Controller design for flow networks of switched servers with setup times

Erjen Lefeber

35^{th} Benelux Meeting on Systems and Control

Technische Universiteit
Eindhoven
University of Technology

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Where innovation starts

Switching servers with setup times

Switching servers with setup times

Problem

How to control these networks?

Decisions: When to switch, and to which job-type

Goals: Minimal number of jobs, minimal flow time/delay

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Switching servers with setup times

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Switching servers with setup times

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Problem: Even with sufficient capacity, system might become unstable (e.g., re-entrant systems: Kumar-Seidman in IEEE Trans.Autom.Contr.'90)

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Switching servers with setup times

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Problem: Even with sufficient capacity, system might become unstable (e.g., re-entrant systems: Kumar-Seidman in IEEE Trans.Autom.Contr.'90)

Current status (after 25 years)

Several policies exist that guarantee stability of the network

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Different approach

Notions from control theory

1. Generate feasible reference trajectory
2. Design (static) state feedback controller
3. Design observer
4. Design (dynamic) output feedback controller

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Parallels with this problem

1. Determine desired system behavior
2. Derive non-distributed/centralized controller
3. Can state be reconstructed?
4. Derive distributed/decentralized controller

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Acknowledgements

Research

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Problem 1: Optimal periodic behavior

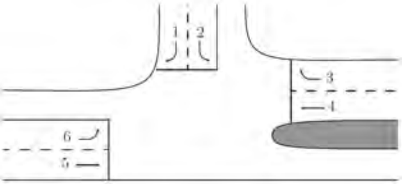
- Stijn Fleuren (PhD student at TU/e)

Problem 2: Feedback control

- Varvara Feoktistova (St. Petersburg University)
- Alexey Matveev (St. Petersburg University)

Problem 1: Optimal periodic behavior

Intersection



Problem 1: Optimal periodic behavior

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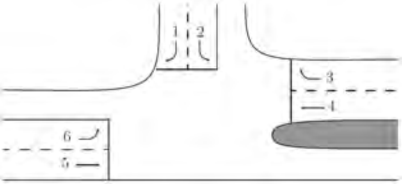
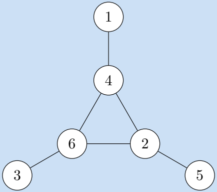


Conflict graph:



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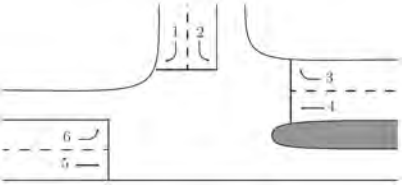

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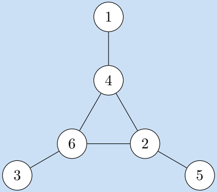
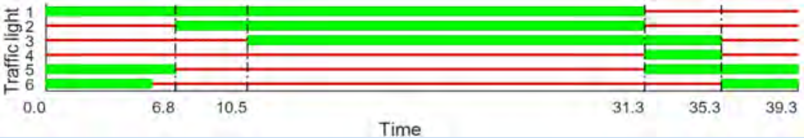


Optimal schedule (data from Grontmij: A2/N279)



Problem 1: Optimal periodic behavior

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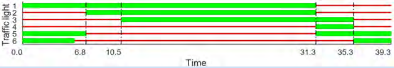
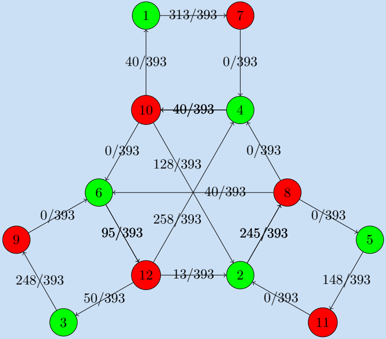


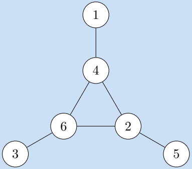
Extended graph



Event times

i	$t(i)$	$t(i+6)$	i+6
1	0.0	31.3	7
2	6.8	31.3	8
3	10.5	35.3	9
4	31.3	35.3	10
5	31.3	6.8	11
6	35.3	5.5	12

Conflict graph



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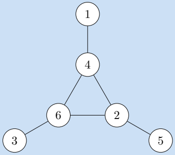
Problem 1: Optimal periodic behavior

Data

- Arrival rates: λ_i
- Service rates: μ_i
- Clearance times: $\sigma_{i,j}$
- Minimal/maximal green time: $g_i^{\min}$, $g_i^{\max}$.
- Minimal/maximal period: $T^{\min}$, $T^{\max}$.
- Conflict graph:



Problem 1: Optimal periodic behavior

Design variables

- $x(i,j)$ fraction of period from event i to event j.
- T' reciprocal of duration of period, i.e. $T' = 1/T$.

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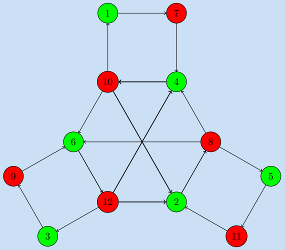
- Conflict: $x(i, i+n) + x(i+n, j) + x(j, j+n) + x(j+n, i) = 1$
- Integer cycle: $\sum_{(i,j)\in C^+} x(i,j) - \sum_{(i,j)\in C^+} x(i,j) = z_C$.

Problem 1: Optimal periodic behavior

Extended graph



Cycle

Cycle: $\{(4, 10), (10, 2), (12, 2), (12, 4)\}$

Problem 1: Optimal periodic behavior

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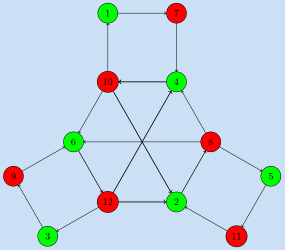

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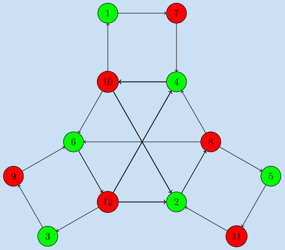

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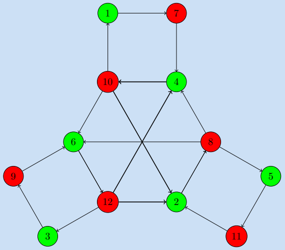

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Only for cycles from integer cycle base.

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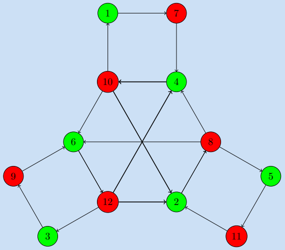

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Objective

Minimize weighted average delay (Fluid, Webster, Miller, v.d. Broek):

$$\sum_{i=1}^{n} \frac{r_i}{2\lambda_i(1-\rho_i)T}\left(r_i\lambda_i + \frac{s_i^2}{1-\rho_i} + \frac{r_i\rho_i^2 s_i^2 T^2}{(1-\rho_i)(T-r_i)^2((1-\rho_i)T-r_i)}\right)$$

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Concluding remarks for Problem 1

- Mixed integer convex optimization problem.

Problem 1: Optimal periodic behavior

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- Mixed integer convex optimization problem.
- Data from real intersection in the Netherlands with 29 directions (data from Grontmij):
  - Straight forward implementation (solver: SCIP 3.2.0):
  - Notebook: Intel i5-4300U CPU 1.90GHZ with 16.0GB of RAM.

Problem 1: Optimal periodic behavior

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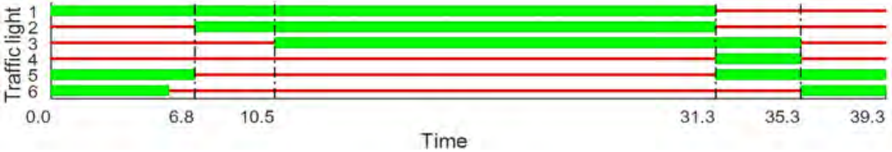
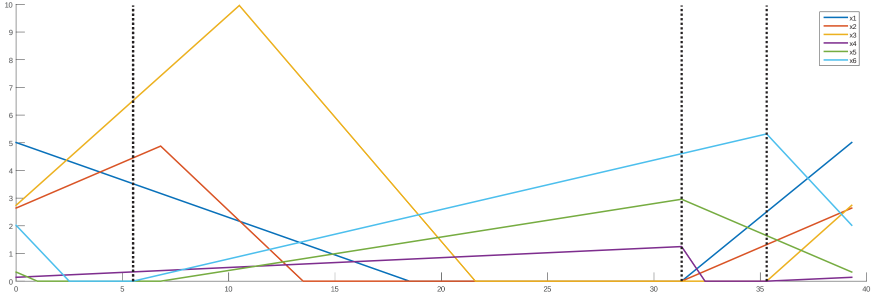
  - Standard group based approach 33.83 seconds
  - Our approach 2.27 seconds
- Can also quickly solve problem with integer durations

Problem 2: Feedback control

Consider the following periodic schedule:

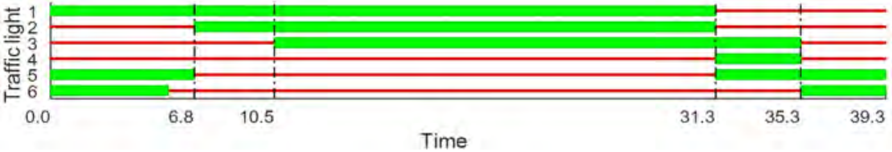


Resulting steady state periodic wip evolution:



Problem 2: Feedback control

Steady state periodic wip evolution:



- Mode 1: directions 1, 2 and 3 served (steady state: 5.5 – 31.3)
- Mode 2: directions 3, 4 and 5 served (steady state: 31.3 – 35.3)
- Mode 3: directions 5, 6 and 1 served (steady state: 35.3 – 5.5)

NB: In mode 1: directions 2 and 3 are served after setup,
and 5 is still served for the first 6.8-5.5=1.3 seconds.
In mode 2: direction 1 is served after setup.

Problem 2: Feedback control

Useful result by Feoktistova, Matveev, Lefeber, Rooda (2012)

Let $\mathcal{T}$ be an operator which:

- is piecewise affine, i.e. $\mathcal{T}x = A_i x + b_i$ for $x \in \{P_i x \leq q_i\}$,

then

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- is monotone, i.e. $A_i \geq 0$,

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- has a fixed point, i.e. there exists x^* such that $x^* = \mathcal{T}x^*$,

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- the fixed point is unique, and

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then

- the fixed point is unique, and
- attracts all solutions of $x_{k+1} = \mathcal{T}x_k$; $x_0 \in \mathbb{R}^n_+$, i.e. $\lim_{k\to\infty} x_k = x^*$.

Problem 2: Feedback control

Useful Lemma's

Composition: $\mathcal{T}_2 \circ \mathcal{T}_1 : A_2(A_1 x + b_1) + b_2 = \underbrace{A_2 A_1}_{A} x + \underbrace{A_2 b_1 + b_2}_{b}.$

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Consequence

If $\mathcal{T}_1, \mathcal{T}_2, \ldots \mathcal{T}_n$ are piecewise affine continuous monotone dominated, then $M = \mathcal{T}_n \circ \cdots \circ \mathcal{T}_2 \circ \mathcal{T}_1$ is piecewise affine continuous monotone dominated.

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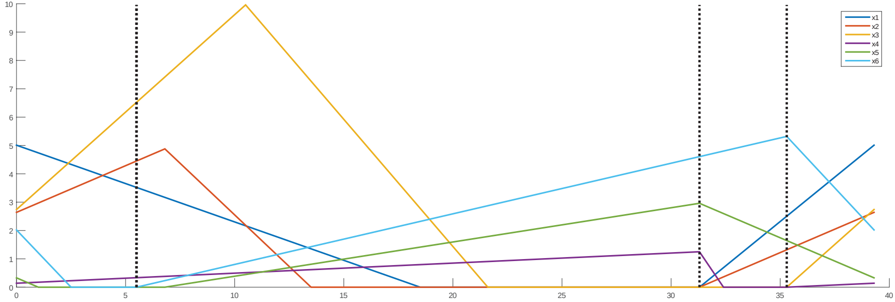
Only need to show that M is strictly dominated and has a fixed point.

Problem 2: Feedback control

Steady state periodic wip evolution:



- Define δ_i as duration of $x_i^* = 0$ for $i = 1, 2, 4, 6$,
- Define $\delta_3 + 4$ as duration of $x_3^* = 0$,
- Define $\delta_5 + 1.3$ as duration of $x_5^* = 0$,
- Define $\theta_1 = x_1^*(5.5)/[x_1^*(35.3) + 4\lambda_1]$,
- Define $\theta_5 = x_5^*(31.3)/x_5^*(35.3)$.

Problem 2: Feedback control (mode 1)

State at start of mode 1: $X = [x_4, x_3, x_2, x_1]^T(t_0)$.
State at end of mode 1: $\mathcal{T}_1 X = [x_6, x_5, x_4]^T(t_1)$.
Stay in mode until $x_1(t_1 - \delta_1) = 0$, $x_2(t_1 - \delta_2) = 0$ and $x_3(t_1 - \delta_3) = 0$.

$$\mathcal{T}_1 X = \begin{cases} \begin{bmatrix} 0 & 0 & 0 & \frac{\lambda_6}{\mu_1 - \lambda_1} \\ 0 & 0 & 0 & \frac{\lambda_5}{\mu_1 - \lambda_1} \\ 1 & 0 & 0 & \frac{\lambda_4}{\mu_1 - \lambda_1} \end{bmatrix} X + \begin{bmatrix} \lambda_6 \delta_1 \\ \lambda_5 \delta_1 \\ \lambda_4 \delta_1 \end{bmatrix} & \text{if } \frac{x_1(t_0)}{\mu_1 - \lambda_1} + \delta_1 \geq \frac{x_2(t_0)}{\mu_2 - \lambda_2} + \delta_2 \vee \frac{x_3(t_0)}{\mu_3 - \lambda_3} + \delta_3 \\ \begin{bmatrix} 0 & 0 & \frac{\lambda_6}{\mu_2 - \lambda_2} & 0 \\ 0 & 0 & \frac{\lambda_5}{\mu_2 - \lambda_2} & 0 \\ 1 & 0 & \frac{\lambda_4}{\mu_2 - \lambda_2} & 0 \end{bmatrix} X + \begin{bmatrix} \lambda_6 \delta_2 \\ \lambda_5 \delta_2 \\ \lambda_4 \delta_2 \end{bmatrix} & \text{if } \frac{x_2(t_0)}{\mu_2 - \lambda_2} + \delta_2 \geq \frac{x_1(t_0)}{\mu_1 - \lambda_1} + \delta_1 \vee \frac{x_3(t_0)}{\mu_3 - \lambda_3} + \delta_3 \\ \begin{bmatrix} 0 & \frac{\lambda_6}{\mu_3 - \lambda_3} & 0 & 0 \\ 0 & \frac{\lambda_5}{\mu_3 - \lambda_3} & 0 & 0 \\ 1 & \frac{\lambda_4}{\mu_3 - \lambda_3} & 0 & 0 \end{bmatrix} X + \begin{bmatrix} \lambda_6 \delta_3 \\ \lambda_5 \delta_3 \\ \lambda_4 \delta_3 \end{bmatrix} & \text{if } \frac{x_3(t_0)}{\mu_3 - \lambda_3} + \delta_3 \geq \frac{x_1(t_0)}{\mu_1 - \lambda_1} + \delta_1 \vee \frac{x_2(t_0)}{\mu_2 - \lambda_2} + \delta_2 \end{cases}$$

Problem 2: Feedback control (mode 2)

State at start of mode 2: $X = [x_6, x_5, x_4]^T(t_0)$.
State at end of mode 2: $\mathcal{T}_2 X = [x_2, x_1, x_6, x_5]^T(t_1)$.
Stay in mode until $x_4(t_1 - \delta_4) = 0$ and $x_5(t_1) = \theta_5 x_5(t_0)$.
NB: Serve x_5 at arrival rate as soon as $x_5 = \theta_5 x_5(t_0)$.

$$\mathcal{T}_2 X = \begin{cases} \begin{bmatrix} 0 & 0 & \frac{\lambda_2}{\mu_4-\lambda_4} \\ 0 & 0 & \frac{\lambda_1}{\mu_4-\lambda_4} \\ 1 & 0 & \frac{\lambda_6}{\mu_4-\lambda_4} \\ 0 & \theta_5 & 0 \end{bmatrix} X + \begin{bmatrix} \lambda_2\delta_4 \\ \lambda_1\delta_4 \\ \lambda_6\delta_4 \\ 0 \end{bmatrix} & \text{if } \frac{x_4(t_0)}{\mu_4-\lambda_4} + \delta_4 \geq \frac{(1-\theta_5)x_5(t_0)}{\mu_5-\lambda_5} \\ \begin{bmatrix} 0 & \frac{\lambda_2(1-\theta_5)}{\mu_5-\lambda_5} & 0 \\ 0 & \frac{\lambda_2(1-\theta_5)}{\mu_5-\lambda_5} & 0 \\ 1 & \frac{\lambda_2(1-\theta_5)}{\mu_5-\lambda_5} & 0 \\ 0 & \theta_5 & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \text{if } \frac{(1-\theta_5)x_5(t_0)}{\mu_5-\lambda_5} \geq \frac{x_4(t_0)}{\mu_4-\lambda_4} + \delta_4 \end{cases}$$

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Problem 2: Feedback control (mode 3)

State at start of mode 3: $X = [x_2, x_1, x_6, x_5]^T(t_0)$.
State at end of mode 3: $\mathcal{T}_3 X = [x_4, x_3, x_2, x_1]^T(t_1)$.
End of mode: $x_5(t_1-\delta_5)=0$, $x_6(t_1-\delta_6)=0$ and $x_1(t_1)=\theta_1[x_1(t_0)+4\lambda_1]$.
NB: Serve x_1 at arrival rate as soon as $x_1 = \theta_1[x_1(t_0) + 4\lambda_1]$.
Then we get for $\mathcal{T}_3 X$:

$$
\begin{cases}
\begin{bmatrix} 0 & 0 & \frac{\lambda_4}{\mu_5-\lambda_5} & 0 \\ 0 & 0 & \frac{\lambda_3}{\mu_5-\lambda_5} & 0 \\ 1 & 0 & \frac{\lambda_2}{\mu_5-\lambda_5} & 0 \\ 0 & \theta_1 & 0 & 0 \end{bmatrix} X + \begin{bmatrix} \lambda_4\delta_5 \\ \lambda_3\delta_5 \\ \lambda_2\delta_5 \\ 4\theta_1\lambda_1 \end{bmatrix} & \text{if } \frac{x_5(t_0)}{\mu_5-\lambda_5}+\delta_5 \geq \frac{x_6(t_0)}{\mu_6-\lambda_6}+\delta_6 \vee \frac{(1-\theta_1)[x_1(t_0)+4\lambda_1]}{\mu_1-\lambda_1} \\
\begin{bmatrix} 0 & 0 & 0 & \frac{\lambda_4}{\mu_6-\lambda_6} \\ 0 & 0 & 0 & \frac{\lambda_3}{\mu_6-\lambda_6} \\ 1 & 0 & 0 & \frac{\lambda_2}{\mu_6-\lambda_6} \\ 0 & \theta_1 & 0 & 0 \end{bmatrix} X + \begin{bmatrix} \lambda_4\delta_6 \\ \lambda_3\delta_6 \\ \lambda_2\delta_6 \\ 4\theta_1\lambda_1 \end{bmatrix} & \text{if } \frac{x_6(t_0)}{\mu_6-\lambda_6}+\delta_6 \geq \frac{x_5(t_0)}{\mu_5-\lambda_5}+\delta_5 \vee \frac{(1-\theta_1)[x_1(t_0)+4\lambda_1]}{\mu_1-\lambda_1} \\
\begin{bmatrix} 0 & \frac{\lambda_4(1-\theta_1)}{\mu_1-\lambda_1} & 0 & 0 \\ 0 & \frac{\lambda_3(1-\theta_1)}{\mu_1-\lambda_1} & 0 & 0 \\ 1 & \frac{\lambda_2(1-\theta_1)}{\mu_1-\lambda_1} & 0 & 0 \\ 0 & \theta_1 & 0 & 0 \end{bmatrix} X + \begin{bmatrix} 4(1-\theta_1)\lambda_4 \\ 4(1-\theta_1)\lambda_3 \\ 4(1-\theta_1)\lambda_2 \\ 4(1-\theta_1)\lambda_1 \end{bmatrix} & \text{if } \frac{(1-\theta_1)[x_1(t_0)+4\lambda_1]}{\mu_1-\lambda_1} \geq \frac{x_6(t_0)}{\mu_6-\lambda_6}+\delta_6 \vee \frac{x_5(t_0)}{\mu_5-\lambda_5}+\delta_5
\end{cases}
$$

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Problem 2: Feedback control

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Since $\mathcal{T}_3$ is strictly dominated, we have $M = \mathcal{T}_3 \circ \mathcal{T}_2 \circ \mathcal{T}_1$ is strictly dominated.

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Even more

Under conditions such as (show only 2 of 18 expressions):

$$(1-\rho_1)(1-\rho_3)(1-\rho_5) > \rho_1\rho_3\rho_5(1-\theta_1)(1-\theta_5)$$
$$(1-\rho_1)(1-\rho_2)(1-\rho_5) > \rho_2\rho_5(1-\theta_5)(1-\rho_1\theta_1)$$

we can show that a fixed point for M exists.

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we can show that a fixed point for M exists.

This guarantees robustness against changes in parameters.

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Problem 2: Feedback control

Practical problem

End of mode 1: $x_1(t_1 - \delta_1) = 0$, $x_2(t_1 - \delta_2) = 0$ and $x_3(t_1 - \delta_3) = 0$.

End of mode 2: $x_4(t_1 - \delta_4) = 0$ and $x_5(t_1) = \theta_5 x_5(t_0)$.
NB: Serve x_5 at arrival rate as soon as $x_5 = \theta_5 x_5(t_0)$.

End of mode 3: $x_5(t_1 - \delta_5) = 0$, $x_6(t_1 - \delta_6) = 0$ and $x_1(t_1) = \theta_1[x_1(t_0) + 4\lambda_1]$.
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NB: Serve x_1 at arrival rate as soon as $x_1 = \theta_1[x_1(t_0) + 4\lambda_1]$.

Solution

End of mode 1: Minimal duration: τ_1^*. Furthermore, $x_1(t_1 - \delta_1) = 0$, $x_2(t_1 - \delta_2) = 0$ and $x_3(t_1 - \delta_3) = 0$.

End of mode 2: $x_4(t_1 - \delta_4) = 0$ and $x_5(t_1) \leq \theta_5 x_5(t_0)$. Set $\tau_3^* = \frac{\theta_5 x_5(t_0)}{\mu_5 - \lambda_5}$

End of mode 3: Minimal duration: τ_3^*. Furthermore, $x_5(t_1 - \delta_5) = 0$, $x_6(t_1 - \delta_6) = 0$, $x_1(t_1) \leq \theta_1[x_1(t_0)+4\lambda_1]$. Set $\tau_1^* = \frac{\theta_1[x_1(t_0)+4\lambda_1]}{\mu_1 - \lambda_1}$.

Conclusions

Conclusions

- Control of traffic lights can be tackled as two separate problems:
  - Determining optimal periodic behavior
  - Feedback control towards given periodic behavior
- Optimal periodic behavior can be determined
- Presented feedback controller
  - Robust against changes in arrival and service rate.
  - Can be implemented in practice.

Future work

- Properly defining modes for large intersections/arbitrary networks
- Proving methodology works for arbitrary networks

Stability constraints

Let τ_1, τ_2, τ_3 denote durations of modes.

Stability constraints

Let τ_1, τ_2, τ_3 denote durations of modes. Then states evolve as:

	Begin Mode 1	Begin Mode 2	Begin Mode 3
x_1:	$\lambda_1(\tau_2 + \tau_3) - \mu_1(\tau_3 - 4)$	0	$\lambda_1\tau_2$
x_2:	$\lambda_2(\tau_2 + \tau_3)$	0	$\lambda_2\tau_2$
x_3:	$\lambda_3\tau_3$	0	0
x_4:	$\lambda_4\tau_3$	$\lambda_4(\tau_3 + \tau_1)$	0
x_5:	0	$\lambda_5\tau_1$	$\lambda_5(\tau_1 + \tau_2) - \mu_5\tau_2$
x_6:	0	$\lambda_6\tau_1$	$\lambda_6(\tau_1 + \tau_2)$

Stability constraints

Resulting in

$$\tau_1 = \max\left[\frac{\rho_1\tau_2 + 4}{1-\rho_1} - \tau_3 + \delta_1, \frac{\rho_2(\tau_2+\tau_3)}{1-\rho_2} + \delta_2, \frac{\rho_3\tau_3}{1-\rho_3} + \delta_3\right]$$
$$\tau_2 = \max\left[\frac{\rho_4(\tau_3+\tau_1)}{1-\rho_4} + \delta_4, \frac{(1-\theta_5)\rho_5\tau_1}{1-\rho_5}\right]$$
$$\tau_3 = \max\left[\frac{\rho_5\tau_1}{1-\rho_5} - \tau_2 + \delta_5, \frac{\rho_6(\tau_1+\tau_2)}{1-\rho_6} + \delta_6, \frac{\rho_1(1-\theta_1)(\tau_2+4)}{1-\rho_1}\right]$$
$$T = \tau_1 + \tau_2 + \tau_3$$

Results in

$$\tau_1 = \frac{\rho_3}{1-\rho_3}\frac{\rho_1(1-\theta_1)}{1-\rho_1}\frac{(1-\theta_5)\rho_5}{1-\rho_5}\tau_1 + C$$

So we need

$$1 - \frac{\rho_3}{1-\rho_3}\frac{\rho_1(1-\theta_1)}{1-\rho_1}\frac{(1-\theta_5)\rho_5}{1-\rho_5} > 0$$

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Large intersection