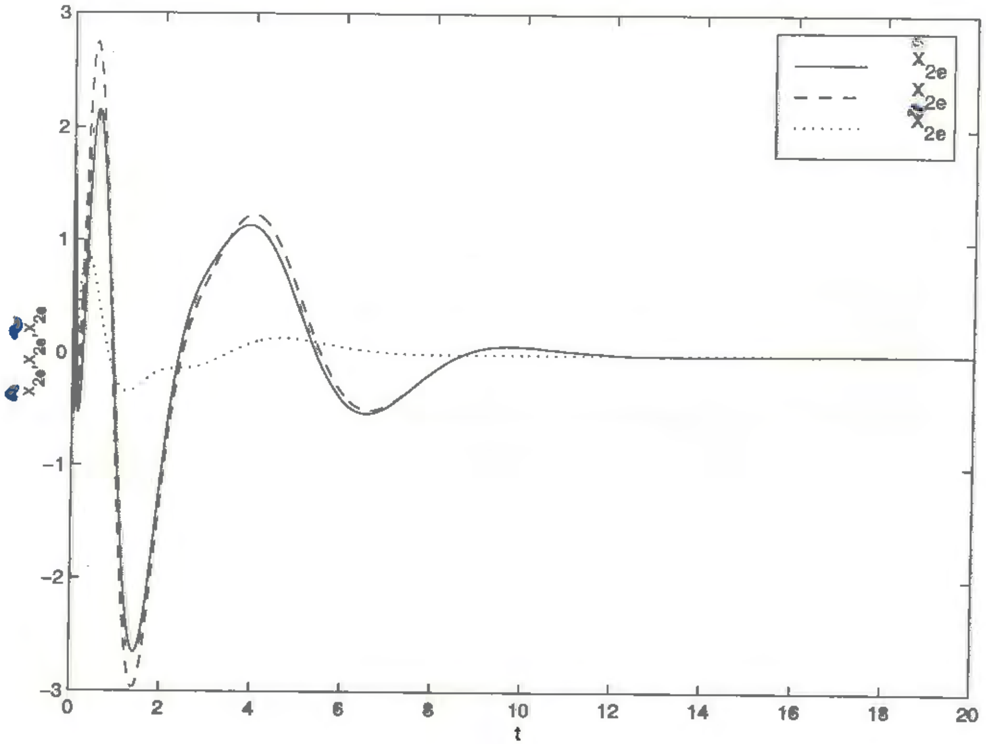
$$c_1 = 1, c_2 = 4, c_3 = 6, c_4 = 4, c_5 = 1, l_2 = 8, l_3 = 24, l_4 = 32, l_5 = 16$$

We looked at the errors $x_e = x_c - x_{c,r}$ and $y_e = y_c - y_{c,r}$.

Output-feedback



Output-feedback



Conclusions

- Linear (time-varying) controllers for nonlinear chained form system.
- Separate controller design by viewing the system as a cascade.
- Both state and output feedback.
- Global results (not based on linearization)
- Similar approach can be used with saturated control inputs.