Optimizing fixed-time control at isolated intersections

Part II: Optimizing the number of green intervals

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Abstract

This paper concerns the optimization of fixed-time schedules for isolated intersections. Such a fixed-time schedule visualizes when each of the traffic lights is green, amber and red during a repeating period. We formulate a mixed-integer programming problem that extends the optimization formulation that is proposed in (Fleuren and Lefeber 2016b). Novel to the proposed formulation is that each traffic light is allowed to have multiple green intervals; the proposed optimization problem optimizes simultaneously: the period duration, the number of green intervals of each traffic light, the durations of these green intervals, and when these green intervals start. We consider three different objective functions: minimization of the period duration of a fixed-time schedule, maximization of the capacity of the intersection, and minimization of the delay that road users experience at the intersection. With a numerical study we assess the effect of multiple green intervals. For the former two objective function, allowing traffic lights to have multiple green intervals has little to no effect on the objective functions. The computational results indicate that the delay that road users experience at the intersection can however possibly decrease substantially by allowing traffic lights to have multiple green intervals.

Keywords. fixed-time control; isolated intersection; optimization framework; cycle periodicity formulation; mixed-integer programming problem

1 Introduction

In today's society, the demand for mobility is constantly increasing. Consequences of this increasing demand for mobility may be congestion, pollution and accidents. An important area where traffic accumulates is an intersection. Such an intersections is often equipped with traffic lights to safely guide the different traffic streams across the intersection. It is important that these traffic lights are controlled such that the discomfort of road users is as little as possible, e.g., by minimizing the waiting times of traffic at these intersections. Two different types of traffic light control exist that attempt to minimize discomfort: fixed-time control and vehicle actuated control. Vehicle actuated control uses sensor information to control the traffic lights. Fixed-time control does not use such sensor information; for fixed-time control the green, amber and red lights are timed periodically.

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We have a renewed interest in fixed-time control due to the development of intelligent navigation systems. Traffic lights cause unpredictable delays when navigating through a network of traffic lights. These intelligent navigation systems can take the future state of the traffic lights into account when calculating the optimal route, possibly selecting a route for which the road user experiences a green wave. These intelligent navigation systems may work roughly as follows. As more floating car data becomes available, this data can be used to predict the future state of traffic lights, see for example (Krijger 2013). As Krijger shows, these future state predictions are especially accurate whenever a fixed-time controller is used at these intersections. These predictions can be used to give feedback to road users, who can use this information to adjust their speed accordingly, reduce their waiting time at traffic lights and save fuel. Furthermore, these predictions can be used by navigation systems to calculate a smart route, possibly selecting an inventive route for which the road user experiences little delay at traffic lights.

Other motivations for fixed-time control may be the following. Vehicle actuated control is often based on a fixed-time controller but is allowed to make some slight changes to the fixed-time controller, e.g., extend or shorten green intervals depending on the traffic situation. A vehicle-actuated controller may behave as a fixed-time controller in certain situations. This may for example happen in a heavily congested situation. Furthermore, it is easier to create harmonizations (green waves) between intersections when using fixed-time controllers. Such harmonizations can be created by designing a fixed-time controller for each intersection individually (by considering them in isolation) and, subsequently, synchronising these fixed-time schedules by solving a so called coordination problem, e.g., with (Gartner et al. 1975) or with (Wünsch and Köhler 1990). Some intersections are not be highly affected by neighbouring intersections. For these intersections it may be convenient and justified to consider them in isolation. This motivates the topic of this paper *fixed-time control at isolated intersections*.

A fixed-time controller can be visualized in a fixed-time schedule. Such a schedule visualizes when each traffic light displays a green, amber and a red light during a repeating period. In (Fleuren and Lefeber 2016b) an efficient formulation is proposed to find the optimal fixed-time schedule for an isolated intersection. In that paper each traffic light is assumed to have only one distinct green interval during a (repeating) period. The starting times and the durations of these green intervals are optimized simultaneously with the period duration. In (Fleuren and Lefeber 2016b) the proposed problem formulation is compared to other formulations; the proposed formulation seems to be superior. In this paper we extend the approach of (Fleuren and Lefeber 2016b) so that it also optimizes the number of green intervals that each traffic light receives. Thus, in this paper we optimize simultaneously: the period duration, the number of green intervals that each traffic light receives, the durations of these green intervals, and when these green intervals start. To the knowledge of the writers, this is the first work concerning the optimization of fixed-time schedules that allows each traffic light to have multiple green intervals. Hence, this is also the first work that considers the number of green intervals that each traffic light receives to be a design variable. Although this paper is self-contained we strongly recommend to read (Fleuren and Lefeber 2016b) before reading this paper.

Mathematically it suffices to model a traffic light with only two modes: effective green and effective red, see for example (Gartner et al. 1975); traffic departs during an effective green interval traffic, however, it does not depart during an effective red period traffic. A fixed-time schedule that uses the effective green and effective red modes can easily be transformed into a fixed-time schedule that uses the actual display colors (and vise versa), see for example (Gartner et al. 1975). To show the effect of multiple effective green intervals, we consider the same example as the one that was used in (Fleuren and Lefeber 2016b). We again visualize this T-Junction in Figure 1; for all relevant data of this intersection we refer to (Fleuren and Lefeber 2016b). When each traffic light is allowed to have only one effective green interval, then the fixed-time schedule that is given in Figure 2a minimizes the average (approximated) delay that road users experience at the intersection; for this fixed-time schedule road users experience an average (approximated) delay of 26.416 seconds. When we allow each traffic light to have one additional effective green interval, then we obtain the fixed-time schedule that is given in Figure 2b; this fixed-time schedule can be obtained by solving a mixed-integer (convex) programming problem. The switching times of this fixed-time schedule are given in Table 1. For this fixed-time schedule, traffic lights 1 and 4 have two effective green intervals and the average delay that road users experience is 25.106 seconds; this is an improvement of ~ 5 percent. In Section 4 we assess the effect of allowing signal groups to have multiple green intervals in more detail; for some intersections we have seen a decrease in the average delay that road users experience of over 10 percent.

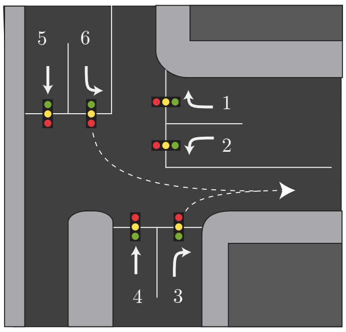
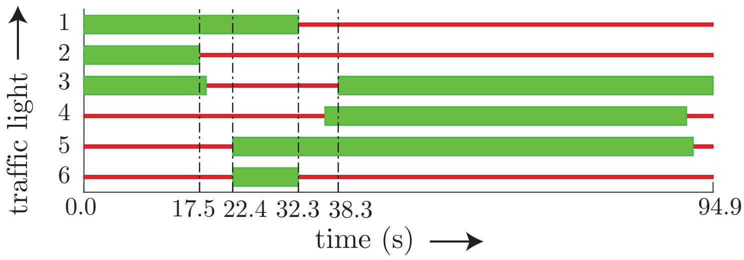


Figure 1: Visualisation of a T-junction. For each traffic light we have given the number of the corresponding traffic light.

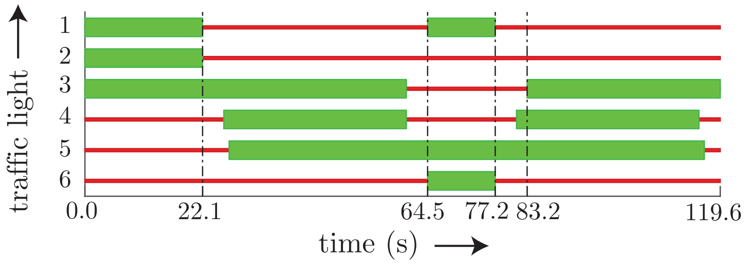
This paper has the following structure. In Section 2, we fix the number of (distinct) effective green intervals that each traffic light receives. In that section we formulate a mixed-integer programming problem to find the optimal fixed-time schedule; in contrast to (Fleuren and Lefeber 2016b), this number of effective green intervals is allowed to exceed one. Subsequently, in Section 3 we adjust this formulation to also optimize the number of effective green intervals that each traffic light has. In Section 4 we perform a numerical study and in Section 5 we give our conclusions.

2 Fixed Number of Effective Green Intervals

In this section we consider the number of effective green intervals of each traffic light to be fixed. First, in Section 2.1 we introduce the inputs that are required to formulate the optimization problem. Subsequently in Section 2.2 we introduce the real-valued design variables of the optimization problem.



(a) The fixed-time schedule that minimizes the average delay when each traffic light is allowed to have only one green interval. This schedule has a period duration of 94.87 seconds. On average a road user experiences a delay of 26.416 seconds for this fixed-time schedule.



(b) The fixed-time schedule that minimizes the average delay when each traffic light is allowed to have one additional effective green interval. This schedule has a period duration of 119.58 seconds. On average a road user experiences a delay of 25.106 seconds for this fixed-time schedule.

Figure 2: Two fixed-time schedules that visualize when each traffic light of the T-junction that is depicted in Figure 1 is effective green and effective red.

Thereupon, we give the linear constraints of this optimization problem in Section 2.3. Finally, in Section 2.4 we consider different objective functions.

2.1 Required Inputs

In this section we elaborate on the input data that is required for the optimization of fixed-time schedules.

Signal Groups. The traffic lights at the intersection are divided amongst signal groups. Each two traffic lights that are part of the same signal groups display the same color at all times. Let $\mathcal{S}$ be the set of signal groups for which we desire an optimized fixed-time schedule. The traffic that is waiting at the intersection is modelled by using a set of first-in-first-out (FIFO) queues $\mathcal{Q}$. The signal group $i \in \mathcal{S}$ controls the access to the intersection for the queues $q \in \mathcal{Q}_i$.

traffic light (i)	green interval (k)	$t^g_{i,k}$ (s)	$t^r_{i,k}$ (s)
1	1	0	22.14
	2	64.49	77.23
2	1	0	22.14
3	1	83.23	60.49
4	1	26.14	60.49
	2	81.23	115.58
5	1	27.14	116.58
6	1	64.49	77.23

Table 1: The times (rounded to hundreds of a second) at which each traffic light $i = 1, 2, 3, 4, 5, 6$ switches to effective green ($t^g_{i,k}$), and effective red ($t^r_{i,k}$) for the fixed-time schedule that is given in Figure 2b.

Number of Effective Green Intervals. For each signal group $i \in \mathcal{S}$ we require the number of effective green intervals K_i that this signal group has.

Arrival Rates and Saturation Flow Rates. For each of the queues $q \in \mathcal{Q}$ we require the arrival rate λ_q, which specifies how much traffic arrives at this queue per second. Moreover, we require the saturation flow rate μ_q, $q \in \mathcal{Q}$, which specifies how much traffic can depart from queue q per second during an effective green interval. We define $\rho_q := \lambda_q/\mu_q$ to be the load of queue $q \in \mathcal{Q}$.

Conflicts. Some traffic streams cannot safely cross the intersection simultaneously. We call their corresponding signal groups conflicting. We require a set $\Psi_{\mathcal{S}}$ of conflicting signal groups.

Minimum Clearance Times. For safety reasons we require minimum clearance times $\underline{c}_{i,j}$ and $\underline{c}_{j,i}$ for each pair of conflicting signal groups $\{i, j\} \in \Psi_{\mathcal{S}}$; signal group j may only become effective green after conflicting signal group i has been effective red for at least $\underline{c}_{i,j}$ seconds. In this paper we do allow such a minimum clearance time to be negative; when $\underline{c}_{i,j}$, $\{i, j\} \in \Psi_{\mathcal{S}}$ is negative, then signal group j may become effective green at most $\mathrm{abs}(\underline{c}_{i,j})$ seconds before signal group i becomes effective red, where $\mathrm{abs}(x)$ is the absolute value of x. For a motivation for these negative minimum clearance times we refer to (Fleuren and Lefeber 2016b). In this paper we use the following terminology. We refer to the interval between signal group i switching to effective red and a conflicting signal group j switching to effective green as a clearance interval; we refer to its duration as a clearance time.

Minimum and Maximum Period Duration. The period duration is restricted by a (strictly positive) lower bound ($\underline{T} > 0$) and an upper bound ($\overline{T}$).

Bounds on Effective Green Times and Effective Red Times. Each effective green interval of signal group $i \in \mathcal{S}$ is bounded from below by the minimum effective green time $\underline{g}_i$ and bounded from above by the maximum effective green time $\overline{g}_i$. Similarly, each effective red interval of signal group $i \in \mathcal{S}$ is bounded from below by $\underline{r}_i > 0$ and from above by $\overline{r}_i$.

2.2 Real-valued Design Variables

In this section we introduce the real-valued design variables of the optimization problem. Before we introduce these variables, we first introduce some notation. We define $\mathcal{K}_i := \{1, \ldots, K_i\}$ as the set of effective green intervals of signal group $i \in \mathcal{S}$. Furthermore, we define Ψ_I as the set of conflicting effective green intervals:

$$\Psi_I := \{\{(i,k),(j,k')\} \mid \{i,j\} \in \Psi_{\mathcal{S}}, k \in \mathcal{K}_i, k' \in \mathcal{K}_j\}.$$

Signal group $i \in \mathcal{S}$ has K_i effective green intervals. We number these intervals according to the periodic order in which they occur, i.e., the effective green intervals of signal group $i \in \mathcal{S}$ are scheduled in the periodically repeating order $1, 2, \ldots, K_i$. Let $\textcircled{$i$}_k$ ($\colorbox{gray}{$i$}_k$) denote the start (end) of effective green interval $k \in \mathcal{K}_i$ of signal group $i \in \mathcal{S}$, i.e., $\textcircled{$i$}_k$ ($\colorbox{gray}{$i$}_k$) represents a switch to effective green (effective red). For ease of notation we define $\colorbox{gray}{$i$}_0 := \colorbox{gray}{$i$}_{K_i}$. We use the following terminology in this paper. We refer to the interval between the event $\textcircled{$i$}_k$ and the event $\colorbox{gray}{$i$}_k$ as *effective green interval* k of signal group $i \in \mathcal{S}$. Similarly we refer to the interval between the event $\colorbox{gray}{$i$}_{k-1}$ and the event $\textcircled{$i$}_k$ as *effective red interval* k of signal group $i \in \mathcal{S}$; note that effective red interval k of signal group i precedes effective green interval k of signal group i. Define the set of periodic events $\mathcal{E}$ as follows:

$$\mathcal{E} = \{\textcircled{i}_k \mid i \in \mathcal{S}, k \in \mathcal{K}_i\} \cup \{\colorbox{gray}{i}_k \mid i \in \mathcal{S}, k \in \mathcal{K}_i\}.$$

Let T be the period duration of the fixed-time schedule. Furthermore, define $f(\varepsilon) \in [0,1)$, $\varepsilon \in \mathcal{E}$ as the time (expressed as a fraction of the period duration) at which the event ε occurs. In Table 2 we give the values of $f(\varepsilon)$, $\varepsilon \in \mathcal{E}$ for the fixed-time schedule that is given in Figure 2b. A fixed-time schedule is completely specified by the period duration T and the fractions $f(\varepsilon) \in [0,1)$, $\varepsilon \in \mathcal{E}$. We indirectly optimize these variables by using the following real-valued design variables, which we visualize in bold. Define $\boldsymbol{\gamma}(\varepsilon_1, \varepsilon_2)$ to be the time (expressed as a fraction of the period duration) between an occurrence of periodic event ε_1 and (the previous or the next occurrence of) periodic event ε_2, i.e.,

$$\boldsymbol{\gamma}(\varepsilon_1, \varepsilon_2) := f(\varepsilon_2) - f(\varepsilon_1) + z(\varepsilon_1, \varepsilon_2)$$

for some integer $z(\varepsilon_1, \varepsilon_2) \in \{-1, 0, 1\}$. We optimize the fractions $\boldsymbol{\gamma}(\varepsilon_1, \varepsilon_2)$ that are subject to a safety constraint. Moreover, we optimize the reciprocal of the period duration $\boldsymbol{T'} := 1/T$. From these fractions $\boldsymbol{\gamma}(\varepsilon_1, \varepsilon_2)$ and the reciprocal $\boldsymbol{T'}$ we can obtain a fixed-time schedule, see for example (Fleuren and Lefeber 2016b). Later in this section we prove that the value of each real-valued design variable $\boldsymbol{\gamma}(\varepsilon, \varepsilon)$ is defined unambiguously for each fixed-time schedule.

2.3 Linear Constraints

In this section we formulate the linear constraints of the optimization problem. When the number of effective green intervals of each signal group is fixed, then the optimization problem is very similar to the one from (Fleuren and Lefeber 2016b). The period duration T is bounded from below by $\underline{T}$ and from above by $\overline{T}$. Thus, the reciprocal $\boldsymbol{T'}$ must satisfy the following constraint:

$$1/\overline{T} \leq \boldsymbol{T'} \leq 1/\underline{T}. \tag{1a}$$

signal group (i)	effective green interval (k)	$f(\textcircled{i}_k)$	$f(\colorbox{gray}{$i$}_k)$
1	1	$\frac{0}{11958}$	$\frac{2214}{11958}$
	2	$\frac{6449}{11958}$	$\frac{7723}{11958}$
2	1	$\frac{0}{11958}$	$\frac{2214}{11958}$
3	1	$\frac{8323}{11958}$	$\frac{6049}{11958}$
4	1	$\frac{2614}{11958}$	$\frac{6049}{11958}$
	2	$\frac{8123}{11958}$	$\frac{11558}{11958}$
5	1	$\frac{2714}{11958}$	$\frac{11658}{11958}$
6	1	$\frac{6449}{11958}$	$\frac{7723}{11958}$

Table 2: The fractions $f(\varepsilon)$, $\varepsilon \in \mathcal{E}$ that are associated with the fixed-time schedule in Figure 2b.

Each effective green time of signal group i is bounded from below and from above:

$$0 \leq \underline{g}_i \boldsymbol{T}' \leq \boldsymbol{\gamma}(\textcircled{i}_k, \colorbox{gray}{i}_k) \leq \overline{g}_i \boldsymbol{T}', \qquad i \in \mathcal{S}, \quad k \in \mathcal{K}_i. \tag{1b}$$

Also each effective red time of signal group i is bounded from below and from above:

$$0 < \underline{r}_i \boldsymbol{T}' \leq \boldsymbol{\gamma}(\colorbox{gray}{i}_{k-1}, \textcircled{i}_k) \leq \overline{r}_i \boldsymbol{T}', \quad i \in \mathcal{S}, \quad k \in \mathcal{K}_i, \tag{1c}$$

Recall that $\colorbox{gray}{$i$}_0 := \colorbox{gray}{$i$}_{K_i}$, which we use in the above constraint. A signal group controls the access to the intersection for the queues $q \in \mathcal{Q}_i$. Queue $q \in \mathcal{Q}$ must be effective green for at least a fraction $\rho_q := \lambda_q/\mu_q$ of the period duration as, otherwise, its queue length would grow indefintely; when queue q is effective green for less than a fraction ρ_q, then the average amount of traffic that arrives at queue q during one period exceeds the average amount of traffic that could depart from this queue during one period. To ensure stability for each queue $q \in \mathcal{Q}_i$, signal group i must be effective green for at least a fraction $\rho_i^{\text{SG}} := \max_{q \in \mathcal{Q}_i} \rho_q > 0$ of the period duration:

$$0 < \rho_i^{\text{SG}} \leq \sum_{k \in \mathcal{K}_i} \boldsymbol{\gamma}(\textcircled{i}_k, \colorbox{gray}{i}_k), \qquad i \in \mathcal{S}. \tag{1d}$$

Minimum clearance times have to be satisfied for each pair of conflicting effective green intervals $\{(i,k),(j,k')\} \in \Psi_I$:

$$\underline{c}_{i,j} \boldsymbol{T}' \leq \boldsymbol{\gamma}(\colorbox{gray}{i}_k, \textcircled{j}_{k'}), \qquad \{(i,k),(j,k')\} \in \Psi_I. \tag{1e}$$

We allow such a minimum clearance time $\underline{c}_{i,j}$ to be negative; when $\underline{c}_{i,j} < 0$ then signal group j may become to effective green at most $\text{abs}(\underline{c}_{i,j})$ seconds before signal group i becomes effective red. However, to have a well-posed optimization problem we restrict the duration of a negative clearance time:

$$\boldsymbol{\gamma}(\textcircled{i}_k, \colorbox{gray}{i}_k) + \boldsymbol{\gamma}(\colorbox{gray}{i}_k, \textcircled{j}_{k'}) \geq \epsilon \boldsymbol{T}', \qquad \{(i,k),(j,k')\} \in \Psi_I, \tag{1f}$$

which restricts effective green interval k of signal group i plus the clearance time to effective green interval k' of signal group j to be at least $\epsilon > 0$ seconds. This constraint ensures the inclusion $\boldsymbol{\gamma}(\textcircled{\textbf{i}}_k, \textcircled{j}_{k'}) \in (-1, 1)$. A clearance time $\boldsymbol{\gamma}(\textcircled{\textbf{i}}_k, \textcircled{j}_{k'})T$ then, as desired, refers to the time between an occurrence of the event $\textcircled{\textbf{i}}_k$ and the next or the previous occurrence of the event $\textcircled{j}_{k'}$ depending on the sign of $\boldsymbol{\gamma}(\textcircled{\textbf{i}}_k, \textcircled{j}_{k'})$. Moreover, the constraints (1f) ensure that each variable $\boldsymbol{\gamma}(\textcircled{i}_k, \textcircled{\textbf{i}}_k)$ and its associated integer $z(\textcircled{i}_k, \textcircled{\textbf{i}}_k)$ are defined unambiguously. We prove both statements in Section 2.3.3

The following constraints reduce the symmetry of the proposed mixed-integer programming problem; for each signal group we assume w.l.o.g. that its first effective red time is the largest:

$$\boldsymbol{\gamma}(\textcircled{\textbf{i}}_{K_i}, \textcircled{i}_1) \geq \boldsymbol{\gamma}(\textcircled{\textbf{i}}_{k-1}, \textcircled{i}_k), \qquad i \in \mathcal{S}, \quad k \in \mathcal{K}_i \setminus \{1\}. \tag{1g}$$

This constraint reduces the solution space and reduces the symmetry of the MIP problem. Therefore, including this last constraint is expected to reduce the computation time that is needed to solve the optimization problem.

2.3.1 Circuital Constraints.

The variables $\boldsymbol{\gamma}(\varepsilon_1, \varepsilon_2)$ are also related via cycle periodicity constraints. For more information on these cycle periodicity constraints than is given in this section we refer to (Serafini and Ukovich 1989) and (Fleuren and Lefeber 2016b). The cycle periodicity constraints model the periodicity of the fixed-time schedule. To formulate these circuital constraints we have to introduce the constraint graph $G = (V, A)$. This constraint graph is defined as follows:

$$\begin{aligned}
V &= \{\textcircled{i}_k \mid i \in \mathcal{S}, k \in \mathcal{K}_i\} \cup \{\textcircled{\textbf{i}}_k \mid i \in \mathcal{S}, k \in \mathcal{K}_i\}, \\
A &= A_g \cup A_r \cup A_c,
\end{aligned}$$

where,

$$\begin{aligned}
A_g &:= \{(\textcircled{i}_k, \textcircled{\textbf{i}}_k) \mid i \in \mathcal{S}, k \in \mathcal{K}_i\}, \\
A_r &:= \{(\textcircled{\textbf{i}}_{k-1}, \textcircled{i}_k) \mid i \in \mathcal{S}, k \in \mathcal{K}_i\}, \\
A_c &:= \{(\textcircled{\textbf{i}}_k, \textcircled{j}_{k'}) \mid \{(i, k), (j, k')\} \in \Psi_I\}.
\end{aligned}$$

The set of vertices V equals the set of events $\mathcal{E}$ and, therefore, each vertex represents either a switch to effective green or a switch to effective red. With each vertex $\varepsilon \in V$ we can associate the fraction $f(\varepsilon)$. Furthermore, constraint graph G has a directed arc $(\varepsilon_1, \varepsilon_2) \in A$ for each of the real-valued design variables $\boldsymbol{\gamma}(\varepsilon_1, \varepsilon_2)$; the arcs in A_g represent effective green intervals, the arcs in A_r represent effective red intervals, and the arcs in A_c represent clearance intervals. See Figure 3 for the constraint graph of the T-junction in Figure 1 when $K_1 = K_4 = 2$ and $K_2 = K_3 = K_5 = K_6 = 1$.

Before we introduce the circuital constraints, we introduce some terminology.

Definition 1 (Walk)**.** *A walk is a sequence of vertices $v_1, v_2, \ldots, v_N \in V$ for which each two subsequent vertices v_k and v_{k+1} are connected via a directed arc $(v_k, v_{k+1}) \in A$.*

Where a walk is only allowed to traverse arcs in the forward direction (from tail to head), a path may also traverse arcs in the backward direction (from head to tail):

Definition 2 (Path)**.** *A path is a sequence of vertices $v_1, v_2, \ldots, v_N \in V$ for which each two subsequent vertices v_k and v_{k+1} are connected via either a directed arc $(v_k, v_{k+1}) \in A$ or a directed arc $(v_{k+1}, v_k) \in A$; a path traverses each arc $a \in A$ at most once.*

A cycle is defined in accordance with (Serafini and Ukovich 1989; Kavitha and Krishna 2009):

Definition 3 (Cycle)**.** *A cycle is a closed path, i.e., a path for which $v_1 = v_N$.*

Thus, a cycle is allowed to traverse arcs in the backward direction. We can represent a cycle by the sets $\mathcal{C}^+$ and $\mathcal{C}^-$, which denote the sets of arcs that this cycle traverses in the forward direction (from tail to head) respectively the set of arcs that this cycle traverses in the backward direction (from head to tail). Reorienting the arcs in $\mathcal{C}^-$ results in a closed walk that traverses each arc at most once. We define $\mathcal{C} := \mathcal{C}^+ \cup \mathcal{C}^-$.

Definition 4 (Circuit)**.** *A circuit is a cycle for which the vertices $v_1, v_2, \ldots, v_{N-1}$ are all distinct, i.e., a cycle for which each vertex is visited at most once.*

Consider a cycle $\mathcal{C}$ in this constraint graph G; this cycle traverses the arcs $a \in \mathcal{C}^+$ in the forward direction (from tail to head) and traverses the arcs $a \in \mathcal{C}^-$ in the backward direction (from head to tail); reorienting the backwards arcs (in $\mathcal{C}^-$) results in a closed walk that traverses each arc at most once. From the periodicity of a fixed-time schedule it follows that for each such cycle the following cycle periodicity constraint should be satisfied:

$$\sum_{(\varepsilon_1,\varepsilon_2)\in\mathcal{C}^+} \boldsymbol{\gamma}(\varepsilon_1, \varepsilon_2) - \sum_{(\varepsilon_1,\varepsilon_2)\in\mathcal{C}^-} \boldsymbol{\gamma}(\varepsilon_1, \varepsilon_2) = \boldsymbol{z}_{\mathcal{C}}, \tag{1h}$$

where $\boldsymbol{z}_{\mathcal{C}} \in \mathbb{Z}$. Fortunately, it suffices to formulate this constraint only for the cycles in some integral cycle basis of the constraint graph G ($\boldsymbol{z}_{\mathcal{C}}$ is an integral-valued design variable for these cycles); this constraint is then automatically satisfied for all the cycles in the constraint graph G, see for example (Liebchen and Peeters 2002). Such an integral cycle basis consists of only $d := |A| - |V| + \nu(G) = 2|\Psi_I| + \nu(G)$ cycles, where $\nu(G)$ is the number of connected components of the graph G; no path exists between each two vertices that are in different connected components.

For some cycles in the constraint graph G we must fix the value of $\boldsymbol{z}_{\mathcal{C}}$. Each two conflicting effective green intervals $\{(i,k),(j,k')\} \in \Psi_I$ must occur within the same period. Therefore, for each pair of conflicting effective green intervals $\{(i,k),(j,k')\} \in \Psi_I$ it holds that:

$$\boldsymbol{\gamma}(\textcircled{i}_k, {\color{gray}\textcircled{i}}_k) + \boldsymbol{\gamma}({\color{gray}\textcircled{i}}_k, \textcircled{j}_{k'}) + \boldsymbol{\gamma}(\textcircled{j}_{k'}, {\color{gray}\textcircled{j}}_{k'}) + \boldsymbol{\gamma}({\color{gray}\textcircled{j}}_{k'}, \textcircled{i}_k) = 1, \tag{1i}$$

which implies that each period consists of effective green interval k of signal group i, a clearance interval from this effective green interval to effective green interval k' of signal group j, the k'th effective green interval of signal group j itself, and a clearance interval back to effective green interval k of signal group i. Furthermore, the effective green intervals of signal group $i \in \mathcal{S}$ together with the effective red intervals of signal group i constitute one period, which implies the following circuital constraint:

$$\sum_{k\in\mathcal{K}_i} \Big(\boldsymbol{\gamma}({\color{gray}\textcircled{i}}_{k-1}, \textcircled{i}_k) + \boldsymbol{\gamma}(\textcircled{i}_k, {\color{gray}\textcircled{i}}_k) \Big) = 1, \qquad i \in \mathcal{S}. \tag{1j}$$

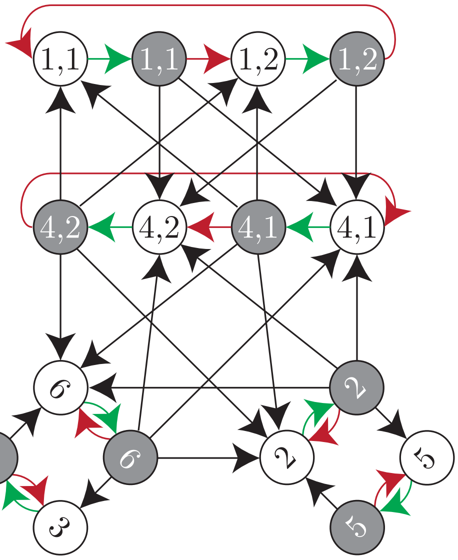


Figure 3: The constraint graph $G = (V, A)$ of the intersection in Figure 1 when $K_1 = K_4 = 2$ and $K_2 = K_3 = K_5 = K_6 = 1$. The white (grey) vertex with the text i, k denotes the event $\textcircled{i}_k$ ($\textcircled{i}_k$). The white (grey) vertex with the text i denotes the event $\textcircled{i}_1$ ($\textcircled{i}_1$). The effective green intervals, effective red intervals and clearance intervals are visualized in green, red respectively black.

The circuital constraints (1h)–(1j) model the periodicity of the fixed-time schedule. In the next section we show how to find an integral cycle basis of the constraint graph; the method that we use to find an integral cycle basis generalizes the method from (Fleuren and Lefeber 2016b).

2.3.2 Obtaining an Integral Cycle Basis.

To formulate the linear constraints (1) we require an integral cycle basis of the constraint graph G; this integral cycle basis is needed to formulate the cycle periodicity constraints (1h). For some integral cycle bases the slack in the integral valued design variables $\boldsymbol{z}_{\mathcal{C}}$, $\mathcal{C} \in \mathcal{B}$ is smaller than for others. To reduce the computation time that is needed to solve the optimization problem, we would like these slacks to be as small as possible; a smaller slack in the integral-valued design variables $\boldsymbol{z}_{\mathcal{C}}$, $\mathcal{C} \in \mathcal{B}$ relates to a smaller computation time needed to solve the MIP problem, which is motivated by the studies from (Liebchen 2003; Wünsch and Köhler 1990). In this section we attempt to find an integral cycle basis for which the slack in the integral variables is small. To this end, we construct an integral cycle basis that includes all the cycles that are associated with the circuital constraints (1i) and (1j); for these cycles the value for $\boldsymbol{z}_{\mathcal{C}}$ is known and equal to one. We construct this integral cycle basis from a strictly fundamental cycle basis. Before we introduce the definition of a strictly fundamental cycle basis we have to define a spanning tree and a spanning forest.

Definition 5 (Spanning tree)**.** *A spanning tree of a graph $G = (V, A)$ is defined as a subset $\mathcal{T} \subseteq A$ such that the graph $G = (V, \mathcal{T})$ contains no cycles and has one connected component.*

Such a spanning tree is defined for an undirected graph as well as for a directed graph. When a graph has multiple connected components, then this graph has no spanning tree. It does however have a spanning forest:

Definition 6 (Spanning forest)**.** *Consider a graph G with $\nu(G) \geq 1$ connected components. Let $\mathcal{T}_i$ be a spanning tree of connected component $i = 1, \ldots, \nu(G)$. Then $\mathcal{F} = \bigcup_{i=1}^{\nu(G)} \mathcal{T}_i$ is a spanning forest of graph G.*

Adding an arc to spanning forest will result in a cycle. Such a cycle is called a *fundamental cycle.* A strictly fundamental cycle basis is comprised of all fundamental cycles that are associated with some spanning forest $\mathcal{F}$:

Definition 7 (Strictly fundamental cycle basis (SFCB))**.** *The set of cycles $\mathcal{B} = \{\mathcal{C}_1, \ldots, \mathcal{C}_d\}$ is a strictly fundamental cycle basis whenever $\mathcal{B}$ is the set of all the fundamental cycles that are associated with some spanning forest $\mathcal{F} \subseteq A$. In other words $\mathcal{B} = \{\mathcal{C}_1, \ldots, \mathcal{C}_d\}$ is a strictly fundamental cycle basis whenever some spanning forest $\mathcal{F} \subseteq A$ exists such that $\mathcal{B} = \{\mathcal{C}_{\mathcal{F}}(a) \mid a \in A \setminus \mathcal{F}\}$, where $\mathcal{C}_{\mathcal{F}}(a)$ is the unique circuit in $\mathcal{F} \cup \{a\}$ that uses the arc a in the forward direction.*

We find the spanning forest $\mathcal{F}$ from a spanning forest $\mathcal{F}'$ of the smaller (undirected) conflict graph $G' = (V', A')$, which is defined as follows:

$$V' := \mathcal{S},$$
$$A' := \Psi_{\mathcal{S}}.$$

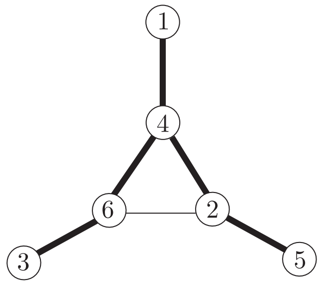
In Figure 4a we have depicted the conflict graph for the T-junction that is depicted in Figure 1. We can obtain a spanning forest $\mathcal{F}$ of the constraint graph G from a spanning forest $\mathcal{F}'$ of the smaller (undirected) conflict graph $G' = (V', A')$ with the following equation, see also Figure 4:

$$\begin{aligned} \mathcal{F} :=& \{(\colorbox{gray}{i}_k, \textcircled{j}_{k'}) \mid \{i,j\} \in \mathcal{F}', i < j\} \cup A_g \\ & \cup \{(\colorbox{gray}{i}_{k-1}, \textcircled{i}_k) \mid i \in \mathcal{S}, k \in \mathcal{K}_i \setminus \{1\}\}. \end{aligned} \tag{2}$$

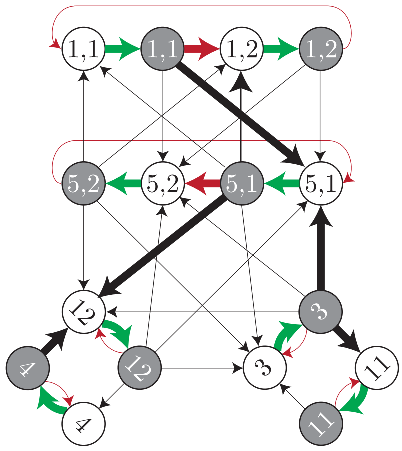
Thus, the spanning forest $\mathcal{F}$ includes all the arcs that represent effective green interval, for each signal group $i \in \mathcal{S}$ it includes $K_i - 1$ of the K_i arcs that represent red intervals, and it includes an arc that represents a clearance interval for each arc in $\mathcal{F}'$. From the spanning forest $\mathcal{F}$ we can obtain a strictly fundamental cycle basis. This SFCB does not necessarily contain all the cycles that are associated with the circuital constraints (1i) and (1j); it contains the cycle that is associated with the circuital constraint (1i) of the conflict $\{(i,k),(j,k')\} \in \Psi_{\mathcal{S}}$ if and only if $k = k' = 1$ and $\{i,j\} \in \mathcal{F}'$. We can however use this SFCB to construct an integral cycle basis that does include all these cycles, see Lemma 1; the resulting integral cycle basis has replaced some of the cycles in the SFCB by cycles that are associated with the circuital constraint (1i); for these cycles the value of $\boldsymbol{z}_{\mathcal{C}}$ is fixed to one.

Lemma 1. *Let $\mathcal{F}'$ be a spanning forest of the conflict graph G' and let $\mathcal{F}$ be the spanning forest of the constraint graph G that is calculated with (2). Define $\mathcal{B} = \{\mathcal{C}_1, \ldots, \mathcal{C}_d\}$ to be the SFCB of graph G that is defined by spanning forest $\mathcal{F}$, and let $\mathcal{B}'$ be the set of cycles that is obtained from $\mathcal{B}$ when, for each arc $(\colorbox{gray}{$i$}_k, \textcircled{j}_{k'}) \notin \mathcal{F}$, $i < j$, we replace the cycle $\mathcal{C}_{\mathcal{F}}((\colorbox{gray}{j}_{k'}, \textcircled{i}_k))$ by the cycle:*

$$\mathcal{C} = \mathcal{C}^+ = \{(\textcircled{i}_k, \colorbox{gray}{i}_k), (\colorbox{gray}{i}_k, \textcircled{j}_{k'}), (\textcircled{j}_{k'}, \colorbox{gray}{j}_{k'}), (\colorbox{gray}{j}_{k'}, \textcircled{i}_k)\}.$$



(a) Spanning forest of the conflict graph G'. Each bold arc is included in the spanning forest.



(b) The spanning forest of the constraint graph G that is induced by a spanning forest of the constraint graph G'. Each bold arc is included in the spanning forest.

Figure 4: Visualization of the spanning forest of the constraint graph G that is obtained from a spanning forest of the conflict graph G'. The conflict graph corresponds to the intersection in Figure 1 and the constraint graph corresponds to $K_1 = K_4 = 2$ and $K_2 = K_3 = K_5 = K_6 = 1$.

The set $\mathcal{B}'$ is an integral cycle basis of the constraint graph G that includes all the cycles that are associated with the circuital constraints (1i) *and* (1j)*.*

Proof. See Appendix A for the proof. □

With Lemma 1 we can construct an integral cycle basis of the constraint graph G. To this end, we require a spanning forest $\mathcal{F}'$ of the conflict graph G'; we calculate this spanning forest with the algorithm of (Amaldi et al. 2004). This algorithm requires a weight for each arc; we take all arc weights to be the same.

For the constraint graph G that is visualized in Figure 3, the resulting integral cycle basis consists of 23 cycles. Six of these cycles correspond to circuital constraints (1i) and eleven of them correspond to circuital constraints (1j). Therefore, six cycles remain for which the value of $\boldsymbol{z}_{\mathcal{C}}$ is unknown (before optimization). Therefore, the optimization problem has six integral-valued design variables for this example.

2.3.3 Well-posedness.

Each variable $\boldsymbol{\gamma}(\varepsilon_1, \varepsilon_2)$ can be written as $\boldsymbol{\gamma}(\varepsilon_1, \varepsilon_2) := f(\varepsilon_2) - f(\varepsilon_1) + z(\varepsilon_1, \varepsilon_2)$. We prove that each real-valued design variable $\boldsymbol{\gamma}(\varepsilon_1, \varepsilon_2)$ and its corresponding integer $z(\varepsilon_1, \varepsilon_2)$ are defined unambiguously, i.e., only one value for $\boldsymbol{\gamma}(\varepsilon_1, \varepsilon_2)$ and only one value for $z(\varepsilon_1, \varepsilon_2)$ can be associated with each fixed-time schedule.

We first prove this unambiguous definition for the arcs that represent effective green and effective red intervals. We do so by proving the inclusion $\boldsymbol{\gamma}(\textcircled{i}_k, \overset{\bullet}{\textcircled{i}}_k) \in [0,1)$ and the inclusion $\boldsymbol{\gamma}(\overset{\bullet}{\textcircled{i}}_k, \textcircled{i}_k) \in [0,1)$; this would imply that the design variables $\boldsymbol{\gamma}(\textcircled{i}_k, \overset{\bullet}{\textcircled{i}}_k)$ and $\boldsymbol{\gamma}(\overset{\bullet}{\textcircled{i}}_k, \textcircled{i}_k)$ (and their associated integers $z(\varepsilon_1, \varepsilon_2)$) are defined unambiguously. These inclusions follow from the fact that the effective green intervals and the effective red intervals of signal group i together constitute one period (1j), the non-negativity of each effective green time (1b), the (strict) positivity of each effective red time (1c), and the (strict) positivity of the sum of the effective green times of signal group i (1d).

Now we prove the unambiguous definition of each variable $\boldsymbol{\gamma}(\overset{\bullet}{\textcircled{i}}_k, \textcircled{j}_{k'})$, $\{(i,k),(j,k')\} \in \Psi_I$; we do so by proving the following inclusion:

$$\boldsymbol{\gamma}(\overset{\bullet}{\textcircled{i}}_k, \textcircled{j}_{k'}) \in (-\boldsymbol{\gamma}(\textcircled{i}_k, \overset{\bullet}{\textcircled{i}}_k), 1 - \boldsymbol{\gamma}(\textcircled{i}_k, \overset{\bullet}{\textcircled{i}}_k)).$$

Since, as already proved, $\boldsymbol{\gamma}(\textcircled{i}_k, \overset{\bullet}{\textcircled{i}}_k)$ is defined unambiguously, this inclusion would prove that the variable $\boldsymbol{\gamma}(\overset{\bullet}{\textcircled{i}}_k, \textcircled{j}_{k'})$ and its corresponding integer $z(\overset{\bullet}{\textcircled{i}}_k, \textcircled{j}_{k'})$ are defined unambiguously as well. The desired inclusion follows from the well-posedness constraint (1f) together with the circuital constraint (1i). This inclusion together with $\boldsymbol{\gamma}(\textcircled{i}_k, \overset{\bullet}{\textcircled{i}}_k) \in [0,1)$ also implies the inclusion $\boldsymbol{\gamma}(\overset{\bullet}{\textcircled{i}}_k, \textcircled{j}_{k'}) \in (-1,1)$. As a consequence, a clearance time from effective green interval k of signal group i to effective green interval k' of signal group j refers, as desired, to the time between an occurrence of the event $\overset{\bullet}{\textcircled{i}}_k$ and the next or the previous occurrence of the event $\textcircled{j}_k$.

2.4 Objective Function

In this section we elaborate on three different objective function: minimizing the period duration T, maximizing the capacity of the intersection, and minimizing the average weighted delay that road users experience.

We can minimize the period duration by maximizing its reciprocal $\boldsymbol{T'}$. The resulting problem is a mixed-integer linear programming (MILP) problem. When maximizing the capacity of the intersection, then we search for the fixed-time schedule for which the largest increase in the arrival rates λ_q, $q \in \mathcal{Q}$ is sustainable. To this end, we multiply the left-hand sides (ρ_i^{SG}) of the stability constraints (1d) by a growth factor $\boldsymbol{\beta}$. The objective is to maximize this growth factor. The resulting problem is also an MILP problem. Whenever the maximum growth factor $\beta^{\max}$ is less than one, then this implies that the intersection is overloaded by $(1-\beta^{\max})100$ percent. On the other hand, when this growth factor is greater than one, then the intersection has $(\beta^{\max}-1)100$ percent of overcapacity.

The last objective is to minimize the average weighted delay that road users experience at the intersection:

$$D = \sum_{i \in \mathcal{S}} \sum_{q \in \mathcal{Q}_i} w_q d_q,$$

where d_q is the average delay at queue $q \in \mathcal{Q}$ and w_q is the weight factor that is associated with this queue. We can use the approximations of for example (Miller 1963; van den Broek et al. 2006; Webster 1958) to approximate the delay d_q. However, all these approximations assume that a signal group has a single effective green interval, i.e., these formulae assume $K_i = 1$. As no better alternative is available at the time of writing, we extend these approximations, in a straightforward manner, to allow a signal group to have multiple effective green intervals. For the formulae of (Miller 1963; van den Broek et al. 2006; Webster 1958) the extended approximation is a convex function of the design variables; in Appendix B we prove this convexity for the approximation of (van den Broek et al. 2006), which is the approximation that we use in this paper. As a consequence of this convexity, the resulting problem is a mixed-integer convex programming problem when minimizing the average delay that road users experience.

For ease of notation we define $r'_{i,k} := \boldsymbol{\gamma}(i_{k-1}, i_k)$ and $r_{i,k} := r'_{i,k}/\boldsymbol{T}'$; note that $r_{i,k}$ is the duration of the kth effective red interval of signal group i. Assume that signal group i receives a single effective green interval, i.e., $K_i = 1$, and consider one of its queues $q \in \mathcal{Q}_i$. The aforementioned approximations for the delay that road users experience at the queue $q \in \mathcal{Q}_i$ can be split into a deterministic part and a stochastic part: $d_q = d_q^{\text{det}} + d_q^{\text{stoch}}$. For the approximation of (van den Broek et al. 2006) we have:

$$
\begin{aligned}
d_q^{\text{det}} &:= \frac{r_{i,1}^2}{2T(1-\rho_q)}. \\
&= \frac{r_{i,1}'^2}{2\boldsymbol{T}'(1-\rho_q)} \\
d_q^{\text{stoch}} &= \frac{r_{i,1}}{2\lambda_q(1-\rho_q)T}\left(\frac{\sigma_q^2}{1-\rho_q} + \frac{r_{i,1}\rho_q^2\sigma_q^2 T^2}{(1-\rho_q)(T-r_{i,1})^2((1-\rho_q)T-r_{i,1})}\right), \\
&= \frac{r'_{i,1}}{2\lambda_q(1-\rho_q)\boldsymbol{T}'}\left(\frac{\sigma_q^2}{1-\rho_q} + \frac{r'_{i,1}\rho_q^2\sigma_q^2 \boldsymbol{T}'^2}{(1-\rho_q)(\boldsymbol{T}'-r'_{i,1})^2((1-\rho_q)\boldsymbol{T}'-r'_{i,1})}\right).
\end{aligned}
\tag{3}
$$

In the following sections we extend this deterministic delay term and this stochastic delay term to also allow $K_i > 1$.

Extending the Deterministic Delay Term. The deterministic delay term describes the delay whenever the arrival process and the departure process would be purely deterministic and fluid-like, see also Figure 5; the amount of waiting traffic increases with a rate of λ_q during an effective red interval. During an effective green interval the queue length decreases with a rate of $\mu_q - \lambda_q$ as long as the queue is not emptied. When the queue is emptied then the queue remains empty until the next effective red interval starts. Consider a queue $q \in \mathcal{Q}_i$ and assume that signal group $i \in \mathcal{S}$ has a single effective red interval of $r_{i,1}$ seconds. The deterministic delay term d_q^{det} can then be computed from the average queue length $\overline{x}_q$ by using Little's law (Chhajed and Lowe 2008):

$$
\begin{aligned}
d_{\text{det}} &:= \overline{x}_q/\lambda_q, \\
&:= \frac{r_{i,1}^2}{2T(1-\rho_q)}, \\
&:= \frac{r_{i,1}'^2}{2\boldsymbol{T}'(1-\rho_q)}.
\end{aligned}
$$

This deterministic delay term assumes that the queue is emptied during its effective green interval, i.e., it assumes stability. We can extend this deterministic delay term, in a straightforward manner, to the case of multiple effective green intervals, i.e., to the case $K_i \geq 1$. Consider again the deterministic and fluid-like arrival and departure process. Whenever, for this deterministic system, the queue is emptied during each effective green interval, then we can find the following expression for the deterministic delay term:

$$d_q^{\text{det}} = \sum_{k \in \mathcal{K}_i} \frac{r_{i,k}'^2}{2\boldsymbol{T}'(1-\rho_q)}. \tag{4}$$

When minimizing the average weighted delay, then we can force the queue to be emptied during each effective green interval (for this deterministic arrival and departure process) by adding the following constraints to the mixed-integer programming problem:

$$(1-\rho_i^{\text{SG}})\boldsymbol{\gamma}(\textcircled{\scriptsize i}_k, \textcircled{\scriptsize i}_k) \geq \rho_i^{\text{SG}}\boldsymbol{\gamma}(\textcircled{\scriptsize i}_{k-1}, \textcircled{\scriptsize i}_k), \qquad i \in \mathcal{S}, \quad k \in \mathcal{K}_i. \tag{5}$$

We only have to include this constraint for the signal groups $i \in \mathcal{S}$ for which $K_i > 1$; this inequality is already implied by the stability constraint (1d) for each signal group $i \in \mathcal{S}$ with $K_i = 1$.

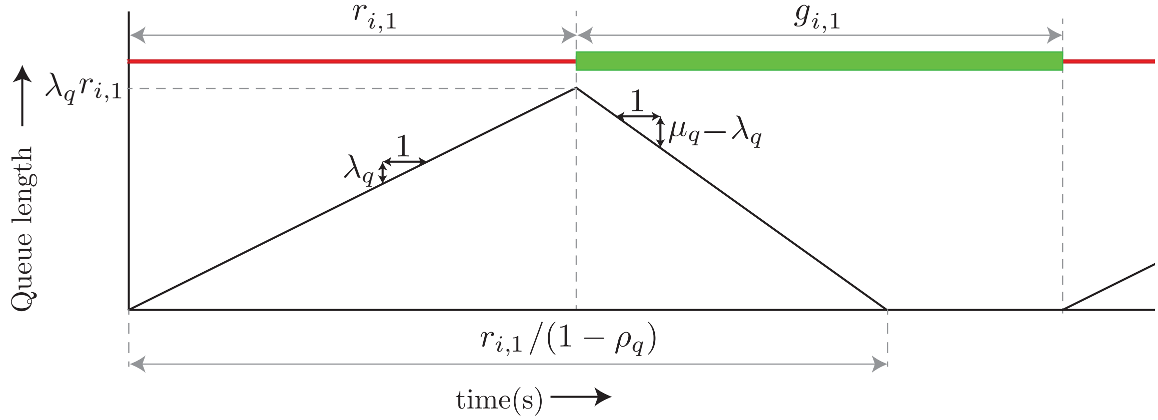


Figure 5: The queueing of passenger cars for the deterministic and fluid-like queueing system associated with the deterministic delay term d_q^{det} of a queue $q \in \mathcal{Q}_i$ when signal group i has only one effective green interval.

Remark. Whenever we do not force the queue q to be emptied during each effective green interval for the deterministic queueing process, then the deterministic delay term d_q^{det} cannot be written as the sum (4); in Appendix C we show that the deterministic delay term d_q^{det} and also the total delay d_q are then not convex.

Extending the Stochastic Delay Term. The stochastic delay term d_q^{stoch} corresponds to the stochastic contributions in the delay, e.g., when the arrivals are stochastic, then the queue might not be empty at the end of an effective green interval. When $K_i = 1$, then this stochastic delay term is a function of the fraction $r'_{i,1}$, which is the total red fraction of signal group i, and the (reciprocal of the) period duration $\boldsymbol{T'}$. In case that a signal group has multiple effective green intervals, then we replace $r'_{i,1}$ by $r'_{i,1} + \ldots, +r'_{i,K_i}$, which is the total red fraction of signal group i when $K_i \geq 1$.

Remark. We do not claim that this straightforward extension of the approximate formulae results in a very good approximation when $K_i > 1$; we merely want to show the different objective functions that we could consider when we allow signal groups to receive multiple effective green intervals. However, this extended approximation does have some desirable properties. Consider the extended approximation of (van den Broek et al. 2006). It satisfies the following properties:

- Consider the case that signal group i has a single effective green interval, i.e., $K_i = 1$. This extended approximation then reduces to the original approximation of (van den Broek et al. 2006).
- Consider the case that signal group i receives multiple effective green intervals, i.e., $K_i > 1$. Let $g_{i,k}$ be the duration of effective green interval k of signal group i. Assume that $g_{i,k} = 0$ for each effective green interval $k \in \mathcal{K}_i \setminus \{1\}$. From equation (5) it then follows that $r_{i,k} = 0$ for each effective green interval $k \in \mathcal{K}_i \setminus \{1\}$. Therefore, effectively, signal group i has only one effective green interval ($g_{i,1}$) and one effective red interval ($r_{i,1}$). The extended approximation then reduces to the original approximation of (van den Broek et al. 2006) with an effective red time of $r_{i,1}$ seconds and a period duration of $g_{i,1} + r_{i,1}$ seconds.
- Consider the case that signal group i receives multiple effective green intervals, i.e., $K_i > 1$. Let g_i (r_i) be the total effective green (effective red) time of signal group i. Assume that signal group i alternates between an effective green time of ${}^{g_i}/_{K_i}$ seconds and an effective red time of ${}^{r_i}/_{K_i}$ seconds. The signal timings of signal group i then repeat every ${}^{r_i}/_{K_i} + {}^{g_i}/_{K_i}$ seconds. The extended approximation then reduces to the original approximation of (van den Broek et al. 2006) with a period duration of ${}^{r_i}/_{K_i} + {}^{g_i}/_{K_i}$ seconds and a (single) effective red interval of ${}^{r_i}/_{K_i}$ seconds.

3 Variable Number of Effective Green Intervals

In the previous section we have formulated the optimization problem for the situation that the number of effective green intervals K_i of each signal group $i \in \mathcal{S}$ is fixed. In this section we consider the number of effective green intervals K_i of each signal group $i \in \mathcal{S}$ to be a design variable. To this end, for each signal group $i \in \mathcal{S}$ we require a minimum number of effective green intervals $\underline{K}_i \geq 1$ and a maximum number of effective green intervals $\overline{K}_i$.

We adjust the optimization problem that is formulated in Section 2 so that also the number of effective green intervals is optimized for each signal group $i \in \mathcal{S}$. In this section we first elaborate on the differences with respect to the optimization problem that is proposed in the previous section. Thereupon, we give the complete mixed-integer programming formulation.

3.1 Notation

Before we elaborate on the differences with the previously formulated optimization problem, we first introduce some notation. We define $\overline{\mathcal{K}}_i := \{1, \ldots, \overline{K}_i\}$ to be the maximum set of effective green intervals of signal group $i \in \mathcal{S}$. Furthermore, we define $\underline{\mathcal{K}}_i := \{1, \ldots, \underline{K}_i\}$ to be the minimum set of effective green intervals of signal group $i \in \mathcal{S}$, and we denote the difference between $\overline{\mathcal{K}}_i$ and $\underline{\mathcal{K}}_i$ by $\mathcal{K}_i^d$, i.e., $\mathcal{K}_i^d := \overline{\mathcal{K}}_i \setminus \underline{\mathcal{K}}_i$. Thus, signal group $i \in \mathcal{S}$ is certain to have the effective green intervals $k \in \underline{\mathcal{K}}_i$. Via optimization we decide whether signal group $i \in \mathcal{S}$ has effective green interval $k \in \mathcal{K}_i^d$. We consider the constraint graph $G = (V, A)$ as defined in Section 2.3 except that we replace the set of effective green intervals $\mathcal{K}_i$ by the maximum set of effective green intervals $\overline{\mathcal{K}}_i$:

$$
\begin{aligned}
V &= \{\textcircled{i}_k \mid i \in \mathcal{S}, k \in \overline{\mathcal{K}}_i\} \cup \{{\color{gray}\textcircled{i}}_k \mid i \in \mathcal{S}, k \in \overline{\mathcal{K}}_i\},\\
A &= A_g \cup A_r \cup A_c,
\end{aligned}
$$

where,

$$
\begin{aligned}
A_g &:= \{(\textcircled{i}_k, {\color{gray}\textcircled{i}}_k) \mid i \in \mathcal{S}, k \in \overline{\mathcal{K}}_i\},\\
A_r &:= \{({\color{gray}\textcircled{i}}_{k-1}, \textcircled{i}_k) \mid i \in \mathcal{S}, k \in \overline{\mathcal{K}}_i\},\\
A_c &:= \{({\color{gray}\textcircled{i}}_k, \textcircled{j}_{k'}) \mid \{(i,k),(j,k')\} \in \Psi_I\},\\
\Psi_I &:= \{\{(i,k),(j,k')\} \mid \{i,j\} \in \Psi_{\mathcal{S}}, k \in \overline{\mathcal{K}}_i, k' \in \overline{\mathcal{K}}_j\}.
\end{aligned}
$$

where ${\color{gray}\textcircled{i}}_0$ is defined to equal ${\color{gray}\textcircled{i}}_{\overline{K}_i}$ and Ψ_I is a set of conflicting effective green intervals.

3.2 Additional Design Variables

With respect to the optimization problem that was proposed in the previous section, the optimization problem that is proposed in this section has one additional binary design variable $b_{i,k}$ for each of the effective green intervals $k \in \mathcal{K}_i^d$ of signal group $i \in \mathcal{S}$; the binary design variable $\boldsymbol{b}_{i,k}$, $k \in \mathcal{K}_i^d$ equals one whenever signal group i has a kth effective green interval and it equals zero otherwise. This implies that these binary variables are related according to the following constraint:

$$
\boldsymbol{b}_{i,k+1} \leq \boldsymbol{b}_{i,k}, \qquad i \in \mathcal{S},\ \ k \in \mathcal{K}_i^d \setminus \{\overline{K}_i\}. \tag{6a}
$$

This constraint implies that signal group i has no $k+1$st effective green interval when it has no kth effective green interval. We use the binary variable $\boldsymbol{b}_{i,k}$ to 'switch' effective green interval k of signal group $i \in \mathcal{S}$ on ($\boldsymbol{b}_{i,k} = 1$) or off ($\boldsymbol{b}_{i,k} = 0$). In other words, we force effective green interval k of signal group i and its preceding effective red interval to have a duration of zero seconds whenever signal group i has no such kth effective green interval, i.e., we force $\boldsymbol{\gamma}({\color{gray}\textcircled{i}}_{k-1}, \textcircled{i}_k) = 0$ and $\boldsymbol{\gamma}(\textcircled{i}_k, {\color{gray}\textcircled{i}}_k) = 0$ when $\boldsymbol{b}_{i,k} = 0$. As a consequence, signal group $i \in \mathcal{S}$ practically has no kth effective green interval when $\boldsymbol{b}_{i,k} = 0$; from constraint (6a) it then follows that signal group $i \in \mathcal{S}$ also has no $k+1$st effective green interval, et cetera.

3.3 Modified Constraints

Some constraints of optimization problem (1) may obstruct $\boldsymbol{\gamma}({\color{gray}\textcircled{i}}_{k-1}, \textcircled{i}_k)$ and $\boldsymbol{\gamma}(\textcircled{i}_k, {\color{gray}\textcircled{i}}_k)$ from becoming zero when $\boldsymbol{b}_{i,k} = 0$, e.g., the lower bound on each effective green time may prevent $\boldsymbol{\gamma}(\textcircled{i}_k, {\color{gray}\textcircled{i}}_k)$ from becoming zero. Therefore, we modify such constraints so that they allow the variables $\boldsymbol{\gamma}(\overset{\bullet}{\textcircled{i}}_{k-1}, \textcircled{i}_k)$ and $\boldsymbol{\gamma}(\textcircled{i}_k, \overset{\bullet}{\textcircled{i}}_k)$ to become zero when $\boldsymbol{b}_{i,k} = 0$. We replace the bounds on the effective green times (1b) as follows; we distinguish between the (certain) effective green intervals $k \in \underline{\mathcal{K}}_i$ and the (uncertain) effective green intervals $k \in \mathcal{K}_i^d$:

$$\underline{g}_i \boldsymbol{T}' \leq \boldsymbol{\gamma}(\textcircled{i}_k, \overset{\bullet}{\textcircled{i}}_k) \leq \overline{g}_i \boldsymbol{T}', \qquad i \in \mathcal{S}, k \in \underline{\mathcal{K}}_i, \tag{6b}$$

$$\underline{g}_i \boldsymbol{T}' - (1 - \boldsymbol{b}_{i,k})L \leq \boldsymbol{\gamma}(\textcircled{i}_k, \overset{\bullet}{\textcircled{i}}_k) \leq \overline{g}_i \boldsymbol{T}', \qquad i \in \mathcal{S}, k \in \mathcal{K}_i^d, \tag{6c}$$

where L is some large number; in this case $L = \underline{g}_i / \underline{T}$ suffices. The latter constraint becomes redundant when $\boldsymbol{b}_{i,k} = 0$. Similarly, we replace the bounds on the effective red times (1c):

$$\underline{r}_i \boldsymbol{T}' \leq \boldsymbol{\gamma}(\overset{\bullet}{\textcircled{i}}_{k-1}, \textcircled{i}_k) \leq \overline{r}_i \boldsymbol{T}', \quad i \in \mathcal{S}, k \in \underline{\mathcal{K}}_i, \tag{6d}$$

$$\underline{r}_i \boldsymbol{T}' - (1 - \boldsymbol{b}_{i,k})L \leq \boldsymbol{\gamma}(\overset{\bullet}{\textcircled{i}}_{k-1}, \textcircled{i}_k) \leq \overline{r}_i \boldsymbol{T}', \quad i \in \mathcal{S}, k \in \mathcal{K}_i^d, \tag{6e}$$

where $L = \underline{r}_i / \underline{T}$ is sufficiently large. Furthermore, we replace the well-posedness constraint (1f). For each pair of conflicting effective green intervals $\{(i,k),(j,k')\} \in \Psi_I$ with $k \in \underline{\mathcal{K}}_i$ we have the original constraint:

$$\boldsymbol{\gamma}(\textcircled{i}_k, \overset{\bullet}{\textcircled{i}}_k) + \boldsymbol{\gamma}(\overset{\bullet}{\textcircled{i}}_k, \textcircled{j}_{k'}) \geq \epsilon \boldsymbol{T}'. \tag{6f}$$

However, for each pair of conflicting effective green intervals $\{(i,k),(j,k')\} \in \Psi_I$ with $k \in \mathcal{K}_i^d$ we have the following modified constraint:

$$\boldsymbol{\gamma}(\textcircled{i}_k, \overset{\bullet}{\textcircled{i}}_k) + \boldsymbol{\gamma}(\overset{\bullet}{\textcircled{i}}_k, \textcircled{j}_{k'}) + (1 - \boldsymbol{b}_{i,k})L \geq \epsilon \boldsymbol{T}', \tag{6g}$$

where $L = 1 + \epsilon / \underline{T}$ is sufficiently large.

3.4 Additional Constraints

We have modified the constraints that may obstruct the kth effective green interval of signal group $i \in \mathcal{S}$ and its preceding effective red interval from having a duration of zero when $\boldsymbol{b}_{i,k} = 0$. Therefore, we can now force these durations to become zero when $\boldsymbol{b}_{i,k} = 0$. We do so with the following additional constraints:

$$\boldsymbol{\gamma}(\overset{\bullet}{\textcircled{i}}_{k-1}, \textcircled{i}_k) + \boldsymbol{\gamma}(\textcircled{i}_k, \overset{\bullet}{\textcircled{i}}_k) \leq \boldsymbol{b}_{i,k} L, \quad i \in \mathcal{S}, \;\; k \in \mathcal{K}_i^d, \tag{6h}$$

$$0 \leq \boldsymbol{\gamma}(\overset{\bullet}{\textcircled{i}}_{k-1}, \textcircled{i}_k), \quad i \in \mathcal{S}, \;\; k \in \mathcal{K}_i^d, \tag{6i}$$

$$0 \leq \boldsymbol{\gamma}(\textcircled{i}_k, \overset{\bullet}{\textcircled{i}}_k), \quad i \in \mathcal{S}, \;\; k \in \mathcal{K}_i^d, \tag{6j}$$

where $L = 1$ is sufficiently large. Consider the case that $\boldsymbol{b}_{i,k} = 0$. The above constraints then force the following events to occur simultaneously: the end of effective green interval $k-1$ ($\overset{\bullet}{\textcircled{i}}_{k-1}$), the start of effective green interval k ($\textcircled{i}_k$), and the end of effective green interval k ($\overset{\bullet}{\textcircled{i}}_k$). Since the events $\overset{\bullet}{\textcircled{i}}_{k-1}$ and $\overset{\bullet}{\textcircled{i}}_k$ occur simultaneously, this implies the following equality when $\boldsymbol{b}_{i,k} = 0$:

$$\boldsymbol{\gamma}(\overset{\bullet}{\textcircled{i}}_k, \textcircled{j}_{k'}) = \boldsymbol{\gamma}(\overset{\bullet}{\textcircled{i}}_{k-1}, \textcircled{j}_{k'}).$$

Thus, a clearance time from the kth effective green interval of signal group i to the k'th effective green interval of signal group j then equals the clearance time from the $k-1$st effective green interval of signal group i to the k'th effective green interval of signal group j. Therefore, for each pair of conflicting effective green intervals $\{(i,k),(j,k')\} \in \Psi_I$ with $k \in \mathcal{K}_i^d$ we include the following constraint:

$$-\boldsymbol{b}_{i,k}L \leq \boldsymbol{\gamma}(\textcircled{\textbf{i}}_k, \textcircled{j}_{k'}) - \boldsymbol{\gamma}(\textcircled{\textbf{i}}_{k-1}, \textcircled{j}_{k'}) \leq \boldsymbol{b}_{i,k}L, \tag{6k}$$

where $L = 2$ is sufficiently large. Moreover, when $\boldsymbol{b}_{i,k} = 0$ then we can find the following equality for each conflict $\{(i,k),(j,k')\} \in \Psi_I$, see Figure 6:

$$\boldsymbol{\gamma}(\textcircled{\textbf{j}}_{k'}, \textcircled{i}_k) = \boldsymbol{\gamma}(\textcircled{\textbf{j}}_{k'}, \textcircled{i}_{k-1}) + \boldsymbol{\gamma}(\textcircled{i}_{k-1}, \textcircled{\textbf{i}}_{k-1}),$$

Therefore, for each conflict $\{(i,k),(j,k')\} \in \Psi_I$ with $k \in \mathcal{K}_i^d$ we include the following constraint:

$$-\boldsymbol{b}_{i,k}L \leq \boldsymbol{\gamma}(\textcircled{\textbf{j}}_{k'}, \textcircled{i}_k) - \Big(\boldsymbol{\gamma}(\textcircled{\textbf{j}}_{k'}, \textcircled{i}_{k-1}) + \boldsymbol{\gamma}(\textcircled{i}_{k-1}, \textcircled{\textbf{i}}_{k-1})\Big) \leq \boldsymbol{b}_{i,k}L. \tag{6l}$$

where $L = 2$ is sufficiently large.

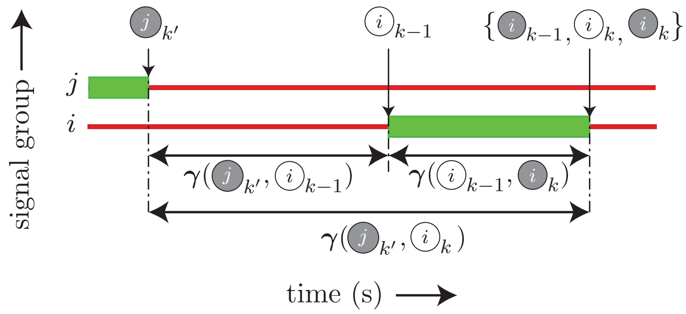


Figure 6: Relation between the clearance times when the optimization has decided that signal group i should not have a kth effective green interval. The events $\textcircled{\textbf{i}}_{k-1}$, $\textcircled{i}_k$, and $\textcircled{\textbf{i}}_k$ then occur simultaneously.

Remark. The constraints (6k) and (6l) are not crucial to the formulated MIP problem; when we omit these constraints then we would find the same fixed-time schedule as when these constraints are included. However, we include these constraints for the following two reasons. First, to speed up the the computation times. With these additional constraints we improve the quality of the LP relaxations. Such an LP relaxation relaxes the integral-valued design variables of an MILP problem to be real-valued design variables. Solving such an LP relaxation results in a lower bound on the corresponding MILP problem. These LP relaxations are important in for example the solvers: CPLEX (International Business Machines Corp 2015), GUROBI (Gurobi Optimization, Inc. 2015), and SCIP (Achterberg 2009). A tighter optimization problem (better LP relaxations) is expected to reduce computation times (Maranas and Zomorrodi 2016). Second, these constraints ensure that each variable $\boldsymbol{\gamma}(\textcircled{\textbf{i}}_k, \textcircled{j}_{k'})$ is defined unambiguously. The well-posedness constraints (6f)– (6g) ensure that this variable is defined unambiguously when $\boldsymbol{b}_{i,k} = 1$. However, they do not ensure this unambiguous definition when $\boldsymbol{b}_{i,k} = 0$; then the constraints (6k)-(6l) ensure this unambiguous definition.

3.5 Complete MIP Problem

Below we summarize the complete MIP problem for the situation that each signal group $i \in \mathcal{S}$ is (possibly) allowed to have multiple effective green intervals.

3.5.1 Objective Function.

We could optimize any of the objective functions that have been introduced in Section 2.4, i.e., we can minimize the period duration T, maximize the capacity of the intersection, or minimize the average weighted delay that road users experience. The objective function can be written as follows:

$$\underset{\boldsymbol{T}',\boldsymbol{\gamma},\boldsymbol{z},\boldsymbol{b}}{\text{minimize}} \;\; J(\boldsymbol{T}', \boldsymbol{\gamma}, \boldsymbol{z}, \boldsymbol{b}), \tag{7a}$$

where the vector $\boldsymbol{\gamma}$ contains all arc lengths $\boldsymbol{\gamma}(\varepsilon_1, \varepsilon_2)$, $(\varepsilon_1, \varepsilon_2) \in A$, the vector $\boldsymbol{z}$ contains all integral-valued design variables $\boldsymbol{z}_{\mathcal{C}}$, $\mathcal{C} \in \mathcal{B}$, and the vector $\boldsymbol{b}$ contains all binary variables $\boldsymbol{b}_{i,k}$, $k \in \mathcal{K}_i^d$, $i \in \mathcal{S}$.

Remark. When maximizing the capacity of the intersection, then we have an additional (real-valued) design variable $\boldsymbol{\beta}$, which is the growth factor of the arrival rates.

3.5.2 linear Constraints.

The MIP problem has the following constraints.

Bounds on the Period Duration. The period duration is bounded from below and from above:

$$1/\overline{T} \leq \boldsymbol{T}' \leq 1/\underline{T}. \tag{7b}$$

Bounds on Effective Green Times and Effective Red Times. Each effective green time is bounded from below and bounded from above. The lower bound on the duration of the kth effective green time of signal group $i \in \mathcal{S}$ becomes redundant whenever $b_{i,k} = 0$.

$$\underline{g}_i \boldsymbol{T}' \leq \boldsymbol{\gamma}(\textcircled{i}_k, \textcolor{gray}{\textcircled{i}}_k) \leq \overline{g}_i \boldsymbol{T}', \qquad i \in \mathcal{S}, k \in \underline{\mathcal{K}}_i, \tag{7c}$$

$$\underline{g}_i \boldsymbol{T}' - (1 - \boldsymbol{b}_{i,k})L \leq \boldsymbol{\gamma}(\textcircled{i}_k, \textcolor{gray}{\textcircled{i}}_k) \leq \overline{g}_i \boldsymbol{T}', \qquad i \in \mathcal{S}, k \in \mathcal{K}_i^d. \tag{7d}$$

Each effective red time is bounded from below and bounded from above. The lower bounds on the duration of the kth effective red time of signal group $i \in \mathcal{S}$ becomes redundant whenever $b_{i,k} = 0$.

$$\underline{r}_i \boldsymbol{T}' \leq \boldsymbol{\gamma}(\textcolor{gray}{\textcircled{i}}_{k-1}, \textcircled{i}_k) \leq \overline{r}_i \boldsymbol{T}', \quad i \in \mathcal{S}, k \in \underline{\mathcal{K}}_i, \tag{7e}$$

$$\underline{r}_i \boldsymbol{T}' - (1 - \boldsymbol{b}_{i,k})L \leq \boldsymbol{\gamma}(\textcolor{gray}{\textcircled{i}}_{k-1}, \textcircled{i}_k) \leq \overline{r}_i \boldsymbol{T}', \quad i \in \mathcal{S}, k \in \mathcal{K}_i^d. \tag{7f}$$

Stability. Each signal group needs to be stable. In other words, signal group i needs to be effective green for at least a fraction ρ_i^{SG}. This ensures that the average amount of traffic that arrives during one period at a queue $q \in \mathcal{Q}_i$ can also depart during one period:

$$0 < \rho_i^{\mathrm{SG}} \leq \sum_{k \in \overline{\mathcal{K}}_i} \boldsymbol{\gamma}(\textcircled{i}_k, \textcolor{gray}{\textcircled{i}}_k), \qquad i \in \mathcal{S}. \tag{7g}$$

Minimum Clearance Times. Minimum clearance times need to be satisfied for each pair of conflicting effective green intervals. This ensures that all conflicting traffic streams can safely cross the intersection:

$$\underline{c}_{i,j}\boldsymbol{T}' \leq \boldsymbol{\gamma}(\textcircled{\mathbf{i}}_k, \textcircled{j}_{k'}), \qquad \{(i,k),(j,k')\} \in \Psi_I. \tag{7h}$$

Number of Effective Green Intervals. The binary variables $\boldsymbol{b}_{i,k}$, $k \in \mathcal{K}_i^d$ are used to optimize the number of effective green intervals of signal group i. These binary variables are related according to the following constraints, which imply that signal group i has no $k+1$st effective green interval whenever it has no kth effective green interval:

$$\boldsymbol{b}_{i,k+1} \leq \boldsymbol{b}_{i,k}, \qquad i \in \mathcal{S},\ \ k \in \mathcal{K}_i^d \setminus \{\overline{K}_i\}. \tag{7i}$$

This binary variable is used to force effective green interval k and effective red interval k of signal group $i \in \mathcal{S}$ to practically not exists whenever $\boldsymbol{b}_{i,k} = 0$. In other words, we force effective green interval k and effective red interval k of signal group $i \in \mathcal{S}$ to have a duration of zero seconds whenever $\boldsymbol{b}_{i,k} = 0$. We do so with the following constraints:

$$\boldsymbol{\gamma}(\textcircled{\mathbf{i}}_{k-1}, \textcircled{i}_k) + \boldsymbol{\gamma}(\textcircled{i}_k, \textcircled{\mathbf{i}}_k) \leq \boldsymbol{b}_{i,k}L, \qquad i \in \mathcal{S},\ \ k \in \mathcal{K}_i^d, \tag{7j}$$

$$0 \leq \boldsymbol{\gamma}(\textcircled{\mathbf{i}}_{k-1}, \textcircled{i}_k), \qquad i \in \mathcal{S},\ \ k \in \mathcal{K}_i^d, \tag{7k}$$

$$0 \leq \boldsymbol{\gamma}(\textcircled{i}_k, \textcircled{\mathbf{i}}_k), \qquad i \in \mathcal{S},\ \ k \in \mathcal{K}_i^d. \tag{7l}$$

Reducing Symmetry. We reduce the symmetry of the MIP problem. To this end, for each signal group $i \in \mathcal{S}$ we force its first effective red interval to be the largest:

$$\boldsymbol{\gamma}(\textcircled{\mathbf{i}}_{K_i}, \textcircled{i}_1) \geq \boldsymbol{\gamma}(\textcircled{\mathbf{i}}_{k-1}, \textcircled{i}_k), \qquad i \in \mathcal{S}, \quad k \in \overline{\mathcal{K}}_i \setminus \{1\}. \tag{7m}$$

Reducing the symmetry of the MIP problem is expected to decrease the time that is needed to solve the optimization problem.

Cycle Periodicity Constraints. The periodicity of the fixed-time schedule is forced with the cycle periodicity constraints:

$$\sum_{(\varepsilon_1,\varepsilon_2)\in\mathcal{C}^+} \boldsymbol{\gamma}(\varepsilon_1,\varepsilon_2) - \sum_{(\varepsilon_1,\varepsilon_2)\in\mathcal{C}^-} \boldsymbol{\gamma}(\varepsilon_1,\varepsilon_2) = \boldsymbol{z}_{\mathcal{C}}, \qquad \forall \mathcal{C} \in \mathcal{B}, \tag{7n}$$

where $\mathcal{B}$ is an integral cycle basis of the constraint graph G. For some cycles we know the multiplicity $\boldsymbol{z}_{\mathcal{C}}$ (and must fix this multiplicity). For each pair of conflicting effective green intervals $\{(i,k),(j,k')\} \in \Psi_I$ it holds that:

$$\boldsymbol{\gamma}(\textcircled{i}_k, \textcircled{\mathbf{i}}_k) + \boldsymbol{\gamma}(\textcircled{\mathbf{i}}_k, \textcircled{j}_{k'}) + \boldsymbol{\gamma}(\textcircled{j}_{k'}, \textcircled{\mathbf{j}}_{k'}) + \boldsymbol{\gamma}(\textcircled{\mathbf{j}}_{k'}, \textcircled{i}_k) = 1, \tag{7o}$$

which implies that effective green interval k of signal group i and effective green interval k' of conflicting signal group j should be scheduled within the same period. Furthermore, for each signal group $i \in \mathcal{S}$ we have:

$$\sum_{k \in \overline{\mathcal{K}}_i} \Big(\boldsymbol{\gamma}(\fcolorbox{black}{gray}{i}_{k-1}, \textcircled{i}_{k}) + \boldsymbol{\gamma}(\textcircled{i}_{k}, \fcolorbox{black}{gray}{i}_{k}) \Big) = 1, \tag{7p}$$

which implies that the effective green intervals and the effective red intervals of signal group $i \in \mathcal{S}$ together constitute one period.

Well-posedness Constraints. The following constraints ensure that the real-valued design variables $\boldsymbol{\gamma}(\varepsilon_1, \varepsilon_2)$, $(\varepsilon_1, \varepsilon_2) \in A$ are defined unambiguously. For each pair of conflicting effective green intervals $\{(i,k),(j,k')\} \in \Psi_I$ with $k \in \underline{\mathcal{K}}_i$ we have:

$$\boldsymbol{\gamma}(\textcircled{i}_{k}, \fcolorbox{black}{gray}{i}_{k}) + \boldsymbol{\gamma}(\fcolorbox{black}{gray}{i}_{k}, \textcircled{j}_{k'}) \geq \epsilon \boldsymbol{T}', \tag{7q}$$

and for each pair conflicting effective green intervals $\{(i,k),(j,k')\} \in \Psi_I$ with $k \in \mathcal{K}_i^d$ we have:

$$\boldsymbol{\gamma}(\textcircled{i}_{k}, \fcolorbox{black}{gray}{i}_{k}) + \boldsymbol{\gamma}(\fcolorbox{black}{gray}{i}_{k}, \textcircled{j}_{k'}) + (1 - \boldsymbol{b}_{i,k})L \geq \epsilon \boldsymbol{T}', \tag{7r}$$

where ϵ is some small positive value.

Relating Clearance Times. If signal group $i \in \mathcal{S}$ does not have a kth effective green interval ($\boldsymbol{b}_{i,k} = 0$) then the kth effective green interval of signal group i is forced to have a duration of zero seconds and immediately follow the $k-1$st effective green interval of signal group $i \in \mathcal{S}$. As a result, the clearance times associated with the kth effective green interval of signal group i are then related to the clearance times associated with the $k-1$st effective green interval of this signal group. This relation is expressed in the following constraints. For each pair of conflicting effective green intervals $\{(i,k),(j,k')\} \in \Psi_I$ with $k \in \mathcal{K}_i^d$ we have:

$$-\boldsymbol{b}_{i,k} L \leq \boldsymbol{\gamma}(\fcolorbox{black}{gray}{i}_{k}, \textcircled{j}_{k'}) - \boldsymbol{\gamma}(\fcolorbox{black}{gray}{i}_{k-1}, \textcircled{j}_{k'}) \leq \boldsymbol{b}_{i,k} L. \tag{7s}$$

Furthermore, for each pair of conflicting effective green intervals $\{(i,k),(j,k')\} \in \Psi_I$ with $k \in \mathcal{K}_i^d$ we have:

$$-\boldsymbol{b}_{i,k} L \leq \boldsymbol{\gamma}(\fcolorbox{black}{gray}{j}_{k'}, \textcircled{i}_{k}) - \Big(\boldsymbol{\gamma}(\fcolorbox{black}{gray}{j}_{k'}, \textcircled{i}_{k-1}) + \boldsymbol{\gamma}(\textcircled{i}_{k-1}, \fcolorbox{black}{gray}{i}_{k-1}) \Big) \leq \boldsymbol{b}_{i,k} L. \tag{7t}$$

3.5.3 Relation to a Fixed Number of Effective Green Intervals.

Consider a fixed-time schedule for which the number of effective green intervals K_i of each signal group $i \in \mathcal{S}$ satisfies $\underline{K}_i \leq K_i \leq \overline{K}_i$. In the following lemma we prove that each such fixed-time schedule satisfies the linear constraints (1) if and only if it satisfies the linear constraints of MIP problem (7). This implies that when we fix the binary variables $\boldsymbol{b}_{i,k}$ as follows:

$$\begin{aligned} \boldsymbol{b}_{i,k} &= 0, \qquad i \in \mathcal{S}, k = \underline{K}_i + 1, \ldots, K_i, \\ \boldsymbol{b}_{i,k} &= 1, \qquad i \in \mathcal{S}, k = K_i + 1, \ldots, \overline{K}_i, \end{aligned}$$

then, as desired, the resulting constraints (7) permit the same fixed-time schedules as the constraints (1).

Lemma 2. *Consider any fixed-time schedule for which signal group $i \in \mathcal{S}$ has K_i effective green intervals. Let MIP_{fix} be the optimization problem* (1) *that considers the number of effective green intervals to be fixed and equal to K_i. Furthermore, let MIP_{var} be the optimization problem* (7) *that considers the number of effective green intervals to be a design variable that may be chosen between $\underline{K}_i \leq K_i$ and $\overline{K}_i \geq K_i$. The fixed-time schedule that we consider satisfies the constraints of MIP_{fix} if and only if it satisfies the constraints of MIP_{var}, i.e., MIP_{fix} has a solution that results in this fixed-time schedule if and only if MIP_{var} has a solution that results in this fixed-time schedule.*

Proof. Define $\mathcal{K}_i := \{1, \ldots, K_i\}$. Signal group $i \in \mathcal{S}$ has K_i effective green intervals for the fixed-time schedule that we consider. Therefore, its associated solution to MIP_{var} satisfies:

$$\begin{aligned} \boldsymbol{b}_{i,k} &:= 1, && i \in \mathcal{S},\ \ k \in \overline{\mathcal{K}}_i \setminus \mathcal{K}_i, \\ \boldsymbol{b}_{i,k} &:= 0, && i \in \mathcal{S},\ \ k \in \overline{\mathcal{K}}_i \setminus \mathcal{K}_i. \end{aligned}$$

When fixing these binary variables as shown above, then the optimization problem MIP_{var} includes all the constraints of optimization problem MIP_{fix}. Therefore, any fixed-time schedule that does not satisfy the constraints of MIP_{fix} also does not satisfy the constraints of MIP_{var}. As a consequence, what remains is to prove that each fixed-time schedule that satisfies the constraints of MIP_{fix} also satisfies the constraints of MIP_{var}. Consider a solution $(\boldsymbol{\gamma}_{\text{fix}}, \boldsymbol{T}'_{\text{fix}}, \boldsymbol{z}_{\text{fix}})$ that satisfies the linear constraints of MIP_{fix}. We construct a solution $(\boldsymbol{T}', \boldsymbol{\gamma}, \boldsymbol{z}, \boldsymbol{b})$ to MIP_{var} that results in the same fixed-time schedule; this would conclude this proof. First, the period duration of both solutions should be the same:

$$\boldsymbol{T}' := \boldsymbol{T}'_{\text{fix}}.$$

Second, all effective green and effective red intervals must have the same duration:

$$\begin{aligned} \boldsymbol{\gamma}(\textcircled{i}_k, \colorbox{gray}{i}_k) &:= \boldsymbol{\gamma}_{\text{fix}}(\textcircled{i}_k, \colorbox{gray}{i}_k), \quad i \in \mathcal{S},\ \ k \in \mathcal{K}_i, \\ \boldsymbol{\gamma}(\colorbox{gray}{i}_{k-1}, \textcircled{i}_k) &:= \boldsymbol{\gamma}_{\text{fix}}(\colorbox{gray}{i}_k, \textcircled{i}_k), \quad i \in \mathcal{S},\ \ k \in \mathcal{K}_i. \end{aligned}$$

The remaining effective green and effective red intervals have a duration of zero, which is forced by (7j)–(7l):

$$\begin{aligned} \boldsymbol{\gamma}(\textcircled{i}_k, \colorbox{gray}{i}_k) &:= 0, \quad i \in \mathcal{S},\ \ k \in \overline{\mathcal{K}}_i \setminus \mathcal{K}_i, \\ \boldsymbol{\gamma}(\textcircled{i}_{k-1}, \colorbox{gray}{i}_k) &:= 0, \quad i \in \mathcal{S},\ \ k \in \overline{\mathcal{K}}_i \setminus \mathcal{K}_i. \end{aligned}$$

We use the definition of Ψ_I as defined in Section 3.1, i.e.,

$$\Psi_I := \{\{(i,k),(j,k')\} \mid \{i,j\} \in \Psi_{\mathcal{S}}, k \in \overline{\mathcal{K}}_i, k' \in \overline{\mathcal{K}}_j\}.$$

Consider a pair of conflicting effective green intervals $\{(i,k),(j,k')\} \in \Psi_I$ with $k \in \mathcal{K}_i$ and $k' \in \mathcal{K}_j$. The clearance time from effective green interval k of signal group i to effective green interval k' of signal group j are the same for both solutions. Thus, for each pair of conflicting effective green intervals $\{(i,k),(j,k')\} \in \Psi_I$ with $k \in \mathcal{K}_i$ and $k' \in \mathcal{K}_j$ we have:

$$\boldsymbol{\gamma}(\colorbox{gray}{i}_k, \textcircled{j}_{k'}) := \boldsymbol{\gamma}_{\text{fix}}(\colorbox{gray}{i}_k, \textcircled{j}_{k'}).$$

Consider a pair of conflicting effective green intervals $\{(i,k),(j,k')\} \in \Psi_I$ with $k \notin \mathcal{K}_i$ and $k' \in \mathcal{K}_j$. It holds that effective green interval k of signal group i has a duration of zero seconds and directly follows effective green interval K_i of signal group i. As a result, for each conflict $\{(i,k),(j,k')\} \in \Psi_I$ with $k \notin \mathcal{K}_i$ and $k' \in \mathcal{K}_j$ it holds that:

$$\boldsymbol{\gamma}(\colorbox{gray}{i}_k, \textcircled{j}_{k'}) := \boldsymbol{\gamma}_{\text{fix}}(\colorbox{gray}{i}_{K_i}, \textcircled{j}_{k'}). \tag{8}$$

Consider a pair of conflicting effective green intervals $\{(i,k),(j,k')\} \in \Psi_I$ with $k' \notin \mathcal{K}_j$. Realization k' of signal group j directly follows effective green interval K_j of signal group j. As a result, it holds that a clearance time to such an effective green interval k' of signal group j equals a clearance time to effective green interval K_j of signal group j plus the duration of this effective green interval, i.e., for each pair of conflicting effective green intervals $\{(i,k),(j,k')\} \in \Psi_I$ with $k \in \mathcal{K}_i$, and $k' \notin \mathcal{K}_j$ we have:

$$\boldsymbol{\gamma}(\colorbox{gray}{i}_k, \textcircled{j}_{k'}) := \boldsymbol{\gamma}_{\text{fix}}(\colorbox{gray}{i}_k, \textcircled{j}_{K_j}) + \boldsymbol{\gamma}_{\text{fix}}(\textcircled{j}_{K_j}, \colorbox{gray}{j}_{K_j}),$$

and for each pair of conflicting effective green intervals $\{(i,k),(j,k')\} \in \Psi_I$ with $k \notin \mathcal{K}_i$, and $k' \notin \mathcal{K}_j$ we have:

$$\boldsymbol{\gamma}(\colorbox{gray}{i}_k, \textcircled{j}_{k'}) := \boldsymbol{\gamma}_{\text{fix}}(\colorbox{gray}{i}_{K_i}, \textcircled{j}_{K_j}) + \boldsymbol{\gamma}_{\text{fix}}(\textcircled{j}_{K_j}, \colorbox{gray}{j}_{K_j}).$$

The values of the integral-valued design variables $\boldsymbol{z}_{\mathcal{C}}$ can be calculated from (7n); their values depend on the cycle basis that is used. We can verify that the proposed solution indeed satisfies the constraints of MIP_{var}, which proves this lemma. $\square$

3.5.4 Well-posedness.

Consider a fixed-time schedule for which signal group i has K_i effective green intervals and define $\mathcal{K}_i := \{1, \ldots, K_i\}$. We prove that each real-valued design variable $\boldsymbol{\gamma}(\varepsilon_1, \varepsilon_2)$ and its associated integer $z(\varepsilon_1, \varepsilon_2)$ are defined unambiguously, i.e., only one value for $\boldsymbol{\gamma}(\varepsilon_1, \varepsilon_2)$ (and only one value for $z(\varepsilon_1, \varepsilon_2)$) can be associated with this fixed-time schedule. In Section 2.3.3 we have already proved that the variables $\boldsymbol{\gamma}(\varepsilon_1, \varepsilon_2)$ and the associated integers $z(\varepsilon_1, \varepsilon_2)$ are uniquely defined for the variables that are associated with effective green interval $k \in \mathcal{K}_i$ of signal group $i \in \mathcal{S}$, effective red interval $k \in \mathcal{K}_i$ of signal group $i \in \mathcal{S}$, and the clearance time between effective green interval $k \in \mathcal{K}_i$ of signal group $i \in \mathcal{S}$ and conflicting effective green interval $k' \in \mathcal{K}_j$ of signal group $j \in \mathcal{S}$. All other variables $\boldsymbol{\gamma}(\varepsilon_1, \varepsilon_2)$ can only attain one value for this fixed-time schedule and are therefore also defined unambiguously. We give these values below. For each signal group $i \in \mathcal{S}$ and each effective green interval $k \in \overline{\mathcal{K}}_i \setminus \mathcal{K}_i$, the effective green time and the preceding effective red time are forced to be zero by constraints (7j)–(7l).

$$\boldsymbol{\gamma}(\textcircled{i}_k, \colorbox{gray}{i}_k) := 0,$$

and

$$\boldsymbol{\gamma}(\textcircled{i}_{k-1}, \boldsymbol{\textcircled{i}}_k) := 0.$$

We use the definition of Ψ_I as defined in Section 3.1, i.e.,

$$\Psi_I := \{\{(i,k),(j,k')\} \mid \{i,j\} \in \Psi_{\mathcal{S}}, k \in \overline{\mathcal{K}}_i, k' \in \overline{\mathcal{K}}_j\}.$$

For each pair of conflicting effective green intervals $\{(i,k),(j,k')\} \in \Psi_I$ with $k \notin \mathcal{K}_i$ and $k' \in \mathcal{K}_j$ we have:

$$\boldsymbol{\gamma}(\boldsymbol{\textcircled{i}}_k, \textcircled{j}_{k'}) := \boldsymbol{\gamma}(\boldsymbol{\textcircled{i}}_{K_i}, \textcircled{j}_{k'}),$$

which is forced by (7s). Furthermore, for each pair of conflicting effective green intervals $\{(i,k),(j,k')\} \in \Psi_I$ with $k \in \mathcal{K}_i$ and $k' \notin \mathcal{K}_j$ we have:

$$\boldsymbol{\gamma}(\boldsymbol{\textcircled{i}}_k, \textcircled{j}_{k'}) := \boldsymbol{\gamma}(\boldsymbol{\textcircled{i}}_k, \textcircled{j}_{K_j}) + \boldsymbol{\gamma}(\textcircled{j}_{K_j}, \boldsymbol{\textcircled{j}}_{K_j}),$$

which is forced by (7t), and for each pair of conflicting effective green intervals $\{(i,k),(j,k')\} \in \Psi_I$ with $k \notin \mathcal{K}_i$ and $k' \notin \mathcal{K}_j$ we have:

$$\boldsymbol{\gamma}(\boldsymbol{\textcircled{i}}_k, \textcircled{j}_{k'}) := \boldsymbol{\gamma}(\boldsymbol{\textcircled{i}}_{K_i}, \textcircled{j}_{K_j}) + \boldsymbol{\gamma}(\textcircled{j}_{K_j}, \boldsymbol{\textcircled{j}}_{K_j})$$

which is forced by (7s) and (7t). Therefore, all variables $\boldsymbol{\gamma}(\varepsilon_1, \varepsilon_2)$ and their associated integers $z(\varepsilon_1, \varepsilon_2)$ are defined unambiguously.

4 Numerical Results

In this section we perform an extensive numerical study. To this end, we use the thirteen real-life intersections from (Fleuren and Lefeber 2016a). These thirteen intersections are categorized by size: small (S), medium (M) and large (L). The set of small intersections consists of five intersections with each only six signal groups. The set of medium intersections consists of four intersections that each have between 9 and 15 signal groups. The set of large intersections consists of four intersections with 27 to 29 signal groups.

For each of these real-life intersections we consider 36 different optimization problems. For all these optimization problems we fix the minimum number of effective green intervals $\underline{K}_i$, $i \in \mathcal{S}$ to one, i.e., each signal group must have at least one effective green interval. We do however vary the maximum number of effective green intervals $\overline{K}_i$, $i \in \mathcal{S}$; we consider three variants: we allow the zero, two or four signal groups with the largest loads $\rho_i^{\mathrm{SG}} := \max_{q \in \mathcal{Q}_i} \rho_q$ to have an additional effective green interval.

We also distinguish between three objective functions: minimizing the period duration, maximizing the capacity of the intersection and minimizing the average delay that road users experience at this intersection. When minimizing the period duration, we consider six different scalings of the arrival rates λ_q: 1.00, 1.05, 1.10, 1.15, 1.20 and 1.25. Together with the three different values for $\overline{K}_i$, $i \in \mathcal{S}$

this constitutes 18 optimization problems that minimize the period duration. When maximizing the capacity of the intersection, we vary only the values for $\overline{K}_i$, $i \in \mathcal{S}$ and, therefore, three optimization problems maximize the capacity of the intersection. The remaining 15 optimization problems minimize the average delay that road users experience at the intersection. For each of these optimization problems we fix the period duration. Let $T^{\min}$ be the minimum period duration of any feasible fixed-time schedule for which each signal group has a single effective green interval; we can obtain $T^{\min}$ by minimizing the period duration with $\underline{K}_i$ and $\overline{K}_i$ both equal to one for each signal group $i \in \mathcal{S}$. We consider five different period durations, which are all scalings of the minimum period duration $T^{\min}$: $1.1T^{\min}, 1.2T^{\min}, 1.3T^{\min}, 1.4T^{\min}, 1.5T^{\min}$. Together with the three variants for the maximum number of effective green intervals this constitutes 18 optimization problems for each of the thirteen intersection.

We formulate each of the optimization problems that minimizes the delay as a mixed-integer linear programming problem. We do so as follows. We approximate the delay d_q, $q \in \mathcal{Q}$ with the (extended version of the) formula of Van den Broek, see Section 2.4; in Section 2.4 we have extended this formula in a straightforward manner to allow signal groups to have multiple effective green intervals, i.e., $\overline{K}_i > 1$. Recall that the average delay equals:

$$D = \sum_{i \in \mathcal{S}} \sum_{q \in \mathcal{Q}_i} w_q d_q = \sum_{i \in \mathcal{S}} d_i,$$

where d_i is the contribution of signal group i to the average delay, which equals:

$$d_i = \sum_{q \in \mathcal{Q}_i} w_q d_q.$$

Since we minimize the average delay that road users experience at the intersection, we take w_q proportional to the arrival rate λ_q, i.e., $w_q = \lambda_q/\Lambda$, where $\Lambda := \sum_{q \in \mathcal{Q}} \lambda_q$ is the total arrival rate at the intersection.

We approximate d_i with piecewise linear functions. We distinguish between two cases: $\overline{K}_i = 1$ and $\overline{K}_i > 1$. Consider the case that $\overline{K}_i = 1$. We then introduce an auxiliary variable $\boldsymbol{d}_i$, which represents d_i. Let the period duration be fixed to T seconds. The delay d_i is then only a function of the design variable $\boldsymbol{\gamma}(\textcircled{i}_1, \textcircled{i}_1)$. We approximate d_i by a piecewise linear function. To this end, we include inequalities of the form: $\boldsymbol{d}_i \geq a\boldsymbol{\gamma}(\textcircled{i}_1, \textcircled{i}_1) + b$. We define this piecewise linear function as follows. Note that each feasible effective red time $\boldsymbol{\gamma}(\textcircled{i}_1, \textcircled{i}_1)T$ must be included in the following interval:

$$[\lfloor \max\{\underline{r}_i, T - \overline{g}_i\} \rfloor, \lceil \min\{\overline{r}_i, T - \underline{g}_i\} \rceil] \cap [0, T - \max_{q \in \mathcal{Q}_i} \rho_q T).$$

We obtain the approximated delay for each integral-valued effective red time $\boldsymbol{\gamma}(\textcircled{i}_1, \textcircled{i}_1)T$ that is included in this interval; we approximate the delay linearly between each two such subsequent points.

When signal group $i \in \mathcal{S}$ possibly receives multiple effective green intervals, i.e., $\overline{K}_i > 1$, then we break d_i into $\overline{K}_i + 1$ different term, see also Section 2.4:

$$d_i = d_{i,1}^{\text{det}} + \ldots + d_{i,\overline{K}_i}^{\text{det}} + d_i^{\text{stoch}},$$

where

$$d_{i,k}^{\text{det}} = \sum_{q \in \mathcal{Q}_i} \frac{w_q \boldsymbol{\gamma}(\colorbox{gray}{\textcircled{i}}_k, \textcircled{i}_k)^2}{2\boldsymbol{T}'(1-\rho_q)},$$
$$d_i^{\text{stoch}} = \sum_{q \in \mathcal{Q}_i} w_q d_q^{\text{stoch}}.$$

We approximate each of these $\overline{K}_i + 1$ different terms with its own piecewise linear function. The deterministic delay term $d_{i,k}^{\text{det}}$, $k = 1, \ldots, \overline{K}_i$ is only a function of the design variable $\boldsymbol{\gamma}(\colorbox{gray}{\textcircled{i}}_k, \textcircled{i}_k)$. We approximate this deterministic delay term with a piecewise linear function by including an auxiliary variable $\boldsymbol{d}_{i,k}^{\text{det}}$, which represents $d_{i,k}^{\text{det}}$, in the optimization problem and including linear constraints of the form: $\boldsymbol{d}_{i,k}^{\text{det}} \geq a\boldsymbol{\gamma}(\colorbox{gray}{\textcircled{i}}_k, \textcircled{i}_k) + b$. We define this piecewise linear function as follows. Note that each feasible effective red time $\boldsymbol{\gamma}(\colorbox{gray}{\textcircled{i}}_k, \textcircled{i}_k)T$ must satisfy the following inequalities:

$$\boldsymbol{\gamma}(\colorbox{gray}{\textcircled{i}}_k, \textcircled{i}_k)T \geq \lfloor \max\{\underline{r}_i, T - \overline{K}_i\overline{g}_i - (\overline{K}_i - 1)\overline{r}_i\} \rfloor,$$
$$\boldsymbol{\gamma}(\colorbox{gray}{\textcircled{i}}_k, \textcircled{i}_k)T \leq \lceil \min\{\overline{r}_i, T - k'\underline{g}_i - (k'-1)\underline{r}_i, T - \max_{q \in \mathcal{Q}_i} \rho_l T - (k'-1)\underline{r}_i\} \rceil.$$

where $k' := \max\{k, \underline{K}_i\}$. We obtain the approximated delay for each integral-valued effective red time $\boldsymbol{\gamma}(\colorbox{gray}{\textcircled{i}}_k, \textcircled{i}_k)T$ that satisfies the above two inequalities; we approximate the delay linearly between each two such subsequent points. Furthermore, we include a lower bound of zero on the variable $\boldsymbol{d}_{i,k}^{\text{det}}$, which prevents $\boldsymbol{d}_{i,k}^{\text{det}}$ from becoming negative when $\boldsymbol{b}_{i,k} = 0$.

The stochastic delay term d_i^{stoch} is a function of the total effective red fraction: $r_i' = \boldsymbol{\gamma}(\colorbox{gray}{\textcircled{i}}_1, \textcircled{i}_1) + \ldots, +\boldsymbol{\gamma}(\colorbox{gray}{\textcircled{i}}_{\overline{K}_i}, \textcircled{i}_{\overline{K}_i})$, see Section 2.4. We define this piecewise linear function as follows. Note that each feasible total effective red time $r_i'T$ must be included in the following interval:

$$\Big[\lfloor \max\{\underline{K}_i\underline{r}_i, T - \overline{K}_i\overline{g}_i\} \rfloor, \lceil \min\{\overline{K}_i\overline{r}_i, T - \underline{K}_i\underline{g}_i\} \rceil \Big] \cap [0, T - \max_{q \in \mathcal{Q}_i} \rho_l T).$$

We obtain the approximated delay for each integral-valued total effective red time $r_i'T$ that is included in this interval; we approximate the delay linearly between each two such subsequent points.

In Table 3 and Table 4 we give the results for the test cases that minimize the period duration. In Table 3 we give the objective values of these test cases, and in Table 4 we give the improvement in the objective value when we allow several signal groups to have multiple effective green intervals. It appears that for our test cases, allowing signal groups to have multiple effective green intervals did not decrease the minimum period duration. An exception is the intersection S4; for this intersection the minimum period duration decreased (up to) 2.9 percent.

In Table 5 we give the objective values for the test cases that maximize the growth factor of the arrival rates. Furthermore, in this table we give the improvement in the objective value when we allow several signal groups to have multiple effective green intervals; again only an improvement (of 1.16 percent) is observed for the intersection S4.

In Table 6 and Table 7 we give the results for the test cases that minimize the average delay that road users experience. In Table 6 we give the objective values of these test cases, and in Table 4 we give the improvement in the objective value when we allow several signal groups to have multiple effective green intervals. When minimizing the delay that road users experience, then a substantial improvement (of often several percent) can be made by allowing multiple signal groups to have multiple effective green intervals. Note that the delays are relatively small at the larger period durations (of for example a period duration that equals $1.5T^{\min}$). For these larger period durations, even smaller delays can be achieved by allowing several signal groups to have multiple effective green intervals; at these larger period duration, there may be more freedom to schedule the effective green intervals and, as a consequence, it may for these larger period durations be easier to fit in additional effective green intervals in the fixed-time schedule. For this test set we allow at most four of the signal groups with the largest loads to have multiple effective green intervals. This already results in substantial improvements. By using either expertise or a trail-and-error approach to determine which signal groups should receive multiple effective green intervals, an even larger improvement may be possible.

In Table 8 we have shown the computation times for the numerical study. We have obtained these computation times for three different solvers: CPLEX version 12.6.1.0 (International Business Machines Corp 2015), GUROBI version 6.0.5 (Gurobi Optimization, Inc. 2015) and SCIP version 3.2.0 (Achterberg 2009). To obtain these computation times, we have solved each optimization problem 10 times and obtained the average computation times over these 10 runs. The results are obtained on a computer with specifications: Intel i5-4300U CPU @1.90GHZ with 16.0GB of RAM. These computation times are increasing in the number of signal groups that is allowed to have an additional effective green interval. This increase in computation times is especially large when minimizing the period duration with the solver GUROBI. The solver GUROBI especially had difficulty with the following test case: minimizing the period duration at intersection L3, when the arrival rates are scaled with a factor 1.2, and four signal groups are allowed to have an additional effective green interval. For this particular test case GUROBI was unable to finish the optimization within 30 minutes; we have solved this optimization problem only once (instead of ten times). For this numerical study, it seems that the solver CPLEX is best able to handle an increase in the number of additional effective green intervals

5 Conclusions

In this paper we have extended the optimization framework that was proposed in (Fleuren and Lefeber 2016b). This extension allows the optimization over the number of effective green intervals of each signal groups. First, in Section 2 we have considered the number of effective green intervals of each signal group to be a fixed and given value. In that section we have formulated an optimization problem to simultaneously optimize: the period duration of the fixed-time schedule, when each of the effective green intervals starts, and when these effective green interval end. The proposed optimization formulation then closely resembles the optimization problem that was proposed in (Fleuren and Lefeber 2016b); however, in contrast to that paper, we do allow signal groups to have multiple effective green intervals. Possible objective functions of the optimization framework are: minimizing the period duration of the fixed-time schedule, maximizing the capacity of the intersection, and minimizing the average delay that road users experience at the intersection. One of the differences with (Fleuren and Lefeber 2016b) is

Scaling	1.00			1.05			1.10			1.15			1.20			1.25		
Ad. inter.	0	2	4	0	2	4	0	2	4	0	2	4	0	2	4	0	2	4
S1	35.9	35.9	35.9	39.3	39.3	39.3	43.5	43.5	43.5	48.7	48.7	48.7	55.3	55.3	55.3	64.0	64.0	64.0
S2	30.0	30.0	30.0	30.0	30.0	30.0	30.0	30.0	30.0	30.0	30.0	30.0	30.8	30.8	30.8	34.3	34.3	34.3
S3	39.1	39.1	39.1	40.7	40.7	40.7	43.1	43.1	43.1	45.7	45.7	45.7	48.7	48.7	48.7	52.0	52.0	52.0
S4	65.8	65.8	65.8	69.0	69.0	69.0	73.0	73.0	72.0	76.8	76.8	74.6	81.2	81.2	78.8	86.2	86.2	83.7
S5	64.1	64.1	64.1	67.5	67.5	67.5	71.3	71.3	71.3	75.5	75.5	75.5	82.9	82.9	82.9	92.4	92.4	92.4
M1	43.7	43.7	43.7	45.6	45.6	45.6	47.7	47.7	47.7	50.0	50.0	50.0	56.3	56.3	56.3	67.9	67.9	67.9
M2	45.5	45.5	45.5	48.1	48.1	48.1	52.2	52.2	52.2	57.2	57.2	57.2	63.2	63.2	63.2	70.5	70.5	70.5
M3	141.0	141.0	141.0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
M4	80.0	80.0	80.0	85.8	85.8	85.8	92.5	92.5	92.5	100.4	100.4	100.4	109.7	109.7	109.7	∞	∞	∞
L1	74.7	74.7	74.7	78.1	78.1	78.1	81.8	81.8	81.8	86.4	86.4	86.4	∞	∞	∞	∞	∞	∞
L2	83.6	83.6	83.6	89.8	89.8	89.8	97.1	97.1	97.1	105.6	105.6	105.6	115.7	115.7	115.7	∞	∞	∞
L3	74.6	74.6	74.6	80.3	80.3	80.3	84.6	84.6	84.6	92.3	92.3	92.3	103.8	103.8	103.8	118.5	118.5	118.5
L4	71.6	71.6	71.6	74.7	74.7	74.7	80.5	80.5	80.5	89.0	89.0	89.0	99.4	99.4	99.4	112.7	112.7	112.7

Table 3: The objective values for each of the $13 \times 6 \times 3$ test cases in the numerical study that minimizes the period duration. The first column indicates which intersection is considered, the first row indicates which scaling of the arrival rates is considered, and the second row indicates how many signal groups are allowed to have an additional effective green interval (Ad. inter.). When this additional number of effective green intervals equals $k = 0, 2, 4$ then the k signal groups with the largest loads are allowed to have an additional effective green interval; for each of these signal groups, the optimization decides whether this signal group should have one or two effective green intervals. This table contains infinite values; these infinite values indicate that the corresponding MILP problems are infeasible.

scaling	1.00		1.05		1.10		1.15		1.20		1.25	
Ad. inter.	2	4	2	4	2	4	2	4	2	4	2	4
S1	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
S2	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
S3	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
S4	0.0%	0.0%	0.0%	0.0%	0.0%	-1.3%	0.0%	-2.8%	0.0%	-2.9%	0.0%	-2.9%
S5	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
M1	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
M2	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
M3	0.0%	-	-	-	-	-	-	-	-	-	-	-
M4	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-	-
L1	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-	-	-	-
L2	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-	-
L3	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
L4	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%

Table 4: Decrease in the objective values when minimizing the period duration and allowing some signal groups to have an additional effective green interval; these improvements are defined with respect to the case that each signal group has only one effective green interval. The first column indicates which intersection is considered, the first row indicates which scaling of the arrival rates is considered, and the second row indicates how many signal groups are allowed to have an additional effective green interval (Ad. inter.). If this additional number of effective green intervals equals $k = 2, 4$, then the k signal groups with the largest loads are allowed to have an additional effective green interval; for each of these signal groups, the optimization decides whether this signal group should have one or two effective green intervals.

Ad. inter.	0	2	4
S1	1.39	1.39 (0.00%)	1.39 (0.00%)
S2	1.56	1.56 (0.00%)	1.56 (0.00%)
S3	1.66	1.66 (0.00%)	1.66 (0.00%)
S4	1.48	1.48 (0.00%)	1.50 (1.16%)
S5	1.35	1.35 (0.00%)	1.35 (0.00%)
M1	1.35	1.35 (0.00%)	1.35 (0.00%)
M2	1.36	1.36 (0.00%)	1.36 (0.00%)
M3	1.02	1.02 (0.00%)	1.02 (0.00%)
M4	1.25	1.25 (0.00%)	1.25 (0.00%)
L1	1.20	1.20 (0.00%)	1.20 (0.00%)
L2	1.21	1.21 (0.00%)	1.21 (0.00%)
L3	1.25	1.25 (0.00%)	1.25 (0.00%)
L4	1.27	1.27 (0.00%)	1.27 (0.00%)

Table 5: The objective values for each of the 13×3 test cases in the numerical study that maximizes the growth factor of the arrival rates that is sustainable. In this table, we also visualize (between brackets) the increase in the objective value with respect to the case that each signal group has only one effective green interval. The first column indicates which intersection is considered, and the first row indicates how many signal groups are allowed to have an additional effective green interval (Ad. inter.). When this additional number of effective green intervals equals $k = 0, 2, 4$ then the k signal groups with the largest loads are allowed to have an additional effective green interval; for each of these signal groups, the optimization decides whether this signal group should have one or two effective green intervals.

the objective function when minimizing the delay that road users experience at the intersection. In that paper, we could compute the delay d_q that road users experience at a queue $q \in \mathcal{Q}$ with the formulae of for example (Miller 1963; van den Broek et al. 2006; Webster 1958). However, all these formulae assume that a signal group receives only one effective green interval. Therefore, we have extended these approximations, in a straightforward manner, to allow for multiple effective green intervals.

Subsequently, in Section 3 we have considered the number of effective green intervals of each signal group to be a design variable and formulated an optimization problem to simultaneously optimize: the period duration of the fixed-time schedule, the number of effective green intervals of each signal group, when each of these effective green intervals starts, and when these effective green intervals end. This optimization formulation uses binary variables to optimize the number of effective green intervals of each signal group. Each such binary variable is used to switch on (or off) a specific effective green interval; when this binary equals zero, then this effective green interval (and its preceding effective red time) is forced to have a duration of zero seconds and, as a consequence, this effective green intervals then practically does not exist.

Finally, in Section 4 we have performed an extensive numerical case study. For this numerical study, we have concluded that allowing several signal groups to have multiple effective green intervals had little (or no) effect when minimizing the period duration and also had little (to no) effect when maximizing the capacity of the intersection. However, results from the numerical case study indicate that the average delay that road users experience may substantially decrease by allowing signal groups

scaling	1.1			1.2			1.3			1.4			1.5		
Ad. inter.	0	2	4	0	2	4	0	2	4	0	2	4	0	2	4
S1	22.71	22.71	22.71	17.13	17.13	17.13	15.16	15.16	15.16	14.20	14.20	14.20	13.75	13.75	13.72
S2	14.98	14.98	14.98	14.61	14.61	14.61	14.38	14.38	14.38	14.31	14.31	14.31	14.50	14.50	14.50
S3	47.93	47.93	47.94	31.72	31.72	31.72	26.58	26.58	26.58	24.55	24.55	24.55	23.62	23.62	23.62
S4	37.60	37.60	30.80	27.26	27.26	24.80	24.94	24.94	23.35	24.33	24.33	22.36	24.45	23.74	21.36
S5	44.33	44.33	44.33	34.09	34.09	34.09	31.80	31.80	31.80	30.91	30.91	30.91	30.89	30.89	30.89
M1	34.17	34.17	33.53	25.57	25.57	24.85	23.70	23.70	22.98	22.79	22.79	22.08	22.26	22.26	21.59
M2	41.67	41.67	40.70	29.73	29.73	28.85	26.70	26.70	25.88	25.55	25.55	24.78	25.18	25.18	24.46
M3	86.37	86.37	79.53	64.44	64.44	61.65	58.27	58.27	57.54	57.16	57.16	56.66	58.10	56.38	54.51
M4	47.71	47.71	45.70	36.82	36.82	35.19	33.24	33.24	31.93	31.95	31.95	30.79	31.71	30.38	29.74
L1	40.48	40.48	39.79	38.12	37.90	36.45	38.56	38.27	36.68	39.46	39.17	37.39	40.58	40.12	37.95
L2	61.97	61.97	61.97	50.15	50.15	50.15	48.51	48.51	47.30	48.61	48.61	44.82	49.60	48.09	44.30
L3	37.61	37.61	37.61	33.05	33.05	33.05	32.33	32.33	32.33	32.67	32.40	31.49	33.46	33.09	31.53
L4	32.61	31.74	31.74	29.14	28.19	28.19	28.66	27.62	27.42	29.07	27.96	27.65	29.86	28.68	28.32

Table 6: The objective values for each of the $13 \times 5 \times 3$ *test cases in the numerical study that minimizes the average delay that road users experience. For each of these test cases, the period duration is fixed to some scaling* (> 1) *of the minimum period duration; the minimum period duration can be found in column 2 of Table 3. The first column indicates which intersection is considered, the first row indicates which scaling of the minimum period duration is considered, and the second row indicates how many signal groups are allowed to have an additional effective green interval (Ad. inter.). When this additional number of effective green intervals equals* $k = 0, 2, 4$ *then the* k *signal groups with the largest loads are allowed to have an additional effective green interval; for each of these signal groups, the optimization decides whether this signal group should have one or two effective green intervals.*

scaling	1.1		1.2		1.3		1.4		1.5	
Ad. inter.	2	4	2	4	2	4	2	4	2	4
S1	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-0.3%
S2	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
S3	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
S4	0.0%	-18.1%	0.0%	-9.0%	0.0%	-6.4%	0.0%	-8.1%	-2.9%	-12.7%
S5	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
M1	0.0%	-1.9%	0.0%	-2.8%	0.0%	-3.1%	0.0%	-3.1%	0.0%	-3.0%
M2	0.0%	-2.3%	0.0%	-3.0%	0.0%	-3.1%	0.0%	-3.0%	0.0%	-2.9%
M3	0.0%	-7.9%	0.0%	-4.3%	0.0%	-1.2%	0.0%	-0.9%	-2.9%	-6.2%
M4	0.0%	-4.2%	0.0%	-4.4%	0.0%	-4.0%	0.0%	-3.6%	-4.2%	-6.2%
L1	0.0%	-1.7%	-0.6%	-4.4%	-0.7%	-4.9%	-0.7%	-5.3%	-1.1%	-6.5%
L2	0.0%	0.0%	0.0%	0.0%	0.0%	-2.5%	0.0%	-7.8%	-3.0%	-10.7%
L3	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-0.8%	-3.6%	-1.1%	-5.8%
L4	-2.7%	-2.7%	-3.3%	-3.3%	-3.6%	-4.3%	-3.8%	-4.9%	-3.9%	-5.2%

Table 7: Decrease in the objective values when minimizing the average delay that road users experience and allowing some signal groups to have an additional effective green interval; these improvements are defined with respect to the case that each signal group has only one effective green interval. For each of these test cases, the period duration is fixed to some scaling (> 1) of the minimum period duration; the minimum period duration can be found in column 2 of Table 3. The first column indicates which intersection is considered, the first row indicates which scaling of the minimum period duration is considered, and the second row indicates how many signal groups are allowed to have an additional effective green interval (Ad. inter.). If this additional number of effective green intervals equals $k = 2, 4$, then the k signal groups with the largest loads are allowed to have an additional effective green interval; for each of these signal groups, the optimization decides whether this signal group should have one or two effective green intervals.

solver	intersections	$\min T$			$\max \beta$			$\min D$		
		0	2	4	0	2	4	0	2	4
CPLEX	Small	0.07	0.09	0.10	0.08	0.09	0.14	0.09	0.11	0.25
	Medium	0.08	0.12	0.18	0.10	0.14	0.26	0.18	0.38	0.84
	Large	0.21	0.43	1.00	0.30	0.33	1.17	0.88	2.80	9.05
GUROBI	Small	0.00	0.01	0.04	0.00	0.01	0.06	0.01	0.02	0.12
	Medium	0.02	0.10	0.54	0.02	0.08	0.54	0.07	0.33	1.02
	Large	0.24	3.21	32.64	0.24	0.66	1.68	0.93	3.67	15.78
SCIP	Small	0.02	0.04	0.09	0.02	0.07	0.23	0.03	0.11	0.38
	Medium	0.06	0.18	0.52	0.10	0.37	1.30	0.19	0.65	2.78
	Large	0.60	2.77	10.82	1.07	8.51	30.01	2.96	11.37	67.17

Table 8: The (geometric) average computation times (in seconds) needed by the approach that is proposed in this paper. We have distinguished between three types of optimization problems ($\min T$, $\max \beta$ and $\min D$), three types of intersections (small, medium and large), and three types of solvers (CPLEX 12.6.1.0, GUROBI 6.0.5. and SCIP 3.2.0). Furthermore, we have varied the number of signal groups that is allowed to have an additional effective green interval, see the second row of this table. If the number of signal groups that is allowed to have an additional effective green interval equals $k = 0, 2, 4$, then the k signal groups with the largest loads are allowed to have an additional effective green interval; for each of these signal groups, the optimization decides whether this signal group should have one or two effective green intervals.

to have multiple effective green intervals. For our test case, this decrease was often several percent. For these test cases we optimize the number of effective green intervals for at most four signal groups (which are the four signal groups with the largest loads); each of these signal groups is allowed to have an additional effective green interval. All other signal groups have a single effective green interval. An even larger decrease is probably possible when we use either expertise or a trail-and-error approach to determine which signal groups should receive multiple effective green intervals.

In this paper we have extended the approximate formulae for the delay that road users experience at a traffic light under fixed-time control, in a straight forward manner, to allow each signal group to have multiple realizations. We recommend to assess the quality of the proposed extended formulae. In what situations does this approximation perform very well? Are there situations for which its performance is insufficient? Perhaps this insight results in a better approximation. Recall however that the proposed extension has the following desirable properties. First, this approximation is convex in the variables $\boldsymbol{T}'$ and $\boldsymbol{\gamma}$. Second, we can break this extended formula into different terms. Each of these terms depends only one one design variable (or on a sum of design variables). Hence, we need relatively few piece-wise linear segments to approximate this formula by using piece-wise linear functions. As a consequence, we can formulate the minimization of the average weighted delay that road users experience as a mixed-integer programming problems. We also recommend to experiment with different cycle basis to further reduce the calculation times. Furthermore, it is possible to extend this optimization framework so that the lane markings can also be optimized; this is the topic of a paper to come.

A Proof of Lemma 1

To prove Lemma 1 we first introduce the formal definition of an integral cycle basis. This definition represents each cycle $\mathcal{C}$ with a vector C, which we call the *cycle-arc incidence vector*. For each arc $a \in A$ we have:

$$C(a) = \begin{cases} +1 & \text{if } a \in \mathcal{C}^+, \\ 0 & \text{if } a \notin \mathcal{C}, \\ -1 & \text{if } a \in \mathcal{C}^-. \end{cases}$$

An integral cycle basis is defined as follows:

Definition 8 (Integral cycle basis)**.** *An integral cycle basis is a set of cycles* $\{\mathcal{C}_1, \ldots, \mathcal{C}_d\}$ *such that any cycle* $\mathcal{C}$ *in* G *can be written as:*

$$C = \alpha_1 C_1 + \ldots + \alpha_d C_d,$$

where $\alpha_1, \ldots, \alpha_d \in \mathbb{Z}$.

Lemma 1. *Let* $\mathcal{F}'$ *be a spanning forest of the conflict graph* G' *and let* $\mathcal{F}$ *be the spanning forest of the constraint graph* G *that is calculated with* (2)*. Define* $\mathcal{B} = \{\mathcal{C}_1, \ldots, \mathcal{C}_d\}$ *to be the SFCB of graph* G *that is defined by spanning forest* $\mathcal{F}$*, and let* $\mathcal{B}'$ *be the set of cycles that is obtained from* $\mathcal{B}$ *when, for each arc* $(\colorbox{gray}{\textcircled{i}}_k, \textcircled{j}_{k'}) \notin \mathcal{F}$, $i < j$, *we replace the cycle* $\mathcal{C}_{\mathcal{F}}((\colorbox{gray}{\textcircled{j}}_{k'}, \textcircled{i}_k))$ *by the cycle:*

$$\mathcal{C} = \mathcal{C}^+ = \{(\textcircled{i}_k, \colorbox{gray}{\textcircled{i}}_k), (\colorbox{gray}{\textcircled{i}}_k, \textcircled{j}_{k'}), (\textcircled{j}_{k'}, \colorbox{gray}{\textcircled{j}}_{k'}), (\colorbox{gray}{\textcircled{j}}_{k'}, \textcircled{i}_k)\}.$$

The set $\mathcal{B}'$ *is an integral cycle basis of the constraint graph* G *that includes all the cycles that are associated with the circuital constraints* (1i) *and* (1j)*.*

Proof. The SFCB $\mathcal{B}$ that is obtained from the spanning forest $\mathcal{F}$ includes the cycles that are associated with the circuital constraint (1i); each arc $(\colorbox{gray}{\textcircled{i}}_{K_i}, \textcircled{i}_1)$ is not included in the spanning forest $\mathcal{F}$ and results in one such cycle. However, this SFCB does not include all of the cycles associated with the circuital constraint (1j). Consider a conflict $\{(i,k),(j,k')\} \in \Psi_I$ and assume w.l.o.g. that $i < j$. The cycle that is associated with the circuital constraint (1j) of the conflict $\{(i,k),(j,k')\} \in \Psi_I$ is the cycle $\mathcal{C} = \mathcal{C}^+ = \{(\textcircled{i}_k, \colorbox{gray}{\textcircled{i}}_k), (\colorbox{gray}{\textcircled{i}}_k, \textcircled{j}_{k'}), (\textcircled{j}_{k'}, \colorbox{gray}{\textcircled{j}}_{k'}), (\colorbox{gray}{\textcircled{j}}_{k'}, \textcircled{i}_k)\}$. This cycle is included in the SFCB if and only if $(\colorbox{gray}{\textcircled{i}}_k, \textcircled{j}_{k'}) \in \mathcal{F}$; this cycle is then the cycle $\mathcal{C}_{\mathcal{F}}((\colorbox{gray}{\textcircled{j}}_{k'}, \textcircled{i}_k))$. Consider a conflict $\{(i,k),(j,k')\} \in \Psi_I$ with $i < j$ for which $(\colorbox{gray}{\textcircled{i}}_k, \textcircled{j}_{k'}) \notin \mathcal{F}$. For each such conflict we replace the cycle $\mathcal{C}_{\mathcal{F}}((\colorbox{gray}{\textcircled{j}}_{k'}, \textcircled{i}_k))$ by the cycle $\mathcal{C} = \mathcal{C}^+ = \{(\textcircled{i}_k, \colorbox{gray}{\textcircled{i}}_k), (\colorbox{gray}{\textcircled{i}}_k, \textcircled{j}_{k'}), (\textcircled{j}_{k'}, \colorbox{gray}{\textcircled{j}}_{k'}), (\colorbox{gray}{\textcircled{j}}_{k'}, \textcircled{i}_k)\}$. The resulting set of cycles $\mathcal{B}'$ includes all the cycles associated with the circuital constraints (1i)–(1j). Furthermore, this set of cycles is an integral cycle basis; we prove this via induction in the remaining part of this proof.

Let $\mathcal{B}_k$ be a set of cycles that is obtained when we have done the replacement (that is described above) for k conflicts. We use the induction hypothesis that $\mathcal{B}_k$ is an integral cycle basis and prove that the set $\mathcal{B}_{k+1}$ then also is an integral cycle basis. Note that $\mathcal{B}_0 := \mathcal{B}$ is a SFCB. Therefore, $\mathcal{B}_0$ is by definition an integral cycle basis.

The set of cycles $\mathcal{B}_{k+1}$ can be obtained from a set $\mathcal{B}_k := \{\mathcal{C}_1, \ldots, \mathcal{C}_d\}$ by performing one additional replacement. Let $\{(i,k),(j,k')\} \in \Psi_I$ with $i < j$ be the pair of conflicting effective green intervals for which we perform this additional replacement. For such a conflict it holds that $(\textcircled{i}_k, \boldsymbol{\textcircled{j}}_{k'}) \notin \mathcal{F}$. As a consequence, from the definition of $\mathcal{B}$ if follows that $\mathcal{C}_{\mathcal{F}}((\boldsymbol{\textcircled{i}}_k, \textcircled{j}_{k'})) \in \mathcal{B}$ and $\mathcal{C}_{\mathcal{F}}((\boldsymbol{\textcircled{j}}_{k'}, \textcircled{i}_k)) \in \mathcal{B}$. The previous $k-1$ replacements did not affect these cycles and, as a consequence, $\mathcal{C}_{\mathcal{F}}((\boldsymbol{\textcircled{i}}_k, \textcircled{j}_{k'})) \in \mathcal{B}_k$ and $\mathcal{C}_{\mathcal{F}}((\boldsymbol{\textcircled{j}}_{k'}, \textcircled{i}_k)) \in \mathcal{B}_k$. Assume w.l.o.g. that $\mathcal{C}_{d-1} := \mathcal{C}_{\mathcal{F}}((\boldsymbol{\textcircled{i}}_k, \textcircled{j}_{k'}))$ and that $\mathcal{C}_d := \mathcal{C}_{\mathcal{F}}((\boldsymbol{\textcircled{j}}_{k'}, \textcircled{i}_k))$. We replace the cycle $\mathcal{C}_d$. From the induction hypothesis it follows that $\mathcal{B}_k := \{\mathcal{C}_1, \ldots, \mathcal{C}_d\}$ is an integral cycle basis. Therefore, for each cycle $\mathcal{C}$ in the constraint graph G we can find $\alpha \in \mathbb{Z}^d$ such that:

$$C = \alpha_1 C_1 + \ldots + \alpha_d C_d.$$

The cycles $\mathcal{C}_{d-1}$ and $\mathcal{C}_d$ are visualized in Figure 7. Let $\mathcal{C}'_d$ be the cycle that is associated with the circuital constraint (1i) of the conflict $\{(i,k),(j,l')\} \in \Psi_I$, i.e.,

$$\mathcal{C}'_d := \mathcal{C}'^{+}_d = \{(\textcircled{i}_k, \boldsymbol{\textcircled{i}}_k), (\boldsymbol{\textcircled{i}}_k, \textcircled{j}_{k'}), (\textcircled{j}_{k'}, \boldsymbol{\textcircled{j}}_{k'}), (\boldsymbol{\textcircled{j}}_{k'}, \textcircled{i}_k)\}.$$

Note that the cycle-arc incidence vector of the cycle $\mathcal{C}'_d$ satisfies $C'_d := C_{d-1} + C_d$, see Figure 7; each arc $a \in \mathcal{C}'_d$ is used in the forward direction by either the cycle $\mathcal{C}_{d-1}$ or by the cycle $\mathcal{C}_d$ (not both) and each arc $a \notin \mathcal{C}'_d$ that is used by the cycle $\mathcal{C}_{d-1}$ ($\mathcal{C}_d$) is used by the cycle $\mathcal{C}_d$ ($\mathcal{C}_{d-1}$) in the opposite direction. Hence, for each cycle $\mathcal{C}$ in graph G we can find $\alpha \in \mathbb{Z}^d$ such that:

$$\begin{aligned} C &= \alpha_1 C_1 + \ldots + \alpha_d C_d, \\ &= \alpha_1 C_1 + \ldots + \alpha_{d-2} C_{d-2} + (\alpha_{d-1} - \alpha_d) C_{d-1} + \alpha_d C'_d, \\ &= \alpha'_1 C_1 + \ldots + \alpha'_{d-1} C_{d-1} + \alpha'_d C'_d, \end{aligned}$$

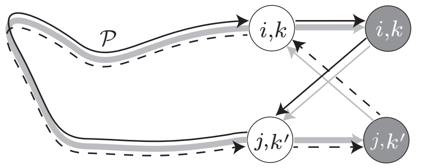
which implies that we can write each cycle C as an integral combination of the cycles in the set $\mathcal{B}_{k+1}$; this implies that $\mathcal{B}_{k+1}$ is an integral cycle basis and concludes the proof.

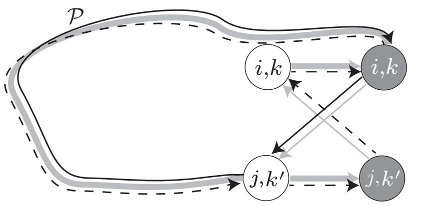
□

B Convexity of the approximation of Van den Broek extended to multiple realizations.

Define $r'_{i,k} := \boldsymbol{\gamma}(\boldsymbol{\textcircled{i}}_{k-1}, \textcircled{i}_k)$ and define $r'_i := r'_{i,1} + \ldots + r'_{i,K_i}$. When the formula of (van den Broek et al. 2006) is extended, in the straightforward manner that is described in Section 2.4, then we obtain:

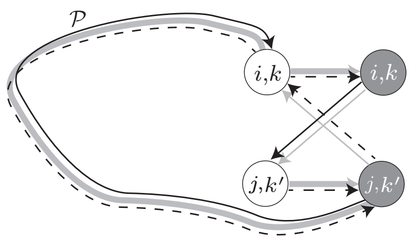
$$d_q := d_q^{\text{stoch}} + \sum_{k \in \mathcal{K}_i} d_{q,k}^{\text{det}},$$

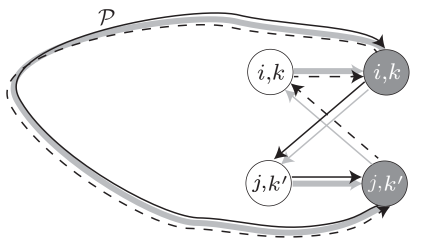




(a) The path $\mathcal{P}$ that connects the vertices i_k and $j_{k'}$

(b) The path $\mathcal{P}$ that connects the vertices i_k and $j_{k'}$





(c) The path $\mathcal{P}$ that connects the vertices i_k and $j_{k'}$

(d) The path $\mathcal{P}$ that connects the vertices i_k and $j_{k'}$

Figure 7: These figures visualize the cycles $\mathcal{C}_{d-1} := \mathcal{C}_{\mathcal{F}}((i_k, j_{k'}))$ and $\mathcal{C}_d := \mathcal{C}_{\mathcal{F}}((j_{k'}, i_k))$. The white (grey) vertex with the text i, k denotes the event i_k (i_k). The solid black lines visualize the cycle $\mathcal{C}_{d-1}$ and the dotted black line visualizes the cycle $\mathcal{C}_d$. Furthermore, the gray lines visualize the relevant arcs of graph G; the arcs that are in the spanning forest $\mathcal{F}$ are visualized in bold. The path $\mathcal{P}$ consists of the arcs in the spanning forest $\mathcal{F}$ that both the cycle $\mathcal{C}_{d-1}$ and the cycle $\mathcal{C}_d$ use (in opposite directions). We visualize four different situations for this path $\mathcal{P}$.

where

$$\begin{aligned}
d^{\text{det}}_{q,k} &:= \frac{r'^2_{i,k}}{2\boldsymbol{T}'(1-\rho_q)}, \\
d^{\text{stoch}}_{q} &:= \frac{r'_i}{2(1-\rho_q)\rho_q}\left(\frac{\sigma_q^2}{\mu_q(1-\rho_q)} + \frac{r'_i\rho_q^2\sigma_q^2}{\mu_q(1-r'_i)^2(1-r'_i-\rho_q)(1-\rho_q)}\right).
\end{aligned}$$

First, we prove that $d^{\text{det}}_{q,k}$ is convex in $r'_{i,k}$ and $\boldsymbol{T}'$ by proving that its Hessian H and its second derivative to $\boldsymbol{T}'$ is non-negative:

$$\begin{aligned}
&\frac{\partial^2 d^{\text{det}}_{q,k}}{\partial \boldsymbol{T}'^2} = \frac{r'^2_{i,k}}{\boldsymbol{T}'^3(1-\rho_q)} \geq 0, \\
&\det(H) = 0.
\end{aligned}$$

The stochastic delay term d_q^{stoch} is only a function of r_i' (not of $\boldsymbol{T}'$). Hence, we prove its convexity by proving that the second derivative to r_i' is non-negative. The stability constraint (1d), the strictly positive effective red times (1c), and circuital constraint (1j) together imply $0 < \rho_q \leq 1 - r_i' < 1$. Define α_q such that $\rho_q := \alpha_q(1 - r_i')$, where $0 < \alpha_q \leq 1$ we then have:

$$\frac{\partial^2 d_q^{\text{stoch}}}{\partial {r_i'}^2} = \frac{\alpha_q \sigma_q^2 \left(((1-\alpha_q) + r_i')^2 + 2(1-\alpha_q)^2 r_i' \right)}{\mu_q (1-\alpha_q)^3 (1 - \alpha_q + r_i')^3 (1 - r_i')^4} \geq 0.$$

The convexity of $d_{q,k}^{\text{det}}$, $k \in \mathcal{K}_i$ and the convexity of d_q^{stoch} imply the convexity of d_q.

C Queue Emptying and Convexity of the Delay Formula

Consider a signal group $i \in \mathcal{S}$ and a queue $q \in \mathcal{Q}_i$ that is controlled by signal group i. In Section 2.4 we split the delay in a deterministic delay term and a stochastic delay term. This deterministic delay term is associated with a purely deterministic and fluid-like arrival and departure process. In that section we force queue q to be emptied during each effective green interval for this purely deterministic queueing process. The resulting approximation is then convex, see Appendix B. In this section we prove that the delay at queue q is not a convex function of the real-valued design variables when we do not force that queue to be emptied during each effective green interval for this purely deterministic queueing process. First, we prove that the deterministic delay term is then not necessarily a convex function. Subsequently we prove that also the approximation of (van den Broek et al. 2006) extended to multiple realizations (as we did in Section 2.4) is not convex.

C.1 Convexity of Deterministic Term

We show that the deterministic delay term is not convex by using an example. We consider a queue $q \in \mathcal{Q}_i$ with a load of $\rho = 0.5$ and an arrival rate of $\lambda = 0.25$.

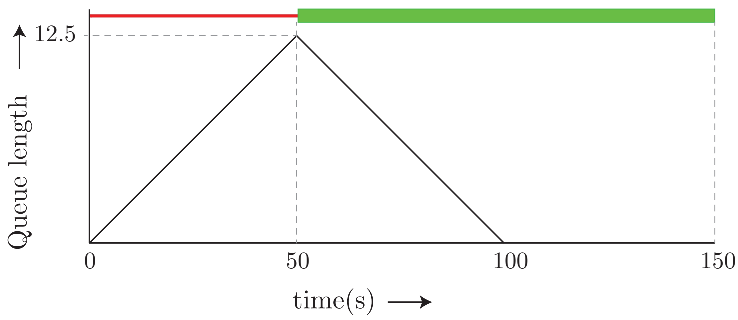
Consider a fixed-time schedule with a period duration of 150 seconds for which signal group i has two realizations ($K_i = 2$). The first effective green interval has a duration of zero seconds, and starts at 50s. The second effective green interval has a duration of 100 seconds and starts at 50 seconds, see also Figure 8a. Define the deterministic delay term d_q^{det} as:

$$d_q^{\text{det}} := \sum_{k=1}^{\mathcal{K}_i} d_{q,k}^{\text{det}}.$$

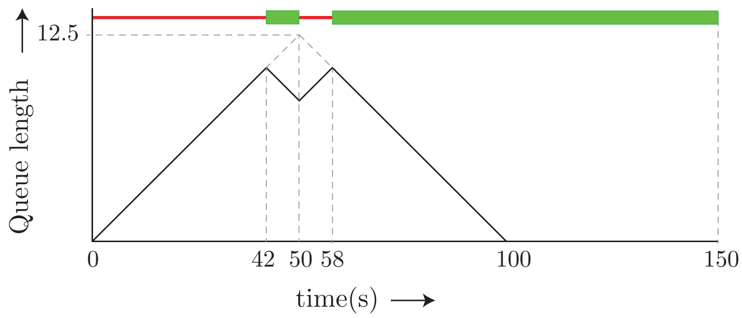
This deterministic delay term can be calculated from the average queue length, which equals $(0.5 \cdot 12.5 \cdot 100)/150 = 25/6$, by applying Little's law (Chhajed and Lowe 2008); this gives a deterministic delay term d_q^{det} of $100/6$ seconds.

Consider the similar fixed-time schedule that is visualized in Figure 8b. Its period duration is again 150 seconds and signal group i again has two realizations ($K_i = 2$). The first effective green interval has a duration of 8 seconds, and starts at 42 seconds. The second effective green interval has a duration of 92 seconds and starts at 58 seconds. For this fixed-time schedule, the deterministic delay term d_q^{det} can be calculated to be $^{1186}/_{75}$ seconds.

Now take the convex combination of the former two fixed-time schedules (each fifty percent). The resulting fixed-time schedule has a period duration of 150 seconds. The first realization of signal group i has a duration of 4 second and starts at 46 seconds. The second effective green interval has a duration of 96 seconds and starts at 54 seconds. If the deterministic delay term would be a convex function, then the deterministic delay term that is associated with this fixed-time schedule would be at most $0.5 \cdot {}^{100}/_{6} + 0.5 \cdot {}^{1186}/_{75} = {}^{1218}/_{75}$. However, similar to the previous two cases we can determine that its deterministic delay term is larger and equal to $^{1234}/_{75}$ seconds, which contradicts convexity.



(a) Signal group i is effective green during the interval $[50, 50]$ and during the interval $[50, 150]$.



(b) Signal group i is effective green during the interval $[42, 50]$ and during the interval $[58, 150]$.

Figure 8: A queue length evolution of queue $q \in \mathcal{Q}_i$ for a deterministic and fluid-like queueing process when $\lambda_q = 0.25$ and $\mu_q = 0.5$.

C.2 Convexity of the Delay

In Section 2.4 we have extended the approximation of (van den Broek et al. 2006) in a straightforward manner to the case of multiple realizations. This approximation d_q can be split into a deterministic d_q^{det} term and a stochastic delay term d_q^{stoch}. In this section we prove that this approximation is not convex whenever the queue is not forced to be emptied during each realization for the deterministic and fluid-like queueing process that is associated with the deterministic delay term. To this end, we consider the same example as in the previous section. Refer to the fixed-time schedules of Figure 8a and Figure 8b as fixed-time schedule 1 respectively fixed-time schedule 2. Furthermore, refer to their convex combination (each 50 percent) as fixed-time schedule 3. Let d_q^k, $d_q^{\text{det},k}$ and $d_q^{\text{stoch},k}$ be the approximated delay of queue q, the deterministic delay term of queue q, respectively the stochastic delay term of queue q for the fixed-time schedule k. We have already showed that:

$$0.5d_q^{\text{det},1} \quad + 0.5d_q^{\text{det},2} \quad < d_q^{\text{det},3}. \tag{9}$$

Furthermore, we can prove that the stochastic delay term is the same for these three signal groups, i.e.,

$$0.5d_q^{\text{stoch},1} + 0.5d_q^{\text{stoch},2} = d_q^{\text{stoch},3}. \tag{10}$$

The equations (9) and (10) together imply:

$$0.5d_q^1 \qquad + 0.5d_q^2 \qquad < d_q^3,$$

which proves that the approximation d_q is not convex. The equality (10) follows from the following observation. The fixed-time schedules of Figure 8a and Figure 8b (as well as each convex combination of these fixed-time schedules) have the same period duration and the same total effective red time. The stochastic delay term depends only on the total effective red time and (possibly) the period duration of the fixed-time schedule, which implies (10).

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