Output Feedback Tracking of Nonholonomic Systems in Chained Form

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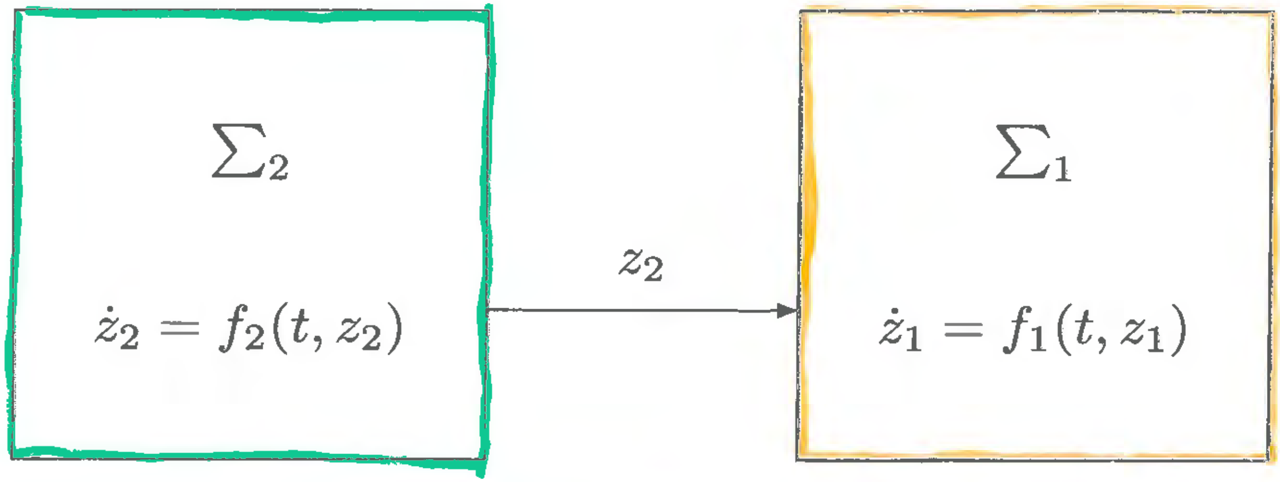
Outline

- Model and Problem formulation
- Cascaded systems
- Derivation of linear tracking controller
- Simulations
- Conclusions

Cascaded systems



$$
\begin{aligned}
\dot{z}_1 &= f_1(t, z_1) + g(t, z_1, z_2)\boxed{z_2} \\
\dot{z}_2 &= f_2(t, z_2)
\end{aligned}
$$

Consider the system

$$
\begin{aligned}
\dot{x}_1 &= -x_1 + x_1^2 \boxed{x_2} \\
\dot{x}_2 &= -\gamma x_2
\end{aligned}
$$

Solution:

$$
\begin{aligned}
x_1(t) &= \frac{2x_1(0)}{x_1(0)x_2(0)e^{-t\gamma} + [2 - x_1(0)x_2(0)]e^{t\gamma}} \\
x_2(t) &= x_2(0)e^{-\gamma t}
\end{aligned}
$$

When $x_1(0)x_2(0) > 2$ we have a finite escape time

$$
t_{esc} = \frac{1}{2\gamma} \ln \frac{x_1(0)x_2(0)}{x_1(0)x_2(0) - 2}
$$

Conditions

E. Panteley en A. Loría (S&CL 33(2), 1998):

Cascade Globally Uniformly Asymptotically Stable (GUAS) when

- Σ_1 GUAS, polynomial 'Lyapunov function'
- $g(t, z_1, z_2)$ at most linear in z_1
- Σ_2 GUAS, $z_2(t)$ integrable

Tracking Problem (state-feedback)

System dynamics:

$$\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= x_2 u_1 \\
&\vdots \\
\dot{x}_n &= x_{n-1} u_1
\end{aligned}$$

Reference dynamics:

$$\begin{aligned}
\dot{x}_{1,r} &= u_{1,r} \\
\dot{x}_{2,r} &= u_{2,r} \\
\dot{x}_{3,r} &= x_{2,r} u_{1,r} \\
&\vdots \\
\dot{x}_{n,r} &= x_{n-1,r} u_{1,r}
\end{aligned}$$

Find control laws

$$u \equiv u(x, x_r, u_r)$$

that yield

$$\lim_{t \to \infty} |x(t) - x_r(t)| = 0$$

Controller Design

Error dynamics ($x_e = x - x_r$):

$$
\begin{aligned}
x_{1,e} &= u_1 - u_{1,r} \\
x_{2,e} &= u_2 - u_{2,r} \\
x_{3,e} &= x_2 u_1 - x_{2,r} u_{1,r} \qquad\qquad + x_2 u_{1,r} - x_2 u_{1,r} \\
&\quad = u_{1,r} x_{2,e} + x_2 (u_1 - u_{1,r}) \\
&\vdots \\
x_{n,e} &= x_{n-1} u_1 - x_{n-1,r} u_{1,r} \qquad\qquad + x_{n-1} u_{1,r} - x_{n-1} u_{1,r} \\
&\quad = u_{1,r} x_{n-1,e} + x_{n-1} (u_1 - u_{1,r})
\end{aligned}
$$

Controller Design

Error dynamics ($x_e = x - x_r$):

$$
\begin{aligned}
\dot{x}_{2,e} &= 0 + (u_2 - u_{2,r}) &&+ 0 \\
\dot{x}_{3,e} &= u_{1,r} x_{2,e} &&+ x_2(u_1 - u_{1,r}) \\
&\vdots &&\vdots \\
\dot{x}_{n,e} &= u_{1,r} x_{n-1,e} &&+ x_{n-1}(u_1 - u_{1,r})
\end{aligned}
$$

$$
\dot{x}_{1,e} = (u_1 - u_{1,r})
$$

Problem has reduced to (seperately) stabilizing the error-dynamics

$$\begin{bmatrix} \dot{x}_{2,e} \\ \dot{x}_{3,e} \\ \dot{x}_{4,e} \\ \vdots \\ \dot{x}_{n,e} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ u_{1,r} & \ddots & & & \vdots \\ 0 & u_{1,r} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & u_{1,r} & 0 \end{bmatrix} \begin{bmatrix} x_{2,e} \\ x_{3,e} \\ x_{4,e} \\ \vdots \\ x_{n,e} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} (u_2 - u_{2,r})$$

and

$$\dot{x}_{1,e} = (u_1 - u_{1,r})$$

Stabilizing the $x_{1,e}$ dynamics:

$$u_1 = u_{1,r} - c_1 x_{1,e} \qquad c_1 > 0$$

For stabilizing the $[x_{2,e}, \ldots, x_{n,e}]^T$ dynamics we need *uniform controllability.*

This turns out to be the case if and only if the vector

$$\begin{bmatrix} 1 \\ \int_{t_0}^{t} u_{1,r}(\sigma) d\sigma \\ \vdots \\ \left(\int_{t_0}^{t} u_{1,r}(\sigma) d\sigma\right)^{n-1} \end{bmatrix}$$

or simpler:
$\exists \varepsilon > 0, \exists \delta > 0$ s.t.
$\forall t \geq 0 \quad \exists s \in [t, t+\delta]$
s.t. $u_{1,r}(s) \geq \varepsilon$

is persistently exciting.

We can use several controllers $u_2 = u_{2,r} - K(t)[x_{2,e}, \ldots, x_{n,e}]^T$ for globally exponentially stabilizing

$$\begin{bmatrix} \dot{x}_{2,e} \\ \dot{x}_{3,e} \\ \dot{x}_{4,e} \\ \vdots \\ \dot{x}_{n,e} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ u_{1,r} & \ddots & & & \vdots \\ 0 & u_{1,r} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & u_{1,r} & 0 \end{bmatrix} \begin{bmatrix} x_{2,e} \\ x_{3,e} \\ x_{4,e} \\ \vdots \\ x_{n,e} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} (u_2 - u_{2,r})$$

- $K(t) = -B^T \left[\int_t^{t+\delta} 2e^{4\alpha(t-\sigma)}\Phi(t,\sigma)BB^T\Phi^T(t,\sigma)d\sigma\right]^{-1}$
- Pole-placement (cf. Valasek & Olgac, needs $\dot{u}_{1,r}, \ldots, \dot{u}_{1,r}^{(n-2)}$)
- $K(t) = \gamma B^T \Phi^{-T}(t,t_0)\Phi^{-1}(t,t_0)$ (M.-S. Chen)

Note that we need $\Phi(t, t_0)$:

$$\Phi(t,t_0) = \begin{bmatrix} f_0(t,t_0) & 0 & \cdots & 0 \\ f_1(t,t_0) & f_0(t,t_0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ f_{n-1}(t,t_0) & \cdots & f_1(t,t_0) & f_0(t,t_0) \end{bmatrix}$$

where

$$f_k(t,t_0) = \frac{1}{k!}\left[\int_{t_0}^{t} u_{1,r}(\sigma)d\sigma\right]^k = \frac{1}{k!}\left[x_{1,r}(t) - x_{1,r}(t_0)\right]^k$$

However, a simpler controller is given by

- $K(t) = [c_2,\ c_3 u_{1,r}(t),\ c_4,\ c_5 u_{1,r}(t),\ \ldots]$ where c_i are such that $\lambda^{n-1} + c_2\lambda^{n-2} + \ldots + c_{n-1}\lambda + c_n$ is Hurwitz.

To summarize

The error dynamics

$$\begin{aligned}
\dot{x}_{1,e} &= u_1 - u_{1,r} \\
\dot{x}_{2,e} &= u_2 - u_{2,r} \\
\dot{x}_{3,e} &= x_2 u_1 - x_{2,r} u_{1,r} \\
&\vdots \\
\dot{x}_{n,e} &= x_{n-1} u_1 - x_{n-1,r} u_{1,r}
\end{aligned}$$

in closed loop with the controller

$$\begin{aligned}
u_1 &= u_{1,r} - c_1 x_{1,e} \\
u_2 &= u_{2,r} - K(t)[x_{2,e}, \ldots, x_{n,e}]^T
\end{aligned}$$

is globally (K-exponentially) asymptotically stable.

Tracking Problem (output feedback)

System dynamics:

$$\begin{aligned}\dot{x}_1 &= u_1\\ \dot{x}_2 &= u_2\\ \dot{x}_3 &= x_2 u_1\\ &\vdots\\ \dot{x}_n &= x_{n-1} u_1\end{aligned}$$

Reference dynamics:

$$\begin{aligned}\dot{x}_{1,r} &= u_{1,r}\\ \dot{x}_{2,r} &= u_{2,r}\\ \dot{x}_{3,r} &= x_{2,r} u_{1,r}\\ &\vdots\\ \dot{x}_{n,r} &= x_{n-1,r} u_{1,r}\end{aligned}$$

Output: $y = [x_1, x_n]^T$

Find control laws

$$u \equiv u(\hat{x}, x_r, u_r) \qquad \dot{\hat{x}} = f(\hat{x}, y, x_r, u_r)$$

that yield $\lim_{t\to\infty} |x(t) - x_r(t)| = 0$

Using again $u_1 = u_{1,r} - c_1 x_{1,e}$ and the cascaded result, we need to stabilize

$$\begin{bmatrix} \dot{x}_{2,e} \\ \dot{x}_{3,e} \\ \dot{x}_{4,e} \\ \vdots \\ \dot{x}_{n,e} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ u_{1,r} & \ddots & & & \vdots \\ 0 & u_{1,r} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & u_{1,r} & 0 \end{bmatrix} \begin{bmatrix} x_{2,e} \\ x_{3,e} \\ x_{4,e} \\ \vdots \\ x_{n,e} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} (u_2 - u_{2,r})$$

$$y = x_{n,e}$$

by means of output feedback.

Using the *separation principle* and *certainty equivalence* we find:

$$
\begin{aligned}
u_2 &= u_{2,r} - K(t)[\hat{x}_{2,e}, \ldots, \hat{x}_{n,e}]^T \\
\begin{bmatrix} \dot{\hat{x}}_{2,e} \\ \dot{\hat{x}}_{3,e} \\ \dot{\hat{x}}_{4,e} \\ \vdots \\ \dot{\hat{x}}_{n,e} \end{bmatrix} &= \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ u_{1,r} & \ddots & & & \vdots \\ 0 & u_{1,r} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & u_{1,r} & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_{2,e} \\ \hat{x}_{3,e} \\ \hat{x}_{4,e} \\ \vdots \\ \hat{x}_{n,e} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} (u_2 - u_{2,r}) + \\
&+ \; H(t)(x_{n,e} - \hat{x}_{n,e})
\end{aligned}
$$

where we can determine $H(t)$ in several ways.

Result

Error dynamics

$$\begin{aligned}
\dot{x}_{1,e} &= u_1 - u_{1,r} \\
\dot{x}_{2,e} &= u_2 - u_{2,r} \\
\dot{x}_{3,e} &= x_2 u_1 - x_{2,r} u_{1,r} \\
&\vdots \\
\dot{x}_{n,e} &= x_{n-1} u_1 - x_{n-1,r} u_{1,r} \\
y &= [x_{1,e}, \ \ x_{n,e}]^T
\end{aligned}$$

Observer

$$\begin{aligned}
\dot{\hat{x}}_{2,e} &= u_2 - u_{2,r} + h_n(t)\tilde{x}_{n,e} \\
\dot{\hat{x}}_{3,e} &= u_{1,r}\hat{x}_{2,e} + h_{n-1}(t)\tilde{x}_{n,e} \\
&\vdots \\
\dot{\hat{x}}_{n-1,e} &= u_{1,r}\hat{x}_{n-2,e} + h_3(t)\tilde{x}_{n,e} \\
\dot{\hat{x}}_{n,e} &= u_{1,r}\hat{x}_{n-1,e} + h_2(t)\tilde{x}_{n,e} \\
\tilde{x}_{n,e} &= x_{n,e} - \hat{x}_{n,e}
\end{aligned}$$

in closed loop with the controller

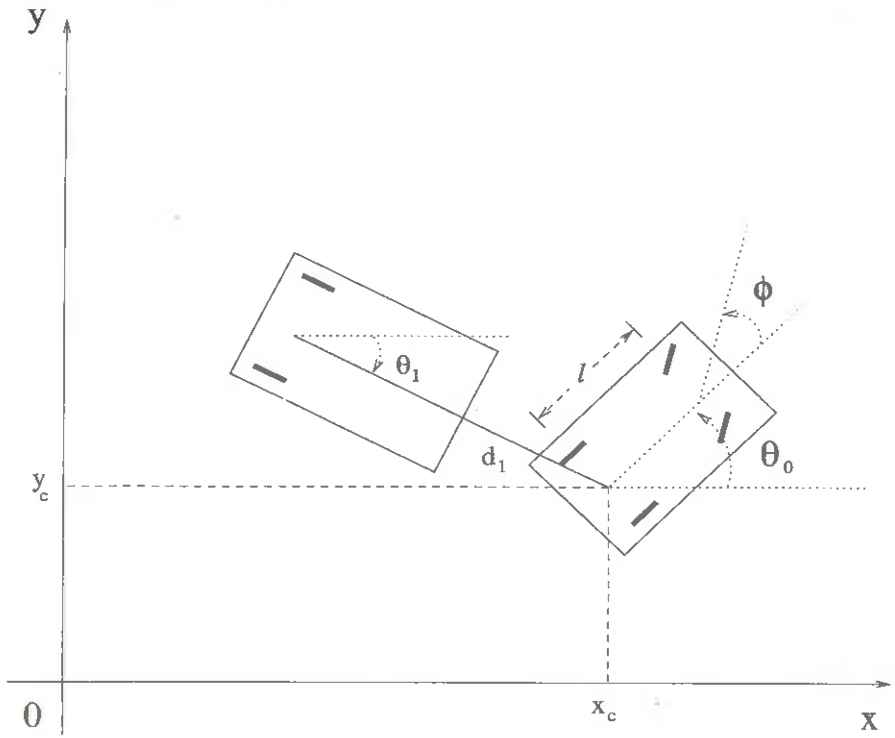
$$\begin{aligned}
u_1 &= u_{1,r} - c_1 x_{1,e} \\
u_2 &= u_{2,r} - K(t)[\hat{x}_{2,e}, \ldots, \hat{x}_{n,e}]^T
\end{aligned}$$

is globally (K-exponentially) asymptotically stable.

Simulations

Car pulling a single trailer:

$$
\begin{aligned}
\dot{x}_c &= v\cos\theta_0 \\
\dot{y}_c &= v\sin\theta_0 \\
\dot{\phi} &= \omega \\
\dot{\theta}_0 &= \frac{1}{l} v\tan\phi \\
\dot{\theta}_1 &= \frac{1}{d_1} v\sin(\theta_0 - \theta_1)
\end{aligned}
$$

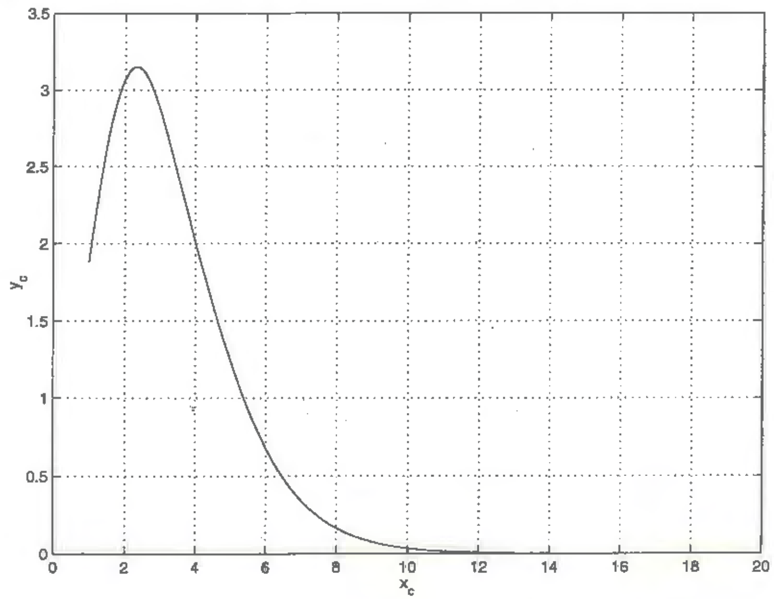


State-feedback

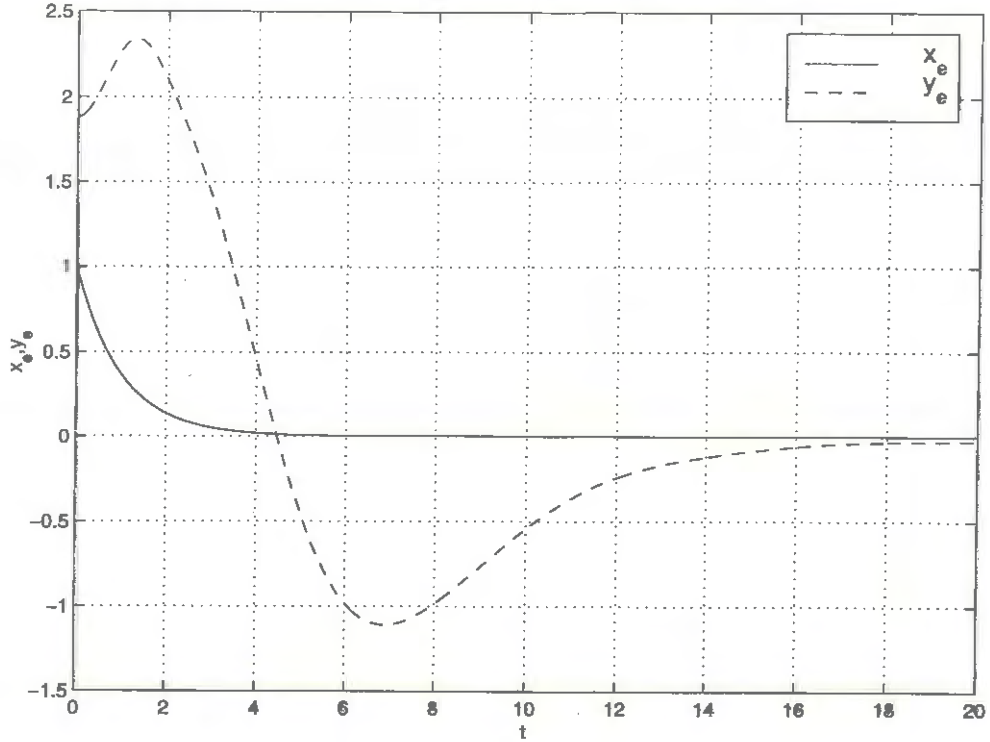
Initial condition: $x_e = [1, 1, 1, 1, 1]$

Reference: Straight line along x_c=axis.

Controller-gains: poles in -1 and $-1, -1, -2, -2$.



State-feedback

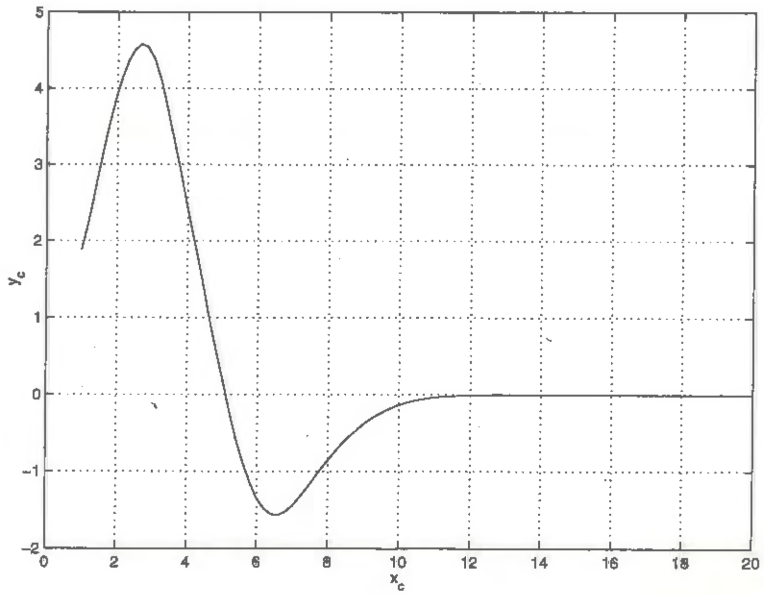


Output-feedback

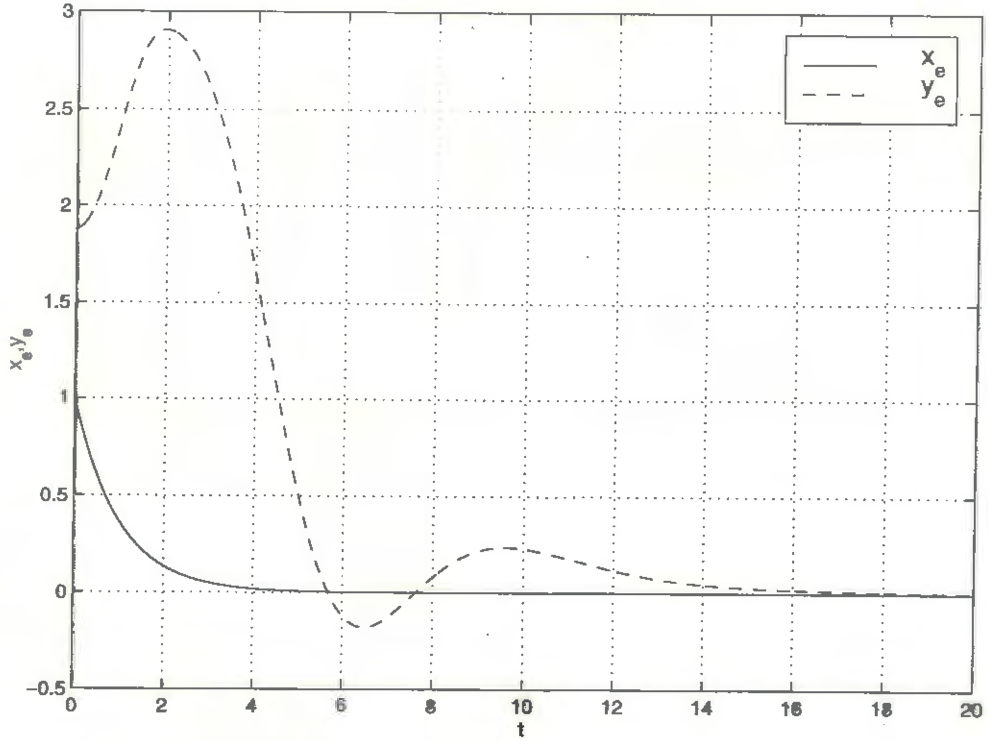
Similar simulation.

Initial observer-errors: $[\hat{x}_{2,e}, \ldots, \hat{x}_{5,e}] = [-1, -1, -1, -1]$.

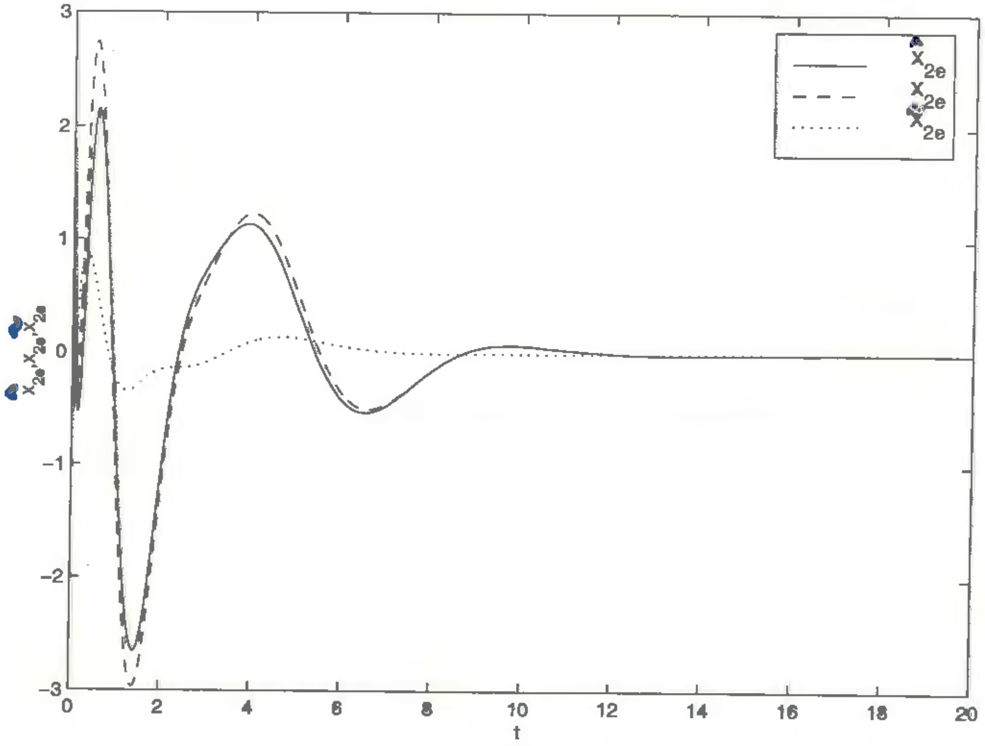
Observer-gains: poles in $-3, -3, -4, -4$.



Output-feedback



Output-feedback



Conclusions

- Linear (time-varying) controllers for nonlinear chained form system.
- Separate controller design by viewing the system as a cascade.
- Both state and output feedback.
- Global results (not based on linearization)
- Similar approach can be used with saturated control inputs.