Technische Universiteit
Eindhoven
University of Technology

/ Department of
Mechanical Engineering
Dynamics and Control

Stabilizing State-Dependent Control of Switched Linear Hybrid Systems

M.F.M. Theunissen — 0725826

DC2016.044

Master's Thesis

Supervisors: Dr.ir. A.A.J. Lefeber
Prof.dr.ir. I.J.B.F. Adan

Eindhoven, June, 2016

Where innovation starts

Abstract

This thesis addresses the derivation of a vehicle actuated stabilizing control policy for an isolated intersection. Isolated intersections can be modelled as switched linear hybrid dynamical systems, assuming traffic flows and traffic processing rates are constant. State-dependent control of such a system provides a vehicle actuated control strategy. First a Lyapunov function candidate is derived for an isolated intersection, based on which control actions are then designed. These control actions are transformed into a control policy. The derived control policy stabilizes the system in predetermined, optimal periodic behaviour. This optimal behaviour is computed from a known, optimal, fixed time schedule of the intersection. The theoretical results are illustrated by means of case studies.

Preface

This thesis is a result of the research I conducted to obtain my masters degree in mechanical engineering. It defines the end of a sometimes challenging but, above all, an amazing student life. During my internship at an international company I gained detailed insight in the field of engineering beyond academia. Within the Dynamics and Control section, at the mechanical engineering department of the Eindhoven University of Technology, I seized my final chance to explore the world of scientific research whilst performing my graduation project.

When traffic control was proposed as a research topic, it appealed to my imagination. Born and raised in Eindhoven I recalled the astonishment when travelling on the Boschdijk, every single traffic light seems to switch on approach, from a green, to a red light. I immediatly thought of driving a car, or even worse, biking in the rain and the peeve of queueing at a vacant intersection. Briefly, the idea that I could contribute to traffic control research, was appealing to me.

If you have a background in mechanical engineering, in control specifically, I suggest to skip the literature review presented in Chapter 2. This chapter contains background information for the interested reader who's knowledge on control systems, in particular traffic control and control of hybrid systems, is limited. For readers with basic knowledge on control systems, Chapter 2 can be used as a work of reference.

Finally, this achievement would have never been possible without the support of some extraordinary people. I would first and foremost like to thank dr.ir. A.A.J. Lefeber for his supervision, guidance and endless patience, in the process evolving me to be worthy of the title engineer. Without his guidance and meticulous feedback I would never have been able to become the person I am today. Furthermore I would like to give a special thanks to my friends and family, their faith in my abilities, their immunity to my complaints, the many laughters and the fun distracting activities which made my final months at this university.

The exceptional guidance enhanced my perseverance to finish this master thesis and to come up with theorems that contributed to the field of control of switched linear systems with setup times. It is therefore that I proudly present my achievements in this thesis.

Eindhoven, The Netherlands — Monica Theunissen

Summary

Traffic congestion is a notorious problem nowadays, it contributes to the level of greenhouse gases and delays road users. Traffic control is studied to a great extent in literature, often these studies aim to optimize traffic control such that congestion can be decreased.
In urban areas merging and crossing roads are in general equally levelled, which creates an intersection. Usually, intersections in urban and suburban areas are controlled by traffic signals to provide safe and orderly crossings. Optimizing this traffic control, by tuning the signal settings for instance, could decrease the overall average waiting time of vehicles at an intersection.

Two main categories of traffic control strategies are discussed; fixed time control and vehicle actuated control. In fixed time control a cycle is defined in which all queues are served once. The setting of each traffic signal, if a traffic light is red or green, are denoted in a so called fixed time schedule. This fixed time schedule states the red and green periods as function of time in one cycle.
In vehicle actuated control the intersections signal settings depend on real-time measured traffic data. The implementation of vehicle actuated control could reduce the overall average waiting time of vehicles at an intersection, provided that this vehicle actuated control is designed correctly.

The research objective is to derive a control policy for an isolated intersection that steers the system to predefined optimal periodic behaviour, regardless the initial values of the system.
An intersection can be modelled as a switched linear hybrid dynamical system. This abstraction is possible assuming that the arrival and process rates at the intersection are known and constant. In this switched linear hybrid dynamical model of an intersection the system states are described by continuous linear equations. The switch between different signal settings is described by a discrete relation, the so called switching rules. Such a switched linear hybrid dynamical system is used in this thesis to model an intersection.

Frequently, in controller design, a control policy is selected first, of which subsequently the resulting controlled system behaviour is studied. Thereafter, the control policy is modified if the resulting controlled system behaviour is undesired. This approach fine tunes policies to improve them, but it does not pave the way to find new policies that might steer the system states to desired system behaviour.
In [1] a different approach to find a control policy is proposed, this approach deviates from the previously discussed frequently practised method of controller design. In [1] the derivation of a control policy starts by studying the desired steady-state behaviour. A Lyapunov function candidate is then defined, based on the work in the system. The difference between the mean work in the system in periodic behaviour and the mean work in the system in optimal periodic behaviour, is defined as the Lyapunov function candidate. A similar approach is used in this study, but contrary to the Lyapunov function proposed in [1], the Lyapunov function candidate is defined in the entire state-space of the system.
To find a general control policy, first a candidate Lyapunov function is derived for example systems of two sizes. Based on these derivations it is assumed that a Lyapunov function candidate can be found for systems of all sizes, provided that specific requirements are met. However, the derivation of a Lyapunov function becomes exhaustive when system dimensions are increased, which makes explicitly denoting a Lyapunov function to find a control policy infeasible.

The control actions are designed with the aim to create a Lyapunov stable controlled system. To

obtain Lyapunov stability, the Lyapunov function should be non-increasing over time, equal zero if the system is in optimal behaviour and exceed zero in all other system behaviour. Therefore, the control actions are designed such that the Lyapunov function value is non-increasing over time and equals zero in the desired periodic behaviour.
The control actions are transformed into a control policy, which is a function of the current system states. This transformation of control actions to control policy is discussed for the example systems described in this thesis. The resulting control policies of the example systems show similarities, based on these similarities a general definition is derived. This yields a control policy that can be implemented for systems of various sizes. The resulting control policy is easy to implement and can directly be obtained from the given fixed time schedule of the particular intersection.

Finally, the above mentioned conclusions are verified by performing two case studies. To verify that the control policy can directly be obtained from the given fixed time schedule, modes need to be defined in this given fixed time schedule. The definition of modes is ambiguous, thus the specific definition of modes used in this thesis requires an explanation. The case studies illustrate the methodology to define modes based on a fixed time schedule. With the definition of modes a control policy is successfully derived for both case studies, based on the measured traffic data and the fixed time schedule of the intersections.

Although the results are promising and the research objective is met in the example intersections, the Netherlands is a relatively small country with compact infrastructure. The number of intersections at which flows are not affected by neighbouring intersections is probably not significant. Consequently, controlling intersections as if they are equivalent to isolated intersections might result in an unwanted increase of overall average waiting time. An extension of the policy that makes it applicable in a network of intersections could prevent the unwanted increase of overall average waiting time. On that account this extension of the vehicle actuated control policy is suggested as a topic of future research.

Contents

Chapter 1. Introduction

Traffic congestion is a notorious problem nowadays. Apart from the obvious increase in travel time for road users, congestion affects the accessibility of the road network [2], it increases environmental pollution and it reduces safety [3]. The latter is contradicting to the aim of the European Union (EU) to reduce the number of road fatalities in the EU, [2]. Furthermore, the European Union 2030 climate and energy framework states that the EU should reduce its emissions in 2030 by at least 40% compared to 1990, [4]. To reduce these emissions alternative fuels are upcoming and the fuel efficiency of vehicles is improving, an additional measure is to decrease traffic congestion, [2].

Traffic congestion occurs if roads with limited capacity are accessed by a large number of road users, [3]. To reduce traffic congestion the road capacity needs to be enhanced. There are various ways to improve road capacity. The most straightforward solution, modifying the road network, takes many years to plan, build and equip [2], is expensive and can be constrained by environmental or spatial limitations. Other attempts to decrease traffic congestion are stimulating public transport or employing road pricing systems in urban areas, [5, 6, 7, 8, 9]. Bottlenecks in the infrastructure are the main cause of congested roads in urban and suburban areas. Addressing the bottlenecks of the infrastructure might result in a less expensive and plain solution to increase the road capacity.

In urban and suburban areas road crossings are in general equally levelled, which creates an intersection. At a signalized intersection traffic flows are controlled by traffic lights, this enables road users to move safely and orderly. Traffic control by traffic signals can be used to maximize intersection capacity and to minimize vehicle delays. The former limitations in the application of traffic lights are eliminated by progression in electronic technology. Currently, signalized intersections are a frequently used strategy to control traffic intersections, because of the the simplicity and reliability of a traffic signal system, [10].

In Section 1.1 previous studies on traffic control are summarized. These studies are conducted at Eindhoven University of Technology (TU/e) and define the basis of this study. In Section 1.2 the research objective is explained. The methodology to achieve this research objective is described in Subsection 1.2.1. The field of application of the theory studied in this thesis is discussed in Section 1.3, it includes a detailed description of the terminology used. Finally, an outline of the remainder of this thesis is presented in Section 1.4.

1.1 Previous Work

Previous studies on signalized intersections conducted at the TU/e, [11, 12, 13, 14, 15], primarily focus on optimizing fixed time control of isolated intersections. Fixed time control is a time-dependent control method, at each point in time the controller specifies which queue is served. The server switches between processing different queues, by switching the colours of the traffic signals. In Section 1.3 the fixed time schedule and corresponding terminology is explained in detail.
The majority of previous studies consider optimizing the fixed time schedule of an isolated intersection, but there are differences. In [11] the number of vehicles queuing in front of a traffic light is approximated by deterministic fluid flow, whereas in [12] a stochastic approximation of the fluid flow is considered to determine the queue. In [14] the start of an ongoing research is shown, a tool

is developed that computes an optimal fixed time schedule for intersections for which the arrival and process rates are predetermined. Whether a system is optimal, depends on the value of one or multiple set performance indicators. In [16] the performance of the tool developed from the one discussed in [14], is compared to the traffic control technology used in practice.

The research on optimizing fixed time control is extended to a network of intersections in [15, 16]. Optimal fixed time control is implemented per intersection with a time shift between the control of all intersections, to minimize the average waiting time for all vehicles.
Furthermore, optimized fixed time control is used as source to find a traffic controller that depends on the queues in front of the traffic lights. In [13, 16, 17] control based on an optimal fixed time schedule of switched single servers with multiple queues is investigated. In [16] a controller is designed consists of fixed blocks and depends on the queue lengths. The sequence of these fixed blocks is automatically derived from the fixed time schedule. This sequence defines the order in which the traffic lights switch signal.

1.2 Research Objective

As summarized in the previous section optimizing the fixed time control of isolated signalized intersections is studied extensively. Studies on queue dependent control policies are currently less extensive.
The aim of this study is to start developing a control policy that dynamically controls the traffic at an intersection, autonomously. The queue lengths at each traffic light should be bounded, so the system should be stabilized. In the derivation of a fixed time control policy, an optimum periodic system behaviour is established. This optimal behaviour depends on set performance indicators. Since the optimal behaviour is predetermined, the desire is to find a controller that stabilizes the system in this optimal periodic behaviour, regardless the initial values of the number of vehicles queued. Concluding, the research objective is:

Derive a control policy that steers the system such that it stabilizes in predefined optimal periodic behaviour, regardless the initial values. The system being an isolated intersection.

The focus is on isolated intersections in this study. It should be noted that in urban and suburban areas approximating traffic networks in isolated intersections might lead to underestimating the overall average waiting time at an intersection.

1.2.1 Methodology

Literature on traffic control is studied to gain general knowledge on traffic control. The subject system of this research, isolated intersections, are modelled as switched linear hybrid dynamical systems with setup times. Therefore, literature on the topic of hybrid dynamical systems and the corresponding control methods is studied. One of the approaches suggested in literature is chosen to derive a control policy for the subject system of this research.

The method used to derive a stabilizing controller requires a Lyapunov function candidate, which is derived first. Subsequently control actions are designed based on this candidate Lyapunov function. Finally, the designed controller is transformed into a control policy. This control policy combines all control actions such that a list in words contains all actions. The derived control policy is then verified by applying the control policy in case studies.
To ensure the developed method to find a stabilizing control policy holds for systems with increasing complexity, various system sizes are studied. Starting with a simple system and increasing the systems size,

- intersection of two directions,

- intersection of three directions,
- case studies.

1.3 Field of application

The example systems throughout this thesis are signalized intersections. However, the control theory discussed in this thesis is applicable to different manufacturing systems. This section gives a clear impression of the field of application suggested in this thesis. The section can be used as a work of reference when reading the remainder of this thesis.

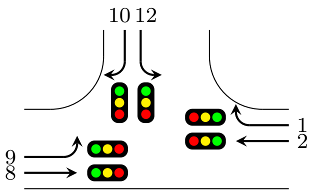

Figure 1.1: Example of an intersection.

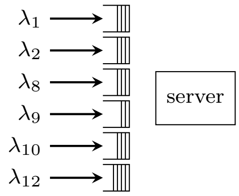

Figure 1.2: Figure 1.1 as manufacturing system.

Figure 1.1 shows a schematic representation of an intersection, throughout this thesis an isolated signalized intersection is modelled as a manufacturing system. An elementary outline of an intersection as a manufacturing system is a system with one server and multiple buffers. In Figure 1.2 the intersection of Figure 1.1 is visualized as a manufacturing system. Other topologies of manufacturing systems can be designed that model the intersection given in Figure 1.1, however Figure 1.1 is the most intuitive. Figure 1.2 does not exactly match the system in Figure 1.1, constraints should be listed to complete the manufacturing system model of an intersection. These constraints depend on the topology of the intersection, and state which buffers can not be served simultaneously, as is explained below.

A combination of directions at an intersection can be conflicting or non-conflicting. Depending on the topology of the intersection, one or multiple directions are non-conflicting and can be served simultaneously. Conflicting directions are directions that need to cross or merge at the intersection. This creates collision probability if the directions are processed simultaneously. For instance in Figure 1.1, direction 9 and direction 2 are conflicting flows, they have to cross paths if served simultaneously. Only non-conflicting directions are allowed to be processed simultaneously by the server.

1.3.1 Terminology

A direction from which vehicles arrive at traffic light i with equal destination, is referred to as flow i. A flow equals a combination of an originating, and a destination direction. The different flows of the example system are schematically represented by numbered arrows in Figure 1.1.
The numbering of flows at an intersection is standard, it starts at the top right of the intersection at the right turn with number 1. Each possible flow is numbered, even if the flow is not present at the intersection. For instance in Figure 1.1 no north to south flow exists, thus number 11 is omitted.
N is a set of natural numbers consisting of all flow numbers, i, in the system. The number of flows in the system is determined by the amount of elements in N.
The vehicles queued in buffer i at time t, are represented by $x_i(t)$, the buffer content. Although this buffer content is a function of time, it is often shortened to x_i to improve the readability of

the equations.
The items in the buffer are called lots, representing the queued vehicles. Lots of type i arrive at buffer i with constant arrival rate λ_i. When buffer i is served, lots are processed at process rate μ_i unless buffer i is empty. In case buffer i is served when $x_i = 0$, lots are processed at their arrival rate, λ_i.
A mode $\boldsymbol{m}$ is a part of the fixed time schedule cycle in which the combination of processed flows does not change. If the system is processing a lot type at its arrival rate in mode $\boldsymbol{m}$ this is called slow mode. A fixed time schedule can contain a slow mode if this results in a lower overall WIP-level of the system. Work in progress (WIP) is the total number of lots in the system. A slow mode can lead to a lower overall average steady-state WIP-level, if arrival and service rates differ significantly per flow.
Work in a buffer is defined as the time it takes to clear the buffer; the number of items in the buffer x_i divided by the process rate. The work in the system, (1.1), equals the sum of the work in all buffers of the system.

$$W = \sum_{i \in N} \frac{x_i}{\mu_i}. \tag{1.1}$$

The assumptions that hold for each intersection discussed in this thesis are presented in detail in Appendix A. Unless explicitly stated otherwise, in this thesis briefly the following assumptions hold for each intersection:

1. All vehicle routes are predetermined.
2. Arrival rates and process rates are known and constant.
3. Given the fact that an optimal fixed time schedule is given, the system has sufficient capacity to serve all arriving lots.

The optimum sequence, combination and duration of red and green signals at an intersection is denoted in a schedule. This schedule is called a fixed time schedule if the schedule is predetermined and only influenced by the elapsed time. For the intersection in Figure 1.1, with given arrival and process rates, the optimal fixed time schedule is presented in Figure 1.3. The duration of one complete sequence of the operation of traffic signals listed in the fixed time schedule, is called the period or cycle of the fixed time schedule. A period or cycle equals T time units.
In a fixed time schedule the time each flow is processed per cycle is denoted by a grey bar. The time per period each flow is not served, is denoted by a black bar in the fixed time schedule. The fixed time schedule contains no amber time, the amber time is approximated as explained in for instance [11, 14]. The amber time is modelled to be part of either the red time, the green time or a combination of both. This results in a fixed time schedule that simply lists the effective green time and effective red time, as shown in Figure 1.3.
Switching from flow p to flow q takes setup time, $\sigma_{p,q}$. Because of the time it takes to clear the intersection of vehicles, the clearance time, the setup time is non-negative when flow p and q are conflicting. In setup of mode $\boldsymbol{B}$ for instance, setup from flow 12 to flow 8, $\sigma_{12,8} = 2$, equals the setup time of mode $\boldsymbol{B}$. Although according to Figure 1.1 flow 12 and flow 10 are also conflicting with flow 2, Figure 1.3 shows that immediately when flow 10 and flow 12 switch to red, flow 2 switches to green. Meaning the setup times from flow 10 to flow 2, and from flow 12 to flow 2 do not determine the duration of B.

The fixed time schedule contains more information, the server can be in one of three modes, in setup of, or processing in mode $\boldsymbol{A}$, $\boldsymbol{B}$ or $\boldsymbol{C}$. If the server is in setup of mode $\boldsymbol{m}$ this is denoted as m. Whilst m refers to serving in mode $\boldsymbol{m}$.
The remaining setup time of the system is denoted as x_0. Consequently, $x_0 = 0$ if the system is processing in mode $\boldsymbol{m}$. When the system is in m with $x_0 > 0$, the server is setting up to serve one of the buffers in m.

Which part of what mode the system is in, and the value of x_0, are denoted in the control action as u_0, with $u_0 \in \{$Ⓐ, 🅐, Ⓑ, 🅑, Ⓒ, 🅒$\}$. Thi u_0 is the first term of the control action, the remaining part of the control action determines at what rate each of the buffers is served. For instance, the server action of the system presented in Figure 1.1 is denoted by $(u_0, u_1, u_2, u_8, u_9, u_{10}, u_{12})$, meaning the server is in u_0 and buffer i is served at rate u_i.

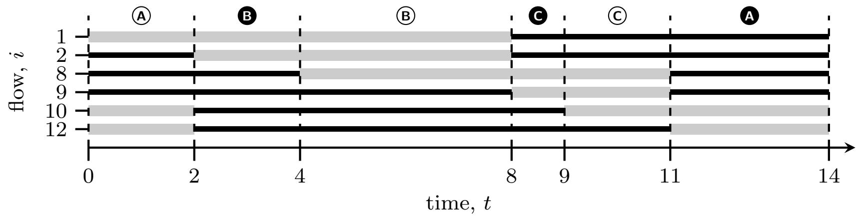

Figure 1.3: Fixed time schedule of the system in Figure 1.1.

1.4 Thesis Outline

Chapter 2 discusses two main strategies to control signalized intersections. Signalised intersections can be classified as a specific class of hybrid dynamical systems, so additionally the chapter elaborates on switched hybrid dynamical systems. Different methods to find a controller for switched linear hybrid systems are reviewed. After Chapter 2, this thesis focusses on the application and extension of one of the control methods reviewed in Chapter 2.
The body of this thesis is organized as follows. Chapter 3 discusses the first step of the method used to find a control policy for a signalized intersection. It describes how candidate Lyapunov functions are derived for systems of various sizes. Chapter 4 defines the control actions that stabilize the system, based on the candidate Lyapunov functions derived in the preceding chapter. Chapter 5 focusses on the transformation of the control actions into a control policy. Additionally in this chapter, a method to directly determine a control policy based on the fixed time schedule is illustrated by case studies.
Finally, in Chapter 6, the research's conclusions are listed and some interesting achievements are summarized. The chapter is finished by recommendations to consider in future research.

Chapter 2. Literature Review

The literature review given in this chapter is twofold. The first part presents an overview of studies on road traffic control. The rapidly increasing number of vehicles, and the importance of transportation, makes traffic control a research topic of interest. The overview gives an impression of how traffic control has progressed over decades and a prospect of ongoing and future research. Section 2.1 focusses on traffic control in urban areas, as this is the field of application of this thesis' subject system. Furthermore, in Subsection 2.1.3, data acquisition is discussed briefly, as data acquisition is of importance for each control strategy.
As stated in Chapter 1, an intersection can be modelled as a switched linear hybrid dynamical system with setup times, a specific class of hybrid dynamical systems (HDS). Modelling intersections as switched linear hybrid dynamical systems creates an interest in control strategies for this type of systems. Hence, hybrid dynamical systems and their controllers are the second topic reviewed in this chapter. A concise overview of different strategies to control and analyse hybrid dynamical systems is given in Subsection 2.2.2 and Subsection 2.2.3 respectively.

Whenever the results from the discussed literature is used in the body of this thesis, the content of that study is summarized, thereby enhancing the self-containment of this thesis. The overview of literature on both topics is by no means a complete overview of all existing material, it is limited to references that were found and showed resemblance with this thesis research topic.

2.1 Traffic Control

Although traffic lights were originally invented to guarantee safely and orderly crossing of conflicting flows, currently traffic lights are mainly used to control traffic in urban areas. As traffic volumes increased the last decades, it was noticed that traffic control by signalization could lead to increasingly or decreasingly efficient capacity use, [3]. Thus, there is an optimal control strategy to be established, a control policy that optimizes the performance according to specified performance indicators. For instance, a performance indicator could be the overall average time spent by vehicles at an intersection. The performance of the system can then be analysed by for example, setting the optimization goal to be the minimum of the performance indicator.

Figure 2.1 illustrates a basic control loop inspired on Figure 1 of [3]. The dashed grey box represents the intersection. It consists of the traffic signals, these are the control devices of the system, and the detectors, which act as the systems sensors. The latter provide data input for the control strategy. With the gathered data, the control strategy computes the optimal traffic signal setting, based on some pre-determined optimization goal. The efficiency of the overall control system is mainly determined by the efficiency of the control strategy. Hence, the control strategy should be chosen wisely. The final components of the control loop are the inputs that can not be manipulated. Although arrivals can be predictable over a time horizon or even measurable, the arrivals can not be modified. Incidents or priority vehicles for example, are a source of disturbances which can naturally not be influenced but do have an effect on the traffic flow.

Different traffic control strategies can be classified as *fixed time strategies* or as *traffic responsive strategies*, according to [3]. For each intersection *fixed time strategies* could be derived that depend on the time of day, based on historical traffic data. If real time measurement data is used to

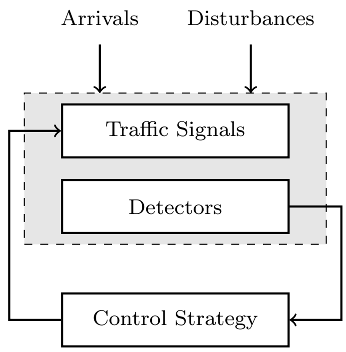

Figure 2.1: Basic control loop of an intersection.

determine traffic light settings, the control method is a type of *traffic responsive strategies*. This type of control method is often referred to as vehicle actuated strategies. Subsection 2.1.1 and Subsection 2.1.2 respectively elaborate on different fixed time strategies and on vehicle actuated control studies.
Control strategies can be categorized in more detail, in [3] traffic control strategies are classified as *isolated strategies*, if the control strategies are applied to a single intersection. If an urban area network of multiple intersections is controlled, the control strategies are referred to as *coordinated strategies* or *control of networks of intersections* in [15, 16]. The travel time between intersections and the possible formation of *platoons*, a group of vehicles that is combined by on of the preceding intersections, becomes of interest when networks of intersections are controlled, [3, 15, 16]. Networks of intersections are outside the scope of this project and therefore not discussed in detail in this literature review. However, to determine the applicability of any isolated intersection strategy, notion of the existence and possible effects of these parameters is of great importance.

If vehicle queues, that are created during the red time of a period time of a signalizing schedule, can be dissolved during the effective green time of a flow, the traffic conditions are called undersaturated in literature, [3]. However if vehicle queues, built during the red period of a signalizing cycle, can not be dissolved in the effective green time of each of the flows, the traffic conditions are called oversaturated. The majority of the discussed strategies in this section suit undersaturated traffic conditions, few strategies are applicable to oversaturated traffic conditions.

2.1.1 Fixed Time Strategies

There are different strategies to determine a fixed time schedule (FTS) of isolated intersections. In [3] these strategies are subdivided in two main categories. The first category contain strategies that optimize the green time of each flow in every mode, and optimize the overall period T of the fixed time schedule. The optimization is based on minimizing the delay at an intersection or maximizing the capacity of an intersection for example. The second category contains strategies in which apart from optimizing the green times of each flow in every mode, and optimizing the overall period of the fixed time schedule, the specification of modes is optimized.
The main advantages of fixed time control methods are that no additional hardware or complex control strategies are required. Fixed time control is easy to implement, especially because it is based on previously acquired data, instead of depending on real time data measurements. However, the lack of real time data can also be a disadvantages of fixed time strategies. Fixed time strategies are based on the simplification that traffic flows are deterministic, in practice though, traffic flows are not constant at a specific time-of-day, can vary over days and might be influenced by disturbances. This variation in traffic data could result in outdated optimized

settings in the fixed time schedule, [3].
In peak hours traffic congestion can be prevented by fixed time control, if the fixed time control is based on the correct optimized settings. But peak hours are not the only cause of congestion in urban areas, traffic interference such as accidents or constructions can be a source of congestion. Additionally, differences in driving velocity can cause traffic congestion, [18, 19]. These sources of congestion is unaccounted for in fixed time schedules, therefore traffic congestion could be increased by fixed time control methods. Vehicle actuated control strategies could prevent outdated optimized settings from occurring. Therefore, vehicle actuated control strategies are potentially more efficient, if designed correctly, [3].

2.1.2 Vehicle Actuated Strategies

In [3] vehicle actuated control (VAC) methods are referred to as traffic responsive strategies. Vehicle actuated strategies use real time measurement data, in which the data is collected by sensors. The detection method and sensors can differ, for more detailed information on different detection methods see Subsection 2.1.3. Besides the advantages of being vehicle actuated, costly disadvantages of vehicle actuated control do exist. A vehicle actuated control method requires installation and maintenance of sensors, which need to be operating properly for the system to function as designed.

One of the least complex vehicle actuated control methods, as explained in [3], is the vehicle interval method. This method is applicable to intersections for which two modes can be defined. For both modes minimum green times are determined, if no vehicle is detected during the minimum green time of a mode the controller switches to the next mode. If however a vehicle is detected in the minimum green time a critical interval is set. During this interval each detected vehicle leads to an extension of the green time. For each vehicle a new critical time is created until the maximum green time is reached, if no vehicle is detected in the critical interval the controller switches to the next mode. In [20] an extension to this control method is presented, it takes into account the traffic flow at directions that are not served in the systems current mode to determine if the system should switch to the next mode. Every pre-set value, c time units, the controller determines to either switch to the next mode based on the traffic data of all flows, or postpones the switch by c time units.

Vehicle actuated traffic control collects real time vehicle arrival data to adjust the traffic signal setting, however the conventional VAC strategies do not evaluate the performance indicators in real time, [21]. In [21] a general dynamic programming algorithm is proposed that optimizes one of the possible performance indicators; total delay time at the isolated intersection, maximum queue length or total number of stops. An advantage of this method is that optimizing the performance indicator automatically yields the optimal mode sequencing. The vehicle actuated control method presented in [22] is a control strategy applicable in oversaturated traffic conditions. It is a combination of a dynamic control algorithm that determines the traffic light settings, taking into account the current and projected queue lengths, and a so called disutility function. The latter measures the system performance by comparing the current performance to pre-set system performance goals. The system is designed for traffic conditions varying each period, resulting in a method that generates suitable traffic light settings to respond to priorities and extensively varying traffic data.
Most classic VAC strategies use classic vehicle detectors at an intersection, detection at a fixed position. A new traffic control algorithm, based on a vehicle detection method using the technology of Wireless Sensor Network (WSN), is developed in [23]. The proposed algorithm results in efficient traffic control regarding the average vehicle waiting time at an isolated intersection. Furthermore the duration of a mode can be determined exactly as the system states are measured dynamically. Another traffic control method, that gathers traffic data via a WSN, is described in [24]. The controller consists of a communication algorithm and a signal manipulation algorithm. The manipulation algorithm determines the system settings such that the average queue length and

the average waiting time are minimal. Combined the communication and signal manipulation algorithm provide an adaptive traffic estimation and dynamic change in the traffic light sequencing and signal settings. For an isolated intersection the efficiency of the proposed controller is illustrated by simulation results.
The development of various real time control strategies is enabled by the improvement in sensor and wireless networks technology, [18]. A traffic control method that results in smooth traffic flows at intersections, is proposed in [18]. The algorithm computes traffic volumes and the degree of congestion, by assigning vehicles to the group of each direction at an intersection. The data needed for this procedure is gathered via vehicle to vehicle communication, by creating Vehicular Ad-hoc Networks (VANETs). The traffic data acquired using VANETs results in a more accurate queue length calculation than loop detector methods. The algorithm determines period time, T, and the green times for each flow, based on the estimated queue length, thereby creating a real time traffic control system.

Traffic responsive control methods make it possible to consider priority vehicles in traffic control. In [25] bus priority is adopted in an adaptive traffic control approach. This approach determines the duration of a mode and the sequence of modes in a non fixed order. The objective of this approach is increasing the bus system performance by putting a minor toll on vehicles that are not prioritized. The difference in optimization approach compared to conventional control strategies is, rather than optimizing the average waiting time of vehicles, the average waiting time of passengers is concerned. The method is developed under the assumption that communication techniques are available to collect real time data.
Prioritized vehicles are also the topic in [26]. In this case it is assumed that WSN is the input source of the controller. Although very basic, in theory it extends the existing dynamic traffic control algorithm and thereby effectively controls prioritized vehicles in all possible modes of the system and even proposes a solution in deadlock condition.
In [27] a model called Signal Priority Procedure for Optimization in Real Time (SPPORT), is developed and evaluated. The influence of public transport vehicles blocking roads when boarding or disembarking passengers is taken into account in the model. It quantifies the effects of prioritizing public transport vehicles on other vehicles. The rule based optimization policy serves queues and platoons, and copes with prioritized vehicles. It is executed such that the traffic control is adapting to current traffic conditions. The algorithm results in reduced overall delay for most traffic conditions, when compared to fixed time and classic vehicle actuated control.

Vehicle Actuated Control in Networks of Intersections

Strategies to control networks of intersections are often affected by adding intersections, the exponential increase in complexity leads to extended computational demands. A model providing an accuracy comparable to standard macroscopic models is proposed in [28]. In this model a linear computational demand is added per intersection and it describes the effect of queues at nearby intersections. Based on this model a hierarchical control method is presented in [28], it significantly reduces traffic congestion in urban areas compared to ordinary decentralized control. Another extension to the traditional fully actuated control is an adaptive control model based on a Markov decision process, presented in [29]. The model finds an optimal decision for the controlled Markov process, considering a platoon dispersion model describing the traffic between intersections in the network. The resulting strategy indicates to be more efficient than the fully actuated control when traffic flow values are high. The disadvantage of the strategy is the effect of dimensionality, the dimensions of the model increase rapidly as the number of intersections in the network increases, affecting the computational velocity.

2.1.3 Detection Methods

In vehicle actuated traffic control accurate detection of vehicles and acquiring this data is of utmost importance. Currently inductive loop detectors, the sensors positioned in the road, are widely used to detect vehicles. The main disadvantage of this detectors type is the hardware, in particular its costly and inconvenient maintenance [18]. There are various types of detection methods using different types of sensors, radio frequency identification, the previously mentioned WSN or image processing units. Wireless Sensor Network technology provides a new vehicle detection method that can monitor vehicles dynamically, as in [23]. The method provides accurate measurements of the number of vehicles and measures the speed of vehicles in real time.
In VANET systems the estimation of vehicle density is useful. In [19] a method is presented that does not require information about the vehicle locations and does not need exchange of information between vehicles. Therein a different approach to predicting vehicle density that solely depends on the mobility pattern of vehicles is developed.
Image detection methods are improving continuously, [30] compares two shadow detection approaches. Image detection algorithms are designed to prevent moving shadows being mistaking for objects or parts of objects. A shadow detection approach to image processing improves the accuracy of the object localization by avoiding merging of objects. Video and image processing methods to compute traffic density are also discussed as a traffic detection method in [31]. A strategy is presented to use feeds from live video monitoring cameras for real time traffic density calculation. It has as main feature that the distance among vehicles can be determined more accurately.
Another detection method is radio frequency identification, using this method as a form of traffic flow detection is proposed in [32]. This method of detection is comparable to the method to detect vehicles used in highway toll systems.

Finally, in this first part of the literature study, data processing is discussed. Even though it is technically not a direct detection method, it is important for both fixed time and vehicle actuated control. Incorrect processing could result in a less efficient traffic control or even incorrect traffic control parameters. In [33] fundamental relationships are derived. Different expressions are found for undersaturated traffic conditions, congested traffic conditions and oversaturated traffic conditions. The relations for the non-congested conditions are functions of the utilization or the average queue length. The number of delayed vehicles increases over time when the traffic conditions are congested. So in that case, it is concluded that the average travel time depends on utilization as well as on the average vehicle queue. Despite the congested traffic conditions traffic control can improve the average travel time by taking the arrival of vehicles into consideration. The result from [33] implies that the measured fundamental diagrams from urban traffic flows can be understood systematically. Furthermore, the existence of this type of studies and the in [33] discussed prior used expressions and diagrams, emphasizes the importance of correct data processing for all traffic control systems.

2.2 Hybrid Dynamical Systems

The essence of hybrid dynamical systems (HDS) is captured in the interaction between discrete and continuous dynamics. In these type of systems there are variables from a continuous set, as well as variables from a discrete set. This occurs a lot in technological systems where physical processes are controlled by logic decision makers [34]. The main feature in the representation and definition of a hybrid system is the interaction between discrete and continuous dynamics [35]. Hybrid systems are represented in different scientific areas, each with their own approach on hybrid systems. In control technology a discrete decision part and a continuous system layer create a hybrid system. Hybrid dynamical systems can be given as a general description of a system. A specification in a subclass of hybrid systems is required to derive specific system characteristics, [35]. The specific class of interest in this thesis is switched linear hybrid dynamical

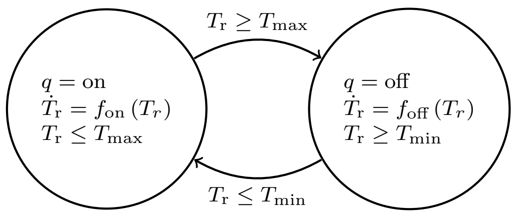

Figure 2.2: Simple HDS, model of a room climate regulator (Figure 1.2 in [34])

systems. Linear refers to the fact that, as described in [1, 14, 15], the continuous time plant of the system is approximated by ordinary differential equations (ODEs) based on a fluid model approach. Furthermore, the emphasis is on single server systems, there exist only one server in the system that switches between the buffers and switching takes setup time.

An example of linear hybrid control is a continuous time plant with states described by differential equations, controlled by a regulator [36]. There are many examples of switched linear hybrid dynamical systems. One simple, clarifying example often mentioned in literature is the regulation of temperature in a room [34, 35, 36]. The system is graphically represented in Figure 2.2, the figure is adopted from [34]. Simplifying the heating system results in a heater controlled by the thermostat which is either switched on or off, the discrete state the system is in. Thus the system can be in two modes, mode on or mode off. The evolution of room temperature T_r, is described by a differential equation depending on this mode. The discrete state q therefore affects the evolution of the temperature, the continuous state. Switching between the modes, on or off, is controlled by a controller, a logical device, in this case the thermostat. Hence the system state is hybrid, it depends on the discrete state $q \in \{\mathrm{on}, \mathrm{off}\}$ and the continuous state value $T_r \in \mathbb{R}$, which results in the hybrid state (q, T_r). Even though technically q and T_r are functions of the time t, this is often omitted in denotations to shorten the expressions. This case is an example of state dependent switching. The value of the temperature T_r combined with pre set conditions, trigger the change of the discrete state.

2.2.1 Hybrid Dynamic Model of an Intersection

As briefly mentioned in amongst others [1, 13, 14, 15, 37], an intersection can be modelled as a switched hybrid dynamical system by approximating it using a fluid model. The system receives incoming items, vehicles, and stores these lots in internal buffers. The server is able to process lots, by moving the lots from the buffer to the destination at a prescribed process rate. The location of the server determines which item type is, or what types are, being processed. Changing the location of the server takes a non negative setup time.

The parameters used throughout this thesis and the framework for the model is adopted from [1]. Herein it is mentioned how an intersection can be modelled as a hybrid dynamical system. Basically a switched linear hybrid system is a system that consists of a finite number of all linear subsystems combined with logical rules that manage switching among the subsystems. Consider the simplest intersection possible, an intersection where two directions, $i \in \{1, 2\}$, need to cross paths. A simple control policy for this intersection is a clearing policy: clear the currently served buffer and switch to the next buffer. The switched linear hybrid dynamical model is graphically represented in Figure 2.3, using the same method as for the heating system. To create this graphical representation the simple control policy is combined with the system dynamics described next.

The system state at time t consists of the buffer content corresponding to direction $i = 1$ at time t, the content of buffer $i = 2$ at time t and the mode the system is in. The system is either in mode $\boldsymbol{A}$ or $\boldsymbol{B}$, respectively serving or setting up for direction $1, 2$. The final part of the system

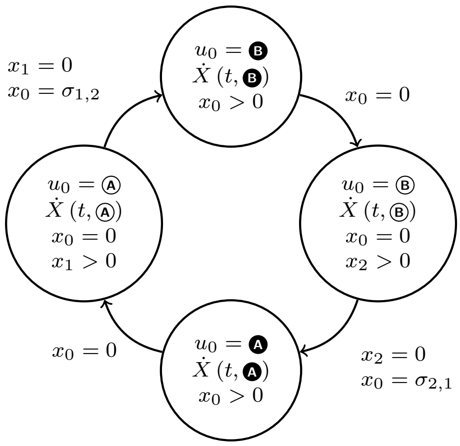

Figure 2.3: HDS model of a simple two way intersection

state is the remaining setup time x_0, which only equals zero if the server is processing items. The input of the system is a discrete input, it can take a finite number of values. The value of u_0 determines the action of the server at the moment, $u_0 \in \{$🅐, Ⓐ, 🅑, Ⓑ$\}$. The value of u_i is the process rate of direction i, which equals zero if the direction is not served in the current mode. If the served buffer i is empty, u_i equals the arrival rate. The value of u_i equals the service rate μ_i, if direction i is served and the buffer is not empty. So the discrete event dynamics can be described by the inputs and:

$$x_0(t) := \begin{cases} \sigma_{2,1} & \text{if } u_0 = \text{🅐 and } m = B, \\ \sigma_{1,2} & \text{if } u_0 = \text{🅑 and } m = A, \end{cases} \tag{2.1}$$

$$m := \begin{cases} A & \text{if } u_0 \in \{\text{🅐}, \text{Ⓐ}\} \text{ and } m = \boldsymbol{B}, \\ B & \text{if } u_0 \in \{\text{🅑}, \text{Ⓑ}\} \text{ and } m = \boldsymbol{A}, \end{cases} \tag{2.2}$$

the hybrid dynamical system model of the intersection is completed by adding the continuous dynamics listed below.

$$\dot{x}_0(t) = \begin{cases} -1 & \text{if } u_0 \in \{\text{🅐}, \text{🅑}\}, \\ 0 & \text{if } u_0 \in \{\text{Ⓐ}, \text{Ⓑ}\}. \end{cases} \tag{2.3}$$

$$\dot{X}(t, u_0) = \begin{bmatrix} \dot{x}_1(t, u_0) \\ \dot{x}_2(t, u_0) \end{bmatrix} = \begin{cases} \begin{bmatrix} \lambda_1 - u_1(t) \\ \lambda_2 \end{bmatrix} & \text{if } u_0 = \text{Ⓐ}, \\ \\ \begin{bmatrix} \lambda_1 \\ \lambda_2 - u_2(t) \end{bmatrix} & \text{if } u_0 = \text{Ⓑ}, \\ \\ \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} & \text{if } u_0 \in \{\text{🅐}, \text{🅑}\}. \end{cases} \tag{2.4}$$

2.2.2 Control of Hybrid Dynamical Systems

There are two different types of switching, the system can switch time-driven or event-driven [13]. A time dependent controller, switches depending on the value of time. In an event-driven switch, also referred to as state dependent switching, whether a switch occurs depends on the system state values. The main research question is how to derive a controller for a switched linear hybrid dynamical system as presented in Figure 1.2. The approach to find a solution differs for each class of hybrid systems. With differences in the formulation of the control problem, as well as different approaches to the optimizing and solution method. A variety of literature considering stabilizing control of switched linear hybrid dynamical systems using Lyapunov arguments and linear matrix inequalities is available, in this review a sample of this literature is cited.

In [36] two main control policies are proposed. A cyclic control policy, which starts with emptying the currently served buffer then switches to the next, subsequently empties this buffer, and moves on to the next buffer, repeating this process cyclically. The second type of strategy is referred to as clear the largest buffer level policy, it clears the buffer with the largest scaled content and switches to the next buffer with the largest scaled content. It either starts at the buffer that is served at $t = 0$ or it immediately switches to the buffer with the largest scaled content depending on the specific setting of the strategy. As mentioned in [1, 36] these control methods are not necessarily stable policies.
A different method of switched linear systems control is mentioned in [34], an approach based on model predictive control (MPC). Each time the system states are sampled an optimal control problem is solved repeatedly via numerical optimization. The control problem and optimization are based on given constraints, weights and the prevailing state values as initial conditions, the latter are either measured or estimated.
In addition to continuous-time hybrid models, discrete-time hybrid models are examined in literature. An off-line method to determine optimal control rules for discrete-time linear hybrid systems is the main result of [38]. Optimal control rules are determined based on the minimization of performance indicators subject to linear constraints on states and inputs. The focus is on piecewise affine models in discrete-time, such as discrete-time switching systems where the dynamic behaviour of the system is described by linear models and logic rules for switching between these models.
Most studies described in literature assume the detection of states and switching signals is instant, but in real systems detection does not occur instantly. In [39] a design method is given for a stabilizing state feedback controller of switched linear systems, the method includes time-delay in detection of switching signals. When the controller is started there is no feedback added to the system. When the value of the switching signal is received the corresponding feedback is added. If a change in the signal is detected the state feedback is adjusted accordingly based on the current state data and the current feedback values. Apart from the in-line state feedback controller that is designed an off-line version is given as well, meaning all state feedback values can be computed beforehand.

Based on Phase Dynamical Operators

In [37] and the literature therein it is mentioned that clearing policies and clear a fraction policies do not necessarily stabilize the subject systems. Often, the proposed policies start as an algorithm and the resulting system behaviour is studied. This type of control limits optimization to fine tuning parameters of a predetermined policy. Optimal scheduling of systems with setups is extensively discussed in literature, however the focus is on open-loop schedules, which do not cope well with disturbances. Therefore, in [37] a general approach is considered in which an optimal steady-state periodic behaviour is determined for a fluid queueing network, after which a feedback policy is designed that over time ensures convergence to the desired periodic behaviour. The designed feedback controller should result in globally attracting periodic behaviour. Deriving a set of phase control rules is the main topic of the research. Phase control rules are rules that determine which mode the system is in, until when it stays in that mode, and the next mode the system should switch to. The operator that maps the system states at the beginning of a mode to the system states at the end of a mode is called the phase dynamical operator. A specific set of properties are required for the phase dynamical operator, compositions of these phase dynamical operators inherit the properties. The composition of all phase dynamical operators of the entire cycle of modes is called the monodromy operator. Because the composition of the phase dynamical operators inherit the required properties, the properties automatically hold for the monodromy operator. The required properties are determined such that if they hold for the monodromy operator, global stability of the desired periodic behaviour is ensured.

Based on the results of [37] a different control policy can be proposed. A policy that creates phase dynamical operators holding the required properties, thus making the desired steady-state behaviour globally attractive. In this policy not only the buffer contents of the buffer to clear per mode are taken into account, but the buffer levels of all other served buffers per mode in the system are of interest. Based on the desired periodic behaviour, threshold values for the other served buffers are provided. If the threshold value is reached for a served buffer that does not need to be cleared during that mode, the buffer is served at the arrival rate. This ensures that the buffer content remains at the threshold value. For unserved buffers the threshold value determines whether switching should be postponed for the buffer to receive more items, or if the buffer content exceeds the threshold value and switching can occur. This policy has a corresponding monodromy operator that meets all the requirements, which means the system is asymptotically stable. Although the policy stabilizes the system in the desired steady-state behaviour, the process rate of traffic cannot be restricted. Vehicles controlled by drivers can only be forced to a certain extend to approach an intersection at a certain rate. Therefore, this controller is not suitable for this thesis subject system.

Based on Lyapunov Functions

Often in controller design the system behaviour, as a result of applying a specific control policy, is studied. If this behaviour is not the desired system behaviour, the policy is changed or adjusted. The approach of [1] is to start by studying the desired steady-state behaviour, then derive a control policy based on this desired periodic behaviour. In [1] this approach is successfully performed for some feasible part of the domain, the feasible part of the domain is not the complete domain.

The example system in [1] contains non zero setup times, hence the steady state of the system is not a fixed point but a periodic cycle. A solution is searched to the problem of controlling the system states to equal the desired given optimal periodic cycle. Given this desired periodic cycle and the system, a controller needs to be designed. The design is based on Lyapunov's direct method. The basic idea of Lyapunov's direct method is that, if a Lyapunov function is continuously decreasing over time, the system states move towards their equilibrium value. In a mechanical system such a function is almost always based on a type of energy function, as energy is continuously decreasing over time when the system moves towards an equilibrium.

To use Lyapunov's direct method, a candidate Lyapunov function that can be associated with the system should be found. Apart from being continuously decreasing when moving towards the equilibrium states, a candidate Lyapunov function should equal zero at the equilibrium value. Furthermore, for all state values that are unequal to the equilibrium value, the Lyapunov function value should be larger than zero. As an energy function cannot be computed for a manufacturing system it is suggested in [1] to take a function with similar properties, namely the additional amount of mean work in the system in periodic behaviour. Work in this case does not refer to the amount of items in the system, but the time it takes the server to clear all items. With additional amount of mean work being the average amount of work in the system in periodic behaviour, compared to the average amount of work in the system during the desired steady-state behaviour. The value of the extra mean work in the system equals the minimum mean work in the system

during periodic behaviour. This minimum mean work in the system depends on the duration of the first mode when the fixed time schedule is repeatedly executed. The minimum mean work in the system is achieved when the duration of the first mode is chosen such that, the extra mean work in the system in periodic behaviour is as low as possible for the starting values. The extra work is then defined as the difference between the minimum mean work in the system, and mean work in the system in the optimal periodic behaviour.
The definition of work in the system means the extra work in the system is always larger than zero unless the states are in the periodic cycle. Whether the function is continuously decreasing depends on the input of the system. The input can be chosen such that the extra work is always decreasing. Since the input depends on the current state the resulting controller is a feedback controller.

The candidate Lyapunov function is only defined for a large subset of the state space. An extension is needed to design a controller for the complete domain. Finally it is shown that the derived controller makes the system converge to the periodic cycle. For more details on the approach or results, the interested reader is referred to [1].

2.2.3 Analysis of Hybrid Dynamical Systems

The most important part in the analysis of hybrid dynamical systems is studying the stability of the system. An unstable open loop system can be stabilized by adding a feedback control action. A systematic procedure to derive hybrid feedback stabilization methods does currently not exist. Thinking of LaSalle's extension to Lyapunov stability theory; along trajectories the value of the Lyapunov function should always be non-increasing and an invariant set should exist where the Lyapunov function value remains constant. When taking this Lyapunov approach the difficulty is proving that the energy of the system is indeed in all cases non-increasing and that in the end all energy is dissipated [35].

In [40] a basic systematic approach to stability analysis of a controlled hybrid system is taken. The transient behaviour of the controlled system can be reviewed by a graphical representation of the modes. In the modes graph each mode is presented as a circle with the mode labelled in it and the state values. The arrows between the modes represent possible transitions between modes, it is a graphical method of representing maps and images. Studying this graph repeatedly one might conclude that after a while some modes cannot be entered any more ending up with periodic behaviour, if this periodic behaviour is the desired periodic behaviour the system is converging.

It is known that different switching policies result in different system behaviour, hence different switching strategies can result in different system performance [41]. If there exist periodic behaviour that attracts all trajectories the system is called globally periodic [36], in that case the system is called stabilizable. In case the switched linear system is force free [41] shows that asymptotic stabilizability, exponential stabilizability and switched convergence are equivalent for switched linear systems. Furthermore, for a variety of cost indices it is shown in [41] that reaching a finite optimum for the cost in the infinite-time horizon optimization problem is equivalent to the system being asymptotically stabilizable.
In theory very fast switching between modes could lead to stabilization of the system, imagine holding a stick on your finger and trying to balance it. This phenomena, stabilizing by fast switching is called chattering. Contrary to the before mentioned example, this phenomena is generally undesired system behaviour. There are different methods to prevent chattering [35], however these are not discussed in this review. Due to the setup times, the non zero time needed to switch between modes, chattering is automatically suppressed.
Finally, in [42] an overview of studies on stability of switched linear systems is given. This paper outlines various switching stabilization methods given in literature. It also reviews the multiple Lyapunov function theory and the stability analysis based on that. A rather intuitive example of such a theory is a stabilizable system of which stabilizability is determined based on a multiple

Lyapunov theory. This is a system is stabilizable if the Lyapunov like function decreases when the mode this function corresponds to, is active.

From the literature review presented in this chapter, it is concluded that vehicle actuated control policies are potentially more efficient than fixed time control strategies. However, fixed time control strategies are applicable in congested situations. An ideal solution would be to have a control policy that converges to the optimal fixed time control policy over time. Based on the hybrid dynamical model of an intersection discussed in this chapter, an attempt is made to derive such a policy in this thesis. In Chapter 3, the method discussed as 'Based on Lyapunov Functions' is used to derive candidate Lyapunov functions of example intersections.

Chapter 3. Lyapunov Function Candidate

A subsection of Subsection 2.2.2 briefly summarized the approach proposed in [1] to derive a control policy. The candidate Lyapunov function derived in [1] is defined for a large subset of the state space. To find a candidate Lyapunov function that covers the entire domain, an adjustment is made to the candidate Lyapunov function derivation in this chapter. Furthermore, this chapter indirectly illustrates the complexities when the system dimensions are increased.
Basically a similar approach as mentioned in [1] is used to derive a candidate Lyapunov function. The derivation of a candidate Lyapunov function is explained by discussing an illustrative example system for each system. A less intuitive result is the algorithm of finding a candidate Lyapunov function for a general system of two directions is presented in Appendix C.

In Section 3.1 a candidate Lyapunov function is derived for an example of an intersection of two directions. In Section 3.2 the dimension of the example system is increased by adding a direction to the intersection. The derivation of a candidate Lyapunov function for the three direction system is discussed relatively concisely, since the derivation of a candidate Lyapunov function is similar to the derivation explained in Section 3.1. However, the derivation is explained in more detail when the complexity of the derivation grows or significantly deviates from the derivation presented in Section 3.1. Finally, the observed effects of the increase in system size are discussed.

3.1 Intersection of Two Directions

The system discussed in this section equals the first example system discussed in [1]. The system can be visualized as a simple intersection of two conflicting flows, as illustrated in Figure 3.1. The optimal fixed time schedule that corresponds to this system is given in Figure 3.2 and defines the desired steady-state behaviour. The fixed time schedule is an optimal schedule for the given arrival and service rates, which equal $\lambda_1 = 3$, $\lambda_2 = 1$, $\mu_1 = 8$ and $\mu_2 = 9$ in this particular example system.

The total number of flows in the example system equals two, $N = \{1, 2\}$. Vehicles corresponding to direction $i = 1$ or $i = 2$ are respectively referred to as type 1 and type 2. Switching from processing type 1 to processing type 2 requires $\sigma_{B} = 3$ time units, whereas switching vice versa takes $\sigma_{A} = 1$ time unit.
The server is in one of two modes, in $\boldsymbol{m} = \boldsymbol{A}$ performing setup or processing type 1, or in $\boldsymbol{m} = \boldsymbol{B}$ performing setup or processing type 2. Flow 1 and flow 2 are conflicting flows, as illustrated in Figure 3.1. Hence the server can only serve one type at the time when processing. If the server is in setup of a mode, it can process neither type 1, nor type 2.

The definition of work in the system, priorly stated in (1.1), is a function of the buffer contents x_i. The optimal work in the system as function of time is computed and illustrated in Figure 3.3. This figure is used as a reference throughout this section. Notice that during the slow mode of A the total amount of items in the system increases, as can be seen from $t = 4$ to $t = 5$ in Figure 3.3.

It is assumed that the arrival and process rates are constant, and equivalent to the rates for which the fixed time schedule is designed. If this assumption holds, repeatedly executing the fixed time schedule stabilizes the system in some periodic behaviour, the proof of which can be found in Appendix B. The steady-state behaviour obtained by repeatedly executing the fixed time schedule,

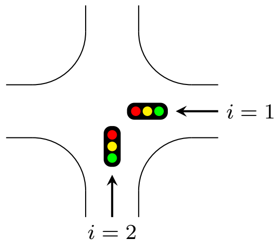

Figure 3.1: Example intersection, $N = \{1, 2\}$.

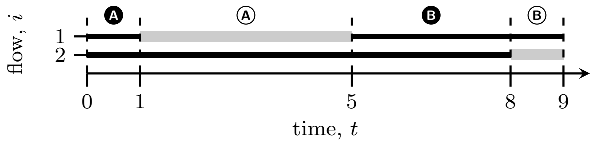

Figure 3.2: Fixed time schedule, $N = \{1, 2\}$.

does not necessarily equal the desired steady-state behaviour. Boundedness of the buffer contents as a system property is however achieved by executing the fixed time schedule repeatedly.
In Figure 3.4 the desired buffer contents are plotted as function of time, the values are computed from the fixed time schedule. The buffer contents of Figure 3.4 are illustrated as periodic cycle in Figure 3.5. These buffer contents in the desired steady-state are referred to as optimal buffer contents. At the start of setup in a mode or processing in a mode, the values of x_1, x_2 can be denoted as x_1^{s*}, x_2^{s*}, with $s \in \{$🅐, Ⓐ, 🅑, Ⓑ$\}$. For instance, $x_1^{Ⓐ*} = 15$ and $x_2^{Ⓐ*} = 1$, other values of x_1^{s*}, x_2^{s*} can straightforwardly be obtained from Figure 3.5. The plots, and all upcoming illustrations in this section, are based on example specific values. These values are computed with data from the fixed time schedule and the system parameters of the illustrative example.

Remark 1. *In the remainder of this chapter the periodic cycle determined by the fixed time schedule is referred to as the optimal periodic cycle or, optimal steady-state behaviour. Although it might not be the actual optimal periodic behaviour for some performance indicators, it is the desired steady-state behaviour computed based on the given optimal fixed time schedule. Therefore, it is stated to be the optimal periodic cycle.*

The value of the candidate Lyapunov function $V(s, x_1, x_2)$, is defined as the minimum mean extra work in the system in steady-state. Extra work is defined as the difference between mean work in steady-state obtained when the fixed time schedule is executed repeatedly, and the mean work in optimal periodic behaviour.
The steady-state used in the derivation of the candidate Lyapunov function is obtained by repeatedly executing the fixed time schedule. Repeatedly executing the fixed time schedule is an action that can be performed in several ways. In any derivation of a candidate Lyapunov function in this thesis the following is meant. The system starts in a given mode, it can either stay in this start-up mode or switch to the successive mode listed in the fixed time schedule. The time spent in each mode, subsequent to the start-up mode, is defined in the fixed time schedule and does not depend on the buffer contents. Except for the start-up mode, the server is obliged to stay in a mode for the duration of the mode registered in the fixed time schedule.
Furthermore, repeatedly executing the fixed time schedule is performed such that the slow mode is performed for at least the duration of the slow mode listed in optimal periodic behaviour. The duration of the slow mode equals the duration of $x_1 = 0$ in Ⓐ or $x_2 = 0$ in Ⓑ in Figure 3.4.

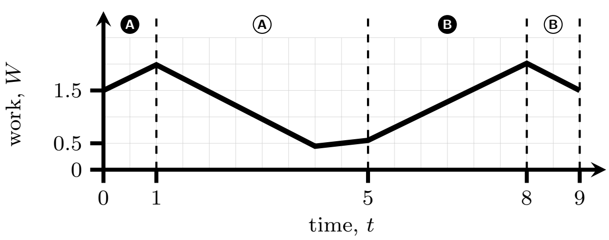

Figure 3.3: Optimal work in the system, $N = \{1, 2\}$.

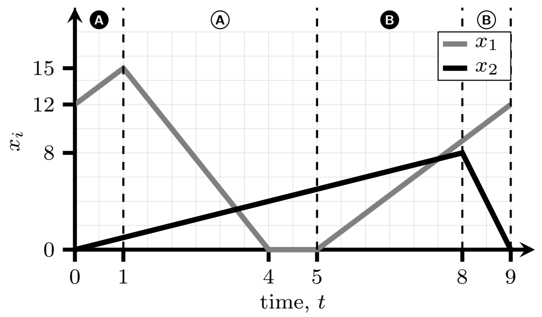

Figure 3.4: x_i in optimal cycle, $N = \{1, 2\}$.

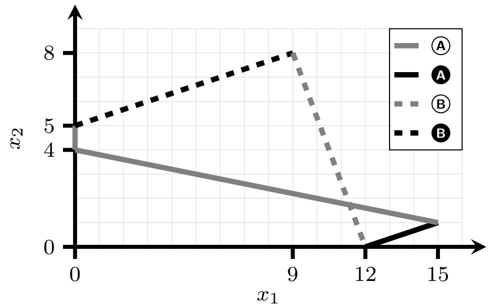

Figure 3.5: Optimal cycle, $N = \{1, 2\}$.

The duration of the slow mode is extended in case the minimum duration of the slow mode has elapsed, whilst the time spent in the entire mode is still unequal to the duration of the mode denoted in the fixed time schedulee. The maximum duration of the slow mode in $\boldsymbol{m} = \boldsymbol{A}$ or $\boldsymbol{m} = \boldsymbol{B}$ respectively equals the duration of Ⓐ, Ⓑ registered in the fixed time schedule. If the system is not in (extended) slow mode, the served buffer content x_i is processed at rate μ_i. Thus the slow mode is required to be executed for at least the duration of the slow mode in optimal periodic behaviour. This completes the definition of repeatedly executing the fixed time schedule.

When the fixed time schedule is started, the server starts in setup or processing of one of the systems modes. The start-up mode refers to the system settings when the server is started.
The first time instant the server is switched on, there is no knowledge of the elapsed time. Hence, the remaining time to spend in the start-up mode is yet to be determined. Therefore, it can be chosen such that the resulting steady-state contains the least extra work in the system possible. However, the time spent in start-up mode needs to match the fixed time schedule. For instance, if the start-up mode is Ⓐ, the time to stay in that mode can be set to any value between zero and the duration of Ⓐ registered in the fixed time schedule. An example is given below to clarify what is meant by choosing the optimal remaining time in the start-up mode.
Assume the server started in Ⓐ, with $x_1 = 15$ and $x_2 = 1$, this start is represented by the black dot and arrow in Figure 3.6. If the remaining time in Ⓐ is set to four time units, the systems steady-state is equivalent to the optimal periodic cycle. The black line in Figure 3.6 illustrates the desired steady-state behaviour with corresponding candidate Lyapunov function value,

$$V\left(Ⓐ, 15, 1\right) = \frac{x_1 - x_1^{Ⓐ*}}{\mu_1} = \frac{15 - 15}{8} = 0. \tag{3.1}$$

If the remaining time in Ⓐ was set to two time units, executing the fixed time schedule repeatedly

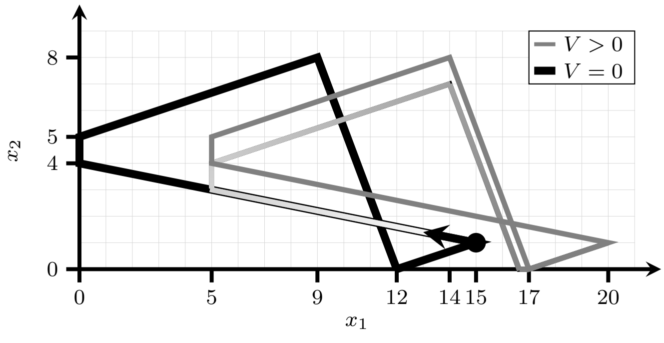

Figure 3.6: Visualisation of time spent in the start-up mode.

first yields the grey gradient line in Figure 3.6. This grey gradient line illustrates the transient behaviour of the system states. The system stabilizes in a periodic cycle represented by the grey solid line in Figure 3.6. The extra items in buffer 1 determine the extra work in periodic behaviour. Therefore, the candidate Lyapunov function equals,

$$V\left(\text{Ⓐ}, 0, 15, 1\right) = \frac{x_1 - x_1^{\text{Ⓐ}*}}{\mu_1} = \frac{20-15}{8} = \frac{5}{8}. \tag{3.2}$$

This example illustrates the fact that the time spent in start-up mode, influences the candidate Lyapunov function value. The optimal choice of the time spent in start-up mode is the time that minimizes the extra work in the system in periodic behaviour. In this particular example, the optimal choice of the time spent in start-up mode is four time units.
The value of the candidate Lyapunov function is derived in mode $\boldsymbol{A}$ in Subsection 3.1.1 and for mode $\boldsymbol{B}$ in Subsection 3.1.2, for all values of $\{x_1, x_2\} \in \mathbb{R}_+^2$, with the subscript + referring to the all non-negative values. The mean extra work in the system in steady-state is derived for the time spent in start-up mode as explained above.

3.1.1 Mode $\boldsymbol{A}$, $N = \{1, 2\}$

First the candidate Lyapunov function is derived for x_1, x_2 and a start-up in Ⓐ. The candidate Lyapunov function depends on the start values, $\{x_1, x_2\} \in \mathbb{R}_+^2$, different start values result in different candidate Lyapunov function values. Figure 3.7 is presented to visualize the different domain parts of the candidate Lyapunov function in Ⓐ. The boundaries are a result of the candidate Lyapunov function derivation, and explicitly denoted in (3.3).

$$\begin{array}{lll}
\mathscr{D}^{\text{I}} & \text{for} \quad x_1 \geq 15, & x_2 \geq 1, \\
\mathscr{D}^{\text{II}} & \text{for} \quad x_1 \geq 15, & x_2 \leq 1, \\
\mathscr{D}^{\text{III}_\text{a}} & \text{for} \quad \frac{40}{37} \leq x_1 \leq 15, & x_2 \geq 5, \\
\mathscr{D}^{\text{III}_\text{b}} & \text{for} \quad x_1 \leq 15, & 4 - \frac{1}{5}x_1 \leq x_2 \leq 4 + \frac{37}{40}x_1 \leq 5, \\
\mathscr{D}^{\text{IV}} & \text{for} \quad x_1 \leq \frac{40}{37}, & x_2 \geq 5, \\
\mathscr{D}^{\text{V}} & \text{for} \quad x_1 \leq \frac{40}{37}, & 4 + \frac{37}{40}x_1 \leq x_2 \leq 5, \\
\mathscr{D}^{\text{VI}} & \text{for} \quad x_1 \leq 15, & x_2 \leq 4 - \frac{1}{5}x_1.
\end{array} \tag{3.3}$$

The derivation of a Lyapunov function candidate is started in $\mathscr{D}^{\text{I}}$ of Ⓐ. Studying the work in Figure 3.3 and varying the duration of the start-up mode results in Figure 3.8. This figure

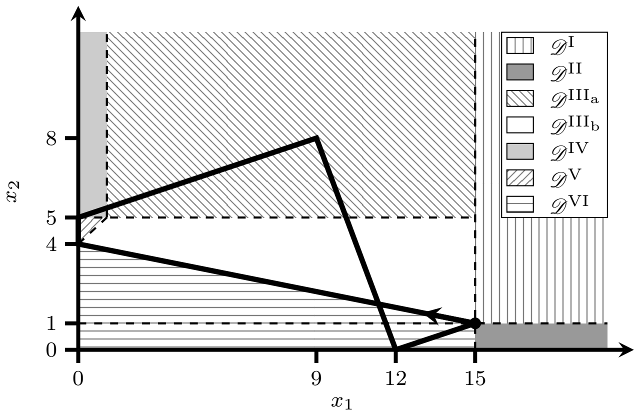

Figure 3.7: Domain parts in Ⓐ.

visualizes that the duration of the start-up mode that results in minimum mean extra work in steady-state, is the maximum duration of Ⓐ. A similar figure is found in [1]. The dashed line in Figure 3.8 is the result of a non-optimal choice for the time spent in the start-up mode. The black line represents the result of the optimal value of the time spent in the start-up mode. The grey line represents the work in the desired periodic behaviour.
Neither x_1 nor x_2 with start values in $\mathscr{D}^{\mathrm{I}}$ of Ⓐ, becomes zero when the fixed time schedule is executed repeatedly. Which leaves to conclude that the system is immediately in its steady-state. The mean extra work in steady-state depends on the values of x_1 and x_2, hence (3.4) states the Lyapunov function candidate in $\mathscr{D}^{\mathrm{I}}$ of Ⓐ.

$$\mathscr{D}^{\mathrm{I}}: \quad V\left(Ⓐ, x_1, x_2\right) = \tfrac{x_1-15}{8} + \tfrac{x_2-1}{9} \quad \text{for } x_1 \geq 15, \quad x_2 \geq 1. \tag{3.4}$$

In $\mathscr{D}^{\mathrm{II}}$ of Ⓐ the minimum mean extra work in periodic behaviour is obtained if the time in start-up mode is set to the duration of Ⓐ registered in the fixed time schedule. The mean extra work in the system depends on the excess content of buffer 1, as illustrated in Figure 3.9. The plot in Figure 3.9 starts in $x_1 = 16$ and $x_2 = 0$. The system states progress, illustrated by the grey gradient line, until it stabilizes in the periodic behaviour, illustrated by the black line in Figure 3.9. This figure also visualizes that x_2 reaches optimal periodic behaviour. Thus, the content of buffer 2 does not contribute to the extra work in periodic behaviour, and the candidate Lyapunov function in $\mathscr{D}^{\mathrm{II}}$ of Ⓐ equals (3.5).

$$\mathscr{D}^{\mathrm{II}}: \quad V\left(Ⓐ, x_1, x_2\right) = \tfrac{x_1-15}{8} \quad \text{for } x_1 \geq 15, \quad x_2 \leq 1. \tag{3.5}$$

The line $x_2 = 4 - \frac{1}{5}x_1$ defines the boundary between $\mathscr{D}^{\mathrm{III_b}}$ and $\mathscr{D}^{\mathrm{VI}}$, as shown in Figure 3.7. $\mathscr{D}^{\mathrm{VI}}$ is not a part of the feasible domain in [1], the derivation of [1] is adjusted to derive a Lyapunov function candidate in the entire domain. $\mathscr{D}^{\mathrm{VI}}$ consists of two parts, the part where $x_2 \geq 1$ and the part where $x_2 \leq 1$. For both parts it is possible to set the time spent in the start-up mode such that the resulting steady-state always equals the desired periodic cycle. Both starting values, $x_2 \leq 1$ and $x_2 \geq 1$, result in optimal periodic behaviour as illustrated in Figure 3.10. The black line in this figure represents the steady-state, the transient behaviour for both starting values is given in a grey gradient line. For all values of x_1, x_2 in $\mathscr{D}^{\mathrm{VI}}$ of Ⓐ, the steady-state equals the desired periodic cycle, the corresponding candidate Lyapunov function is (3.6).

$$\mathscr{D}^{\mathrm{VI}}: \quad V\left(Ⓐ, x_1, x_2\right) = 0 \quad \text{for } x_1 \leq 15, \quad x_2 \leq 4 - \tfrac{1}{5}x_1. \tag{3.6}$$

The yet unexamined part of the domain requires thorough examination. The existence of a slow mode introduces extra complexity in the derivation of a candidate Lyapunov function in this part of the domain. Figure 3.3 illustrates that the work in the system increases during slow mode

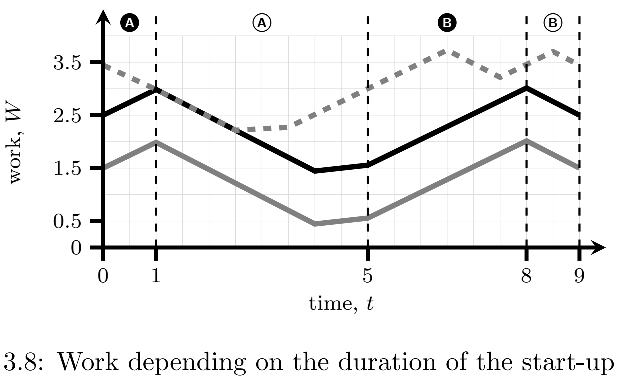

Figure 3.8: Work depending on the duration of the start-up mode.

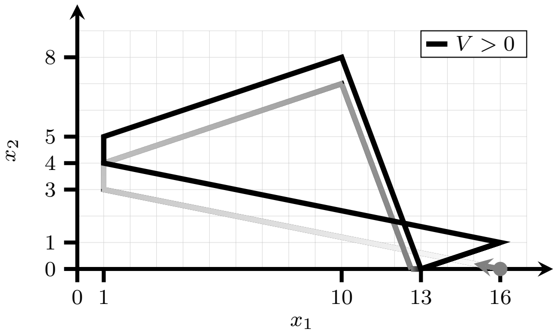

Figure 3.9: A periodic cycle in $\mathscr{D}^{\mathrm{II}}$ of A.

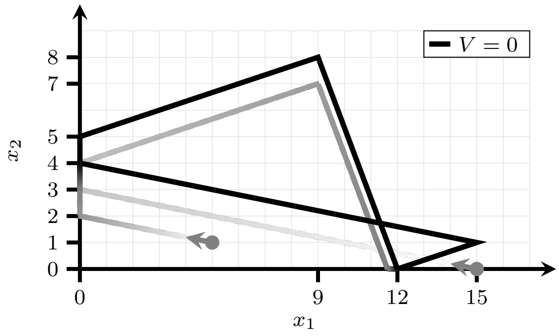

Figure 3.10: A periodic cycle in $\mathscr{D}^{\mathrm{VI}}$ of A.

of A. Which implies that if the value of x_1 becomes small enough, immediately progressing to B might result in minimum mean extra work in steady-state. The domains $\mathscr{D}^{\mathrm{III_a}}$, $\mathscr{D}^{\mathrm{III_b}}$, $\mathscr{D}^{\mathrm{IV}}$ and $\mathscr{D}^{\mathrm{V}}$ are studied merged as $\mathscr{D}^{\mathrm{III}}$. Hereby the domain splits automatically appear in the derivation of a candidate Lyapunov function.
First the part of $\mathscr{D}^{\mathrm{III}}$ where $x_2 \geq 5$ is examined. The candidate Lyapunov function is derived for cases the system starts and continues in A, the resulting periodic behaviour is illustrated in Figure 3.11. This figure illustrates that the optimal duration of the start-up mode, equals the time it takes for x_1 to become equal to zero, plus the minimum duration of the slow mode. This results in the optimal periodic value of x_1. Buffer 2 preserves the extra amount of work it contained when the server started. The candidate Lyapunov function if the system starts and continues in A, is given in (3.7).

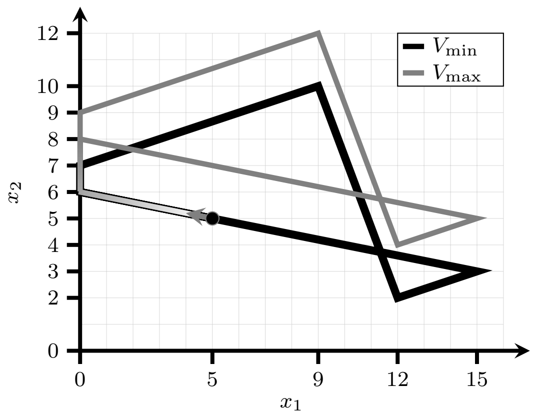

Figure 3.11: Periodic cycles for A, $\mathscr{D}^{\mathrm{III}}$.

Another option in $\mathscr{D}^{\mathrm{III}}$ with $x_2 \geq 5$ is that the system starts in A but immediately switches to B. In that case the duration of the start-up mode equals zero. Because of the fact that $x_2 \geq 5$ and the time spent in the start-up mode equals zero, the system is in its steady-state from the start. The value of x_1 increases and decreases by 15 items each cycle, so the start value of x_1 equals the extra content in buffer 1. If $x_2 = 5$ the content of buffer 2 does not contribute to the mean extra work in the system, whereas in all other cases x_2 does contribute to the value of work in periodic behaviour. Thus, the candidate Lyapunov function in $\mathscr{D}^{\mathrm{III}}$ with $x_2 \geq 5$ equals (3.8), when the system starts in A and immediately switches to B.

$$\mathscr{D}^{\mathrm{III_a}}: \quad V_{\mathrm{process}}\left(\text{Ⓐ}, x_1, x_2\right) = \tfrac{x_2-\left(4-\frac{1}{5}x_1\right)}{9} = \tfrac{x_2+\frac{1}{5}x_1-4}{9} \quad \text{for} \ \ x_1 \leq 15, \quad x_2 \geq 5. \tag{3.7}$$

$$V_{\mathrm{setup}}\left(\text{Ⓐ}, x_1, x_2\right) = \tfrac{x_1}{8} + \tfrac{x_2-5}{9} \quad \text{for } x_1 \leq 15, \quad x_2 \geq 5. \tag{3.8}$$

Equation (3.7) and (3.8) are compared to determine the values of x_1, $x_2 \geq 5$ in $\mathscr{D}^{\mathrm{III}}$, for which immediately switching to setup results in less mean extra work in the system in periodic behaviour. If $x_1 \leq \frac{40}{37}$ switching to mode $\boldsymbol{B}$ results in equal or less mean extra work in steady-state. Which defines a split in $\mathscr{D}^{\mathrm{III}}$ with $x_2 \geq 5$. The part of $\mathscr{D}^{\mathrm{III}}$ with $x_1 \leq \frac{40}{37}$, $x_2 \geq 5$ is referred to as $\mathscr{D}^{\mathrm{IV}}$. The part of $\mathscr{D}^{\mathrm{III}}$ where $x_1 \geq \frac{40}{37}$, $x_2 \geq 5$ is called $\mathscr{D}^{\mathrm{III_a}}$. The candidate Lyapunov function in $\mathscr{D}^{\mathrm{III_a}}$ and $\mathscr{D}^{\mathrm{IV}}$ respectively equal (3.9) and (3.10).

$$\mathscr{D}^{\mathrm{III_a}}: \quad V\left(\text{Ⓐ}, x_1, x_2\right) = \tfrac{x_2+\frac{1}{5}x_1-4}{9} \quad \text{for } \tfrac{40}{37} \leq x_1 \leq 15, \quad x_2 \geq 5. \tag{3.9}$$

$$\mathscr{D}^{\mathrm{IV}}: \quad V\left(\text{Ⓐ}, x_1, x_2\right) = \tfrac{x_1}{8} - \tfrac{x_2-5}{9} \quad \text{for } x_1 \leq \tfrac{40}{37}, \quad x_2 \geq 5. \tag{3.10}$$

Figure 3.7 shows two more domain parts in $\mathscr{D}^{\mathrm{III}}$ for values that meet $x_2 \leq 5$. If the system starts and continues in Ⓐ the candidate Lyapunov function is equal to the candidate Lyapunov function in $\mathscr{D}^{\mathrm{III}}$ with $x_2 \geq 5$, which yields (3.11). When $x_2 \leq 5$ in $\mathscr{D}^{\mathrm{III}}$ and the server immediately switches to 🅑, optimal periodic behaviour is obtained for x_2. The mean extra work in the system in periodic behaviour is determined by the content of buffer 1, the corresponding candidate Lyapunov function equals (3.12).
The domain parts $\mathscr{D}^{\mathrm{III_b}}$ and $\mathscr{D}^{\mathrm{V}}$ are defined by comparing (3.11) and (3.12). Based on that comparison the candidate Lyapunov function of $\mathscr{D}^{\mathrm{III_b}}$ and $\mathscr{D}^{\mathrm{V}}$ are (3.13) and (3.14) respectively.

$$V_{\mathrm{process}}\left(\text{Ⓐ}, x_1, x_2\right) = \tfrac{x_2+\frac{1}{5}x_1-4}{9} \quad \text{for } x_1 \leq 15, \quad 4-\tfrac{1}{5}x_1 \leq x_2 \leq 5. \tag{3.11}$$

$$V_{\mathrm{setup}}\left(\text{Ⓐ}, x_1, x_2\right) = \tfrac{x_1}{8} \quad \text{for} \ \ x_1 \leq 15, \quad 4-\tfrac{1}{5}x_1 \leq x_2 \leq 5. \tag{3.12}$$

$$\mathscr{D}^{\mathrm{III_b}}: \quad V\left(\text{Ⓐ}, x_1, x_2\right) = \tfrac{x_2+\frac{1}{5}x_1-4}{9} \quad \text{for} \ \ x_1 \leq 15, \quad 4-\tfrac{1}{5}x_1 \leq x_2 \leq \tfrac{37}{40}x_1 + 4 \leq 5. \tag{3.13}$$

$$\mathscr{D}^{\mathrm{V}}: \quad V\left(\text{Ⓐ}, x_1, x_2\right) = \tfrac{x_1}{8} \quad \text{for} \ \ x_1 \leq 15, \quad \tfrac{37}{40}x_1 + 4 \leq x_2 \leq 5. \tag{3.14}$$

This completes the derivation of a candidate Lyapunov function in case the system starts in Ⓐ. The final part of the domain in mode $\boldsymbol{A}$ are the values in 🅐. A different definition is used than the definition given in [1] if the system starts in setup of mode $\boldsymbol{A}$. In [1] when the server starts in setup, it is stated there is only one possible periodic cycle. This definition is adjusted in this study to reach optimal periodic behaviour, as time efficiently as possible.
The candidate Lyapunov function in setup is set equal to the candidate Lyapunov function value x_0 time units later in Ⓐ. The values of x_1 and x_2 are replaced in the candidate Lyapunov function of Ⓐ, by their values x_0 time units later. During setup both x_1 and x_2 are not served, thus the candidate Lyapunov function when the system is in setup of mode $\boldsymbol{A}$ equals (3.15).

$$V\left(\text{🅐}, x_1, x_2\right) = V\left(\text{Ⓐ}, x_1 + 3x_0, x_2 + x_0\right). \tag{3.15}$$

This completes the candidate Lyapunov function for all values of x_1, x_2 in mode $\boldsymbol{A}$. A similar derivation is performed for mode $\boldsymbol{B}$ in the next subsection.

3.1.2 Mode $\boldsymbol{B}$, $N = \{1, 2\}$

Similar to mode $\boldsymbol{A}$, the complete domain $\{x_1, x_2\} \in \mathbb{R}^2_+$ is divided in parts. Figure 3.12 visualizes the different domain parts in mode $\boldsymbol{B}$. The domain part boundaries are explicitly denoted in (3.16). The domain parts have equivalent subscripts in mode $\boldsymbol{A}$ and mode $\boldsymbol{B}$ if they have similar properties in both modes. This notation emphasizes the structure of the candidate Lyapunov function derivation.
Mode $\boldsymbol{B}$ does not contain a slow mode, which significantly shortens and simplifies the derivation of a candidate Lyapunov function. $\mathscr{D}^{\mathrm{III_a}}$ and $\mathscr{D}^{\mathrm{III_b}}$ are merged into $\mathscr{D}^{\mathrm{III}}$ whilst $\mathscr{D}^{\mathrm{IV}}$ and $\mathscr{D}^{\mathrm{V}}$ of mode $\boldsymbol{A}$ do not have an equivalent in mode $\boldsymbol{B}$.

$$\begin{array}{lll l} \mathscr{D}^{\mathrm{I}} & \text{for} & x_1 \geq 9, & x_2 \geq 8, \\ \mathscr{D}^{\mathrm{II}} & \text{for} & x_1 \leq 9, & x_2 \geq 8, \\ \mathscr{D}^{\mathrm{III}} & \text{for} & x_1 \geq 12 - \frac{3}{8}x_2, & x_2 \leq 8, \\ \mathscr{D}^{\mathrm{VI}} & \text{for} & x_1 \leq 12 - \frac{3}{8}x_2, & x_2 \leq 8. \end{array} \tag{3.16}$$

The derivation of a candidate Lyapunov function is started in the domain part with the most straightforward result, $\mathscr{D}^{\mathrm{I}}$ of Ⓑ. From Figure 3.3 it is concluded that the mean extra work in periodic behaviour is minimized, if the duration of the start-up mode equals the maximum duration of Ⓑ. Therefore, the candidate Lyapunov function in $\mathscr{D}^{\mathrm{I}}$ of Ⓑ equals (3.17).

$$\mathscr{D}^{\mathrm{I}}: \quad V\left(Ⓑ, x_1, x_2\right) = \tfrac{x_1-9}{8} + \tfrac{x_2-8}{9} \quad \text{for } x_1 \geq 9, \quad x_2 \geq 8. \tag{3.17}$$

Figure 3.3 leaves to conclude that in $\mathscr{D}^{\mathrm{II}}$ of Ⓑ the mean extra work is minimized if the time in the start-up mode is set to the maximum duration of Ⓑ. This is emphasized by the plot in Figure 3.13. This figure illustrates the transient behaviour to steady-state if the duration of the start-up mode is zero, which yields V_{max}. If the time spent in the start-up mode equals the maximum duration of Ⓑ, V_{min} is obtained via transient behaviour.
Figure 3.13 illustrates that the value of x_1 does not contribute to the extra work in the system in $\mathscr{D}^{\mathrm{II}}$. The candidate Lyapunov function in $\mathscr{D}^{\mathrm{II}}$ of Ⓑ equals (3.18), a function of the content of buffer 2.

$$\mathscr{D}^{\mathrm{II}}: \quad V\left(Ⓑ, x_1, x_2\right) = \tfrac{x_2-8}{9} \quad \text{for } x_1 \leq 9, \quad x_2 \geq 8. \tag{3.18}$$

In $\mathscr{D}^{\mathrm{VI}}$ of Ⓑ the time in start up mode is set such that a slow mode of Ⓑ gives $x_1 = 12$ exactly

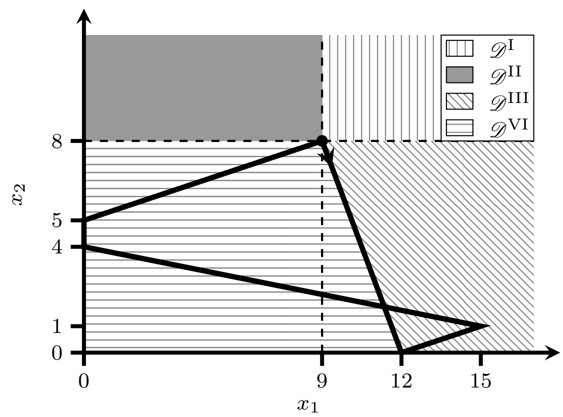

Figure 3.12: Domain parts in Ⓑ.

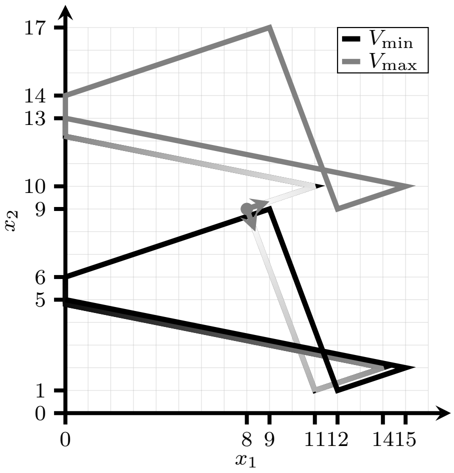

Figure 3.13: Periodic cycles in $\mathscr{D}^{\mathrm{II}}$, Ⓑ.

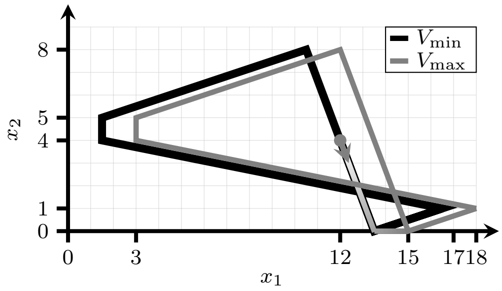

Figure 3.14: Periodic cycles in $\mathscr{D}^{\mathrm{III}}$, Ⓑ.

when 🅐 starts. Therefore, there is no extra work in the system during steady-state and the candidate Lyapunov function in $\mathscr{D}^{\mathrm{VI}}$ of Ⓑ equals (3.19).

$$\mathscr{D}^{\mathrm{VI}}: \quad V\left(\text{Ⓑ}, x_1, x_2\right) = 0 \quad \text{for } x_1 \leq 12 - \tfrac{3}{8}x_2, \quad x_2 \leq 8. \tag{3.19}$$

The final domain part to study is $\mathscr{D}^{\mathrm{III}}$ of Ⓑ. From Figure 3.3 it is concluded that the work is continuously decreasing when the server is processing x_2 at rate μ_2. However, when the server starts processing at rate λ_2 when $x_2 = 0$ the derivative of W becomes (3.20). The work in the system start to increase if the system is in slow mode. Thus the time spent in the start-up mode should equal the time it takes to empty buffer 2.
Figure 3.14 illustrates the mathematical derivation given in (3.20). In this figure the grey line represents the resulting periodic behaviour if the time in the start-up mode is set to the duration of Ⓑ. The black line shows the resulting periodic behaviour if the server switches immediately to setup when $x_2 = 0$. The latter clearly results in less mean extra work in steady-state. The candidate Lyapunov function in $\mathscr{D}^{\mathrm{III}}$ of Ⓑ equals (3.21), it is a function of the extra content of buffer 1.

$$\dot{W} = \frac{3}{8} > 0, \tag{3.20}$$

$$\mathscr{D}^{\mathrm{III}}: \quad V\left(\text{Ⓑ}, x_1, x_2\right) = \frac{x_1 - \left(12 - \frac{3}{8}x_2\right)}{8} = \frac{x_1 + \frac{3}{8}x_2 - 12}{8} \quad \text{for } x_1 \geq 12 - \tfrac{3}{8}x_2, \quad x_2 \leq 8. \tag{3.21}$$

In setup an equivalent definition is chosen for the candidate Lyapunov function as was in mode $\boldsymbol{A}$. Listing the candidate Lyapunov function during setup in mode $\boldsymbol{B}$, (3.22), completes the derivation of a candidate Lyapunov function in the two direction example system. In the upcoming subsection the derived candidate Lyapunov function for the entire state space is presented at once.

$$V\left(\text{🅑}, x_1, x_2\right) = V\left(\text{Ⓑ}, x_1 + 3x_0, x_2 + x_0\right). \tag{3.22}$$

3.1.3 Candidate Lyapunov Function, $N = \{1, 2\}$

Combining expressions of the candidate Lyapunov function for all domain parts in Ⓐ, 🅐, Ⓑ and 🅑, yields an expression for the candidate Lyapunov function in the entire state-space. Although the derivation is extensive, a relatively concise expression of $V(s, x_1, x_2)$, (3.23) is obtained when the derived equations are combined.
The Lyapunov function candidate given in [1] when the system is processing in the feasible part of the domain is equivalent to (3.23). The difference between (3.23) and the result of [1], is the definition of the candidate Lyapunov function in setup. The fact that both results are equivalent for processing in a mode proves that a generalisation of the approach proposed in [1] is established; a candidate Lyapunov function is defined in the entire domain.

$$
\begin{cases}
\frac{x_1-15}{8} + \max\left(\frac{x_2-1}{9}, 0\right) & \text{for Ⓐ}, x_1 \geq 15, \\
\min\left(\frac{x_2+\frac{1}{5}x_1-4}{9}, \frac{x_1}{8} + \frac{x_2-5}{9}\right) & \text{for Ⓐ}, x_1 \leq 15, x_2 \geq 5, \\
\min\left(\frac{x_2+\frac{1}{5}x_1-4}{9}, \frac{x_1}{8}\right) & \text{for Ⓐ}, x_1 \leq 15, 4-\frac{1}{5}x_1 \leq x_2 \leq 5, \\
0 & \text{for Ⓐ}, x_1 \leq 15, x_2 \leq 4-\frac{1}{5}x_1, \\
\frac{x_1+3x_0-15}{8} + \max\left(\frac{x_2+x_0-1}{9}, 0\right) & \text{for 🅐}, x_1 \geq 15-3x_0, \\
\min\left(\frac{x_2+\frac{8}{5}x_0+\frac{1}{5}x_1-4}{9}, \frac{x_1+3x_0}{8} + \frac{x_2+x_0-5}{9}\right) & \text{for 🅐}, x_1 \leq 15-3x_0, x_2 \geq 5-x_0, \\
\min\left(\frac{x_2+\frac{8}{5}x_0+\frac{1}{5}x_1-4}{9}, \frac{x_1+3x_0}{8}\right) & \text{for 🅐}, x_1 \leq 15-3x_0, 4-\frac{1}{5}x_1-\frac{8}{5}x_0 \leq x_2 \leq 5-x_0, \\
0 & \text{for 🅐}, x_1 \leq 15-3x_0, x_2 \leq 4-\frac{1}{5}x_1-\frac{8}{5}x_0, \\
\frac{x_2-8}{9} + \max\left(0, \frac{x_1-9}{8}\right) & \text{for Ⓑ}, x_2 \geq 8, \\
\max\left(0, \frac{x_1+\frac{3}{8}x_2-12}{8}\right) & \text{for Ⓑ}, x_2 \leq 8, \\
\frac{x_2+x_0-8}{9} + \max\left(0, \frac{x_1+3x_0-9}{8}\right) & \text{for 🅑}, x_2 \geq 8-x_0, \\
\max\left(0, \frac{x_1+\frac{27}{8}x_0+\frac{3}{8}x_2-12}{8}\right) & \text{for 🅑}, x_2 \leq 8-x_0.
\end{cases}
\tag{3.23}
$$

3.2 Intersection of Three Directions

An intersection of three directions consists of three flows. Two directions are served simultaneously in all modes, except when the server is performing setup, then only one direction is served. All items that arrive in one cycle, are cleared in that cycle. The fixed time schedule of this example system, Figure 3.15, shows that all flows are conflict free. Therefore, it is possible to serve all directions simultaneously. Because of that, the fixed time schedule in Figure 3.15 does not correspond to an actual traffic setting. The system is merely of interest for research purposes. The results of the derivation of this example system demonstrate the effect on the derivation of a candidate Lyapunov function when the system size is increased.

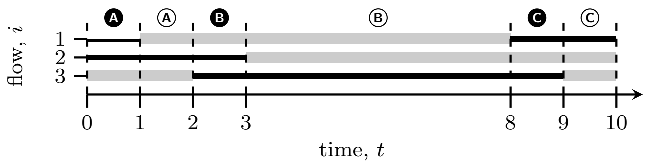

Figure 3.15: Fixed time schedule, $N = \{1, 2, 3\}$.

In this section two new definitions are used, the primarily and secondary served buffers. The primarily served buffer of a mode, is the buffer that was already served in the preceding mode and needs to be cleared during this mode. The secondary served buffer refers to the buffer that is served for the first time in the current mode, of which the service is continued in the successive mode. As can be concluded from Figure 3.15, one buffer per mode is not defined yet. The buffer that is not served in a mode is referred to as the unserved buffer of that mode. During Ⓐ, Ⓑ, Ⓒ, the lots in the primarily and secondary served buffers are processed. In setup of a mode, $u_0 \in \{$🅐,🅑,🅒$\}$ the system is in setup to process respectively x_1, x_2, x_3. Processing the content of the primarily served buffer of the mode is continued in setup.

An extension is needed to the definition of slow mode used in the intersection of two directions, to make the definition of slow mode unambiguous in the intersection of three directions. In the two flow example system slow mode was defined as the served buffer of the mode m being served at its arrival rate. In this example system multiple buffers are served during a mode. Either the primarily served buffer, the secondary served buffer, or both buffers can be in slow mode. Therefore, slow mode is specified as a slow mode of buffer i of mode m. Herein i equals the flow served at its arrival rate and m represents the mode the system is in.

To shorten the expressions of the candidate Lyapunov function in this section, the equations omit $V(s, x_1, x_2, x_3) =$, with $s \in \{$🅐,Ⓐ,🅑,Ⓑ,🅒,Ⓒ$\}$. However, the symbols corresponding to the parts of the domain are listed prior to the equation of the candidate Lyapunov function in each part of the domain, this emphasizes the similarities within modes and between the two flow system and the three flow system. The candidate Lyapunov function described in this section, is derived according to the method explained in detail for the two flow system. Domain parts in which the candidate Lyapunov function derivation is not extended, or did not become more complex compared to the intersection of two directions, are not discussed in this section as they do not provide more insight. The derivation of the candidate Lyapunov function in the complete domain is found in Appendix E.

The system parameters and the fixed time schedule are designed such that each mode contains a slow mode of the primarily served buffer. Although it is not a necessary condition it is possible that the secondary served buffer is cleared during a mode.
The arrival rates are assumed to be constant and known; $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 1$. As are the process rates; $\mu_1 = 2$, $\mu_2 = 3$ and $\mu_3 = 4$. In Table 3.1, the buffer contents are listed at each time instant the service rate values change, during optimal periodic behaviour. These values are important in the candidate Lyapunov function derivation. The stars indicate that these values are the optimal buffer content at the start of setup, or start of processing in a mode.

Figure 3.16 is a graphical representation of the work in the system in optimal periodic behaviour as function of time. Although it is not the case in this example system, it is a possibility that the work in the system decreases during setup since the server continues processing one of the buffers in setup.

Table 3.1: Buffer contents in optimal behaviour.

	$\boldsymbol{t}$	$(\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3})$
$t_{\text{🅐}^*}$	0	$(2,0,4)$
$t_{\text{Ⓐ}^*}$	1	$(3,2,1)$
$t_{\text{Ⓐ}_0^*}$	$\frac{4}{3}$	$(\frac{8}{3},\frac{8}{3},0)$
$t_{\text{🅑}^*}$	2	$(2,4,0)$
$t_{\text{Ⓑ}^*}$	3	$(1,6,1)$
$t_{\text{Ⓑ}_0^*}$	4	$(0,5,2)$
$t_{\text{🅒}^*}$	8	$(0,1,6)$
$t_{\text{Ⓒ}^*}$	9	$(1,0,7)$
$t_{\text{Ⓒ}_0^*}$	10	$(2,0,4)$

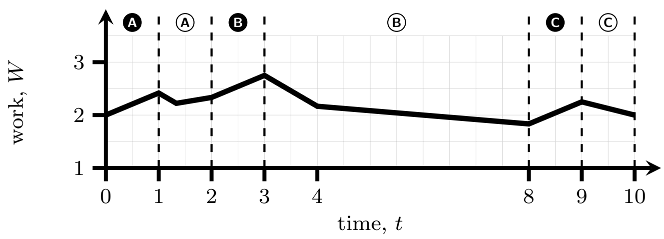

Figure 3.16: Optimal work in the system, $N = \{1, 2, 3\}$.

3.2.1 Mode *A*, $N = \{1, 2, 3\}$

The start values of the buffer contents can be in the complete domain, which in an intersection with three directions equals $\{x_1, x_2, x_3\} \in \mathbb{R}^3_+$. Based on the results of the previously discussed example it can be concluded that the candidate Lyapunov function differs depending on the values of x_1, x_2 and x_3. The differences in the candidate Lyapunov function result in domain partitioning.

In $\mathscr{D}^{\mathrm{I}}$ and $\mathscr{D}^{\mathrm{II}}$ the derivation is not extended. The first part of the domain in which the analogy with the two direction system requires additional explanation is $\mathscr{D}^{\mathrm{III}}$. In the two flow example system the candidate Lyapunov function in $\mathscr{D}^{\mathrm{III}}$, $\mathscr{D}^{\mathrm{IV}}$, $\mathscr{D}^{\mathrm{V}}$, depended on the optimal time to spent in the start-up mode. Only one buffer was served in each mode in the two flow example system, so the optimal time spent in the start-up mode was the time needed to clear the served buffer. In the intersection of three directions, during each mode two buffers are served instead of one, this results in another subdivision of domain parts. For instance during A, it is not only possible for x_1 to be less than the optimal value, but $x_3 \leq 1$ is a possibility as well. The duration of the start-up mode depends on $x_1 - 1$ or x_3, based on which buffer content becomes its respective optimal value in the least amount of time. The buffer content that the time spent in the start-up mode is based on, is denoted in the subscript of the part of the domain.
In the intersection of two directions the extra content of the unserved buffer in $\mathscr{D}^{\mathrm{III}}$, $\mathscr{D}^{\mathrm{IV}}$ and $\mathscr{D}^{\mathrm{V}}$, was a function of the distance between the content in the unserved buffer and the boundary line that described the correlation between the served and unserved buffer. The general approach to derive a correlation between the buffer contents during m, is found in Appendix D. Implementing Appendix D in A, results in the three flow equivalent of the boundary between the unserved and served buffers.

As in the two direction system, the correlation between the unserved and served buffers determine the boundary between created the boundary between $\mathscr{D}^{\mathrm{III}}$ and $\mathscr{D}^{\mathrm{VI}}$. In the latter part of the domain the candidate Lyapunov function value equals zero, however the domain boundaries depend on which buffer content the time spent in the start-up mode is based. If x_1 defines the duration of the start-up mode, Appendix D yields $3x_1 \geq x_3 + 8$, $2x_1 + x_2 \leq 8$ with $x_1 \leq 3$ and $x_3 \leq 1$. Hence, the candidate Lyapunov function in $\mathscr{D}_1^{\mathrm{VI}}$, (3.24). If x_3 determines the remaining time in the start-up mode, Appendix D gives $3x_1 \leq x_3 + 8$, $3x_2 + 2x_3 \leq 8$ should hold, with $x_1 \leq 3$, $x_3 \leq 1$, the candidate Lyapunov function in $\mathscr{D}_3^{\mathrm{VI}}$ equals (3.25).

$$\mathscr{D}_1^{\mathrm{VI}}: \quad 0 \quad \text{for A}, \quad \tfrac{8}{3} \leq x_1 \leq 3, \quad x_2 \leq 8 - 2x_1, \quad x_3 \leq 3x_1 - 8 \leq 1. \tag{3.24}$$

$$\mathscr{D}_3^{\mathrm{VI}}: \quad 0 \quad \text{for A}, \quad \tfrac{8}{3} \leq x_1 \leq 3, \quad x_2 \leq \tfrac{8}{3} - \tfrac{2}{3}x_3, \quad 3x_1 - 8 \leq x_3 \leq 1. \tag{3.25}$$

When $x_1 \leq 3$ and $x_3 \geq 1$ in the three flow equivalent of $\mathscr{D}^{\mathrm{III}}$ in the two direction system, the value of x_2 should exceed the boundary created by the correlation between x_1 and x_2, $2x_1 + x_2 \geq 8$. Since $x_1 \leq 3$, buffer 1 stabilizes in the desired optimal periodic behaviour, x_2 and x_3 contribute to the extra work in the system. The time spent in the start-up mode is based on the value of x_1, yielding the candidate Lyapunov function in $\mathscr{D}_1^{\text{III}}$, (3.26). With Appendix D the boundaries can be computed for the parts of the domain $\mathscr{D}^{\text{III}}$ where either x_1 is less than its optimal value, x_3 is less than its optimal value, or both. If both the content buffer 1 and buffer 3 are less than the optimal value, the start-up mode duration can depend on x_1 or x_3, the buffer content that is defining is listed first in the subscript of the domain part. This completes the candidate Lyapunov function in $\mathscr{D}^{\text{III}}$, (3.26) to (3.29).

$$\mathscr{D}_1^{\text{III}}: \quad \tfrac{x_2-8+2x_1}{3} + \tfrac{x_3-1}{4} \quad \text{for Ⓐ}, \quad \tfrac{8}{3} \le x_1 \le 3, \quad x_2 \ge 8-2x_1, \quad x_3 \ge 1. \tag{3.26}$$

$$\mathscr{D}_3^{\text{III}}: \quad \tfrac{x_1-3}{2} + \tfrac{x_2-\frac{8}{3}+\frac{2}{3}x_3}{3} \quad \text{for Ⓐ}, \quad x_1 \ge 3, \quad x_2 \ge \tfrac{8}{3} - \tfrac{2}{3}x_3, \quad x_3 \le 1. \tag{3.27}$$

$$\mathscr{D}_{1,3}^{\text{III}}: \quad \tfrac{x_2-8+2x_1}{3} + \tfrac{x_3+8-3x_1}{4} \quad \text{for Ⓐ}, \quad \tfrac{8}{3} \le x_1 \le 3, \quad x_2 \ge 8-2x_1, \quad 3x_1-8 \le x_3 \le 1. \tag{3.28}$$

$$\mathscr{D}_{3,1}^{\text{III}}: \quad \tfrac{x_1-\frac{8}{3}-\frac{1}{3}x_3}{2} + \tfrac{x_2-\frac{8}{3}+\frac{2}{3}x_3}{3} \quad \text{for Ⓐ}, \quad \tfrac{8}{3}+\tfrac{1}{3}x_3 \le x_1 \le 3, \quad x_2 \ge \tfrac{8}{3} - \tfrac{2}{3}x_3, \quad x_3 \le 1. \tag{3.29}$$

The split presented above is based on which buffer content defines the remaining time in the start-up mode, it structures the method to derive a candidate Lyapunov function but a review of the results is required. If $x_1 = 3$ and $x_3 > 0$, the work in the system decreases when the server continues in the start-up mode. The server switches if the time spent in the mode equals the maximum duration of Ⓐ. This gives a correction for the candidate Lyapunov function in $\mathscr{D}_1^{\text{III}}$ and $\mathscr{D}_{1,3}^{\text{III}}$. Because x_1 does not terminate the start-up mode, (3.26), (3.28) become respectively (3.30) and (3.31).

$$\mathscr{D}_1^{\text{III}}: \quad \tfrac{x_2-2}{3} + \tfrac{x_3-1}{4} \quad \text{for Ⓐ}, \quad \tfrac{8}{3} \le x_1 \le 3, \quad x_2 \ge 8-2x_1, \quad x_3 \ge 1. \tag{3.30}$$

$$\mathscr{D}_{1,3}^{\text{III}}: \quad \tfrac{x_2-\frac{8}{3}+\frac{2}{3}x_3}{3} \quad \text{for Ⓐ}, \quad \tfrac{8}{3} \le x_1 \le 3, \quad x_2 \ge \tfrac{8}{3} - \tfrac{2}{3}x_3, \quad 3x_1-8 \le x_3 \le 1. \tag{3.31}$$

The work in the system increases during the slow mode of buffer 3 in Ⓐ, see Figure 3.16. If the value of x_1 becomes small enough, immediate switching to ❸ instead of starting in Ⓐ could minimize the mean extra work in the system during periodic behaviour, as explained in the derivation of the candidate Lyapunov function of the intersection of two directions. Therefore, an additional review of the results of the candidate Lyapunov function in $\mathscr{D}^{\text{III}}$ is required. A similar review to the one performed in the derivation of the candidate Lyapunov function of the intersection of two directions that lead to the existence of $\mathscr{D}^{\text{IV}}$.
In the domain parts $\mathscr{D}_1^{\text{III}}$, $\mathscr{D}_3^{\text{III}}$, $\mathscr{D}_{1,3}^{\text{III}}$ and $\mathscr{D}_{1,3}^{\text{III}}$ the candidate Lyapunov function value is determined if the server immediately switches to setup. For instance in $\mathscr{D}_1^{\text{III}}$, switching to setup of the subsequent mode, the content of x_3 contributes to the mean extra work of buffer 3. If $x_1 \le 2$ and $x_2 \le 4$, the buffer contents of buffer 1 and buffer 2 do not contribute to the mean extra work in the system during periodic behaviour. The candidate Lyapunov function in case the system immediately switches to Ⓑ in $\mathscr{D}_1^{\text{III}}$ is (3.32). To determine if immediately switching to setup minimizes the mean extra work in the system, (3.26) is compared to (3.32). This further subdivides the domain, the candidate Lyapunov function in $\mathscr{D}_1^{\text{III}}$ is described by (3.33) to (3.37). This review between switching to setup and continuing in the current mode is performed for the remainder of $\mathscr{D}^{\text{III}}$. In $\mathscr{D}_3^{\text{III}}$ immediately switching to mode $\boldsymbol{B}$ never results in less mean extra

work in the system during periodic behaviour, thus the candidate Lyapunov function is correctly given in (3.27). Similar to $\mathscr{D}_1^{\text{III}}$, $\mathscr{D}_{1,3}^{\text{III}}$ is divided in multiple parts, which subdivides the candidate Lyapunov function in multiple expressions for this part of the domain. Finally, in $\mathscr{D}_{3,1}^{\text{III}}$ immediately switching to mode $\boldsymbol{B}$ results in less mean extra work in the system during periodic behaviour for some system values. Hence, $\mathscr{D}_{3,1}^{\text{III}}$ is divided in several parts, with a corresponding split in the candidate Lyapunov function.

$$\mathscr{D}_{1,\text{setup}}^{\text{III}}: \quad \max\left(0, \tfrac{x_1-2}{2}\right) + \max\left(0, \tfrac{x_2-4}{3}\right) + \tfrac{x_3}{4} \quad \text{for (A)}, \quad x_1 \leq 3, \quad x_2 \geq 8-2x_1, \quad 3x_1 - 8 \leq x_3 \geq 1. \tag{3.32}$$

$$\mathscr{D}_1^{\text{III}_\text{a}}: \quad \tfrac{x_2-2}{3} + \tfrac{x_3-1}{4} \quad \text{for (A)}, \quad \tfrac{17}{6} \leq x_1 \leq 3, \quad x_2 \geq 8-2x_1 \geq 4, \quad 3x_3 \geq 1. \tag{3.33}$$

$$\mathscr{D}_1^{\text{III}_\text{b}}: \quad \tfrac{x_2-2}{3} + \tfrac{x_3-1}{4} \quad \text{for (A)}, \quad 2 \leq x_1 \leq 3, \quad x_2 \leq \tfrac{3}{2}x_1 + \tfrac{1}{4}, \quad x_2 \leq 4, x_3 \geq 1. \tag{3.34}$$

$$\mathscr{D}_1^{\text{IV}_\text{a}}: \quad \tfrac{x_1-2}{2} + \tfrac{x_2-4}{3} + \tfrac{x_3}{4} \quad \text{for (A)}, \quad 2 \leq x_1 \leq \tfrac{17}{6}, \quad x_2 \geq 8-2x_1 \geq 4, \quad x_2 \leq 4, x_3 \geq 1. \tag{3.35}$$

$$\mathscr{D}_1^{\text{IV}_\text{b}}: \quad \tfrac{x_2-4}{3} + \tfrac{x_3}{4} \quad \text{for (A)}, \quad x_1 \leq 2, \quad x_2 \geq 8-2x_1 \geq 4, \quad x_2 \leq 4, x_3 \geq 1. \tag{3.36}$$

$$\mathscr{D}_1^{\text{V}}: \quad \tfrac{x_1-2}{2} + \tfrac{x_3}{4} \quad \text{for (A)}, \quad 2 \leq x_1 \leq 3, \quad \tfrac{3}{2}x_1 + \tfrac{1}{4} \leq x_2 \leq 4, \quad x_2 \leq 4, x_3 \geq 1. \tag{3.37}$$

3.2.2 Mode $\boldsymbol{B}$, $N = \{1, 2, 3\}$

Although (B) contains a slow mode, the work in the system decreases when buffer 1 is in slow mode, see Figure 3.16. When the server starts in mode $\boldsymbol{B}$, immediately switching in the start-up mode to mode $\boldsymbol{C}$, does not decrease the mean extra work in the system in periodic behaviour. Thus in mode $\boldsymbol{B}$ the three flow equivalent of $\mathscr{D}^{\text{III}}$, is not divided in $\mathscr{D}^{\text{III}_\text{a}}$, $\mathscr{D}^{\text{III}_\text{b}}$, $\mathscr{D}^{\text{IV}}$ and $\mathscr{D}^{\text{V}}$. This subsection discusses the derivation of the candidate Lyapunov function in mode $\boldsymbol{B}$ in $\mathscr{D}^{\text{III}}$, as the derivation is not straightforward in this part of the domain.

In $\mathscr{D}^{\text{III}}$ with $x_1 \leq 1$ and $x_2 \geq 6$, x_3 needs to exceed the boundary created by the correlation between buffer 1 and buffer 3, particularly $x_3 \geq 2 - x_1$. Buffer 1 shows the desired behaviour when in steady-state since $x_1 \leq 1$. The value of the excess content of x_2 and x_3, depends on the time spent in the start-up mode. Thus the candidate Lyapunov function in $\mathscr{D}_1^{\text{III}}$ equals (3.38).
If the value of x_2 defines the duration of the start-up mode, $x_1 \geq 1$ and $x_2 \leq 6$, the boundary that determines the extra content regarding the optimal periodic behaviour in buffer 3 becomes $x_3 \geq 7 - x_2$. This results in mean extra work in the system during periodic behaviour in $\mathscr{D}_2^{\text{III}}$ equal to (3.39).

When buffer 1 and buffer 2 contain less than their respective optimal values, and x_1 determines the duration of the start-up mode, both $x_3 \geq 2 - x_1$ and $x_1 \leq x_2 - 5$ hold. The extra content of buffer 2 and buffer 3 depend on the duration of the start-up mode and candidate Lyapunov function equals (3.40). However, when the content of buffer 2 determines the duration of the start-up mode, with Appendix D it is concluded that $x_2 \leq x_1 + 5 \leq 6$ and $x_3 \geq 7 - x_2$. The candidate Lyapunov function in $\mathscr{D}_{2,1}^{\text{III}}$ then equals (3.41).

$$\mathscr{D}_1^{\text{III}}: \quad \tfrac{x_2-6}{3} + \tfrac{x_3+x_1-2}{4} \quad \text{for (B)} \quad x_1 \leq 1, \quad x_2 \geq 6, \quad x_3 \geq 2 - x_1. \tag{3.38}$$

$$\mathscr{D}_2^{\mathrm{III}}: \quad \tfrac{x_1-1}{2} + \tfrac{x_3+x_2-7}{4} \quad \text{for Ⓑ } x_1 \geq 1, \quad x_2 \leq 6, \quad x_3 \geq 7 - x_2. \tag{3.39}$$

$$\mathscr{D}_{1,2}^{\mathrm{III}}: \quad \tfrac{x_2-x_1-5}{3} + \tfrac{x_3+x_1-2}{4} \quad \text{for Ⓑ} \quad x_1 \leq 1, \quad x_1+5 \leq x_2 \leq 6, \quad x_3 \geq 2 - x_1. \tag{3.40}$$

$$\mathscr{D}_{2,1}^{\mathrm{III}}: \quad \tfrac{x_1-x_2+5}{2} + \tfrac{x_3+x_2-7}{4} \quad \text{for Ⓑ} \quad x_1 \leq 1, \quad x_2 \leq x_1+5 \leq 6, \quad x_3 \geq 7 - x_2. \tag{3.41}$$

Starting the derivation of the candidate Lyapunov function in $\mathscr{D}^{\mathrm{III}}$ with subdividing the domain based on which buffer content defines the duration of the start-up mode, structures the derivation. Structuring is useful when the system size is extended. However, the results need to be reviewed to determine if the derived function is complete and correct.
From Figure 3.16 it can be concluded that if $x_1 = 0$ and $x_2 > 0$, the work is continuously decreasing, so the system should continue processing if the time spent in the start-up mode is less than the maximum duration of Ⓑ stated in the fixed time schedule. If $x_2 = 6$ and $x_1 > 0$ and the system continues in Ⓑ, the derivative of work in the system equals (3.42) which proves the work in the system is decreasing. The mean extra work in the system in periodic behaviour is minimized when the server continues in Ⓑ as long as possible and the value of x_2 never defines the end of mode $\boldsymbol{B}$. A correction is needed in $\mathscr{D}_2^{\mathrm{III}}$ and $\mathscr{D}_{2,1}^{\mathrm{III}}$, the candidate Lyapunov function respectively becomes (3.43) and (3.44).

$$\dot{W} = \frac{\lambda_1 - \mu_1}{\mu_1} + \frac{\lambda_2 - \mu_2}{\mu_2} + \frac{\lambda_3}{\mu_3} = \frac{-1}{2} + \frac{-1}{3} + \frac{1}{4} < 0, \tag{3.42}$$

$$\mathscr{D}_2^{\mathrm{III}}: \quad \tfrac{x_1-1}{2} + \tfrac{x_3-1}{4} \quad \text{for Ⓑ} \quad x_1 \geq 1, \quad x_2 \leq 6, \quad x_3 \geq 7 - x_2. \tag{3.43}$$

$$\mathscr{D}_{2,1}^{\mathrm{III}}: \quad \tfrac{x_3+x_1-2}{4} \quad \text{for Ⓑ} \quad x_1 \leq 1, \quad x_2 \leq 6, \quad x_3 \geq 2 - x_1. \tag{3.44}$$

3.2.3 Mode $\boldsymbol{C}$, $N = \{1, 2, 3\}$

At $t = 9$ in the fixed time schedule the slow mode of buffer 2 in mode $\boldsymbol{C}$ starts. Figure 3.16 illustrates that the work in the system decreases when the system is in slow mode of buffer 2. Furthermore, the properties of mode $\boldsymbol{C}$ are slightly different than the properties of mode $\boldsymbol{A}$ and $\boldsymbol{B}$. In mode $\boldsymbol{C}$ the primarily served buffer is cleared during ❸ instead of during Ⓒ.
Because the buffer contents are positive by definition, the boundaries of different domain parts are different compared to the boundaries in mode $\boldsymbol{A}$ and mode $\boldsymbol{B}$, $x_2 \leq 0$ becomes $x_2 = 0$. Even though mode $\boldsymbol{C}$ has different properties, the derivation of a candidate Lyapunov function is performed in a similar structure as used for mode $\boldsymbol{A}$ and $\boldsymbol{B}$. The successful derivation of the candidate Lyapunov function of mode $\boldsymbol{C}$, implies that the definition of mean extra work in the system to derive a candidate Lyapunov function discussed in this chapter is applicable to systems with mode properties equivalent to the properties of mode $\boldsymbol{C}$. In addition to the candidate Lyapunov function derivation of $\mathscr{D}^{\mathrm{III}}$, in this subsection the derivation of the candidate Lyapunov function in $\mathscr{D}^{\mathrm{VI}}$ is discussed. Because there is a difference compared to the candidate Lyapunov function derivation in $\mathscr{D}^{\mathrm{VI}}$ of mode $\boldsymbol{A}$ and $\boldsymbol{B}$. This difference arises due to the non existence of the boundary $x_2 \leq 0$.

In $\mathscr{D}^{\mathrm{VI}}$ the content of both served buffers are $x_2 = 0$ and $x_3 \leq 7$, the value of x_1 is less than the correlation that determines the boundary between $\mathscr{D}^{\mathrm{VI}}$ and $\mathscr{D}^{\mathrm{III}}$. Mode $\boldsymbol{A}$ and mode $\boldsymbol{B}$ showed two options in this part, in mode $\boldsymbol{C}$ the same strategy is used. When the x_2 defines the remaining time in the start-up mode Ⓒ, the result of Appendix D is $3x_2 \leq x_3 - 7$ otherwise the value of x_3 would determine the duration of the mode. If x_3 determines the duration of the start-up mode, then $x_3 \leq 3x_2 + 7 \leq 7$, $x_2 = 0$ and $3x_1 \leq 10 - x_3$. Since the only possible value of x_2 is zero both options result in the same part of the domain and an equal candidate Lyapunov function, (3.45). There is no correction needed in this part of the domain, as mentioned at the start of this subsection when $x_2 = 0$ the work is still decreasing.

$$\mathscr{D}^{\mathrm{VI}}: \quad 0 \quad \text{for Ⓒ}, \quad x_1 \leq 1, \quad x_2 = 0, \quad x_3 = 7. \tag{3.45}$$

Starting the derivation of $\mathscr{D}^{\mathrm{III}}$ for values of x_2 and x_3 that fulfil $x_2 \leq 0$ and $x_3 \geq 7$. In case $x_1 \geq 1 - x_2$ the content of buffer 2 is optimal. The mean extra work in the system during periodic behaviour depends on x_1 and x_3, the candidate Lyapunov function in $\mathscr{D}_2^{\mathrm{III}}$ equals (3.46). When the content of buffer 3 determines the time spent in the start-up mode, $x_2 \geq 0$, $x_3 \leq 7$ and $3x_1 \geq 10 - x_3$, the candidate Lyapunov function equals 3.47.
If both x_2 and x_3 are less than their respective optimal values, $x_2 \leq 0$ and $x_3 \leq 7$, the duration of the start-up mode can be defined by either x_2 or x_3. When the value of x_2 defines the duration of the start-up mode, both $3x_2 \leq x_3 - 7$ and $x_1 \geq 1$ should hold. Because $x_2 = 0$ the value of x_3 should fulfil $7 \leq x_3 \leq 7$ and becomes $x_3 = 7$ which yields the the candidate Lyapunov function in $\mathscr{D}_{2,3}^{\mathrm{III}}$, (3.48). If the content of buffer 3 determines the time spent in the start-up mode, $x_3 \leq 3x_2 + 7$ and $x_1 \geq \frac{10}{3} - \frac{1}{3}x_3$ should hold, the candidate Lyapunov function in $\mathscr{D}_{3,2}^{\mathrm{III}}$ equals (3.49).

$$\mathscr{D}_2^{\mathrm{III}}: \quad \tfrac{x_1-1}{2} + \tfrac{x_3-7}{4} \quad \text{for Ⓒ}, \quad x_1 \geq 1, \quad x_2 = 0, \quad x_3 \geq 7. \tag{3.46}$$

$$\mathscr{D}_3^{\mathrm{III}}: \quad \tfrac{x_1 + \frac{1}{3}x_3 - \frac{10}{3}}{2} + \tfrac{x_2}{3} \quad \text{for Ⓒ}, \quad x_1 \geq \tfrac{10}{3} - \tfrac{1}{3}x_3, \quad x_2 \geq 0, \quad x_3 \leq 7. \tag{3.47}$$

$$\mathscr{D}_{2,3}^{\mathrm{III}}: \quad \tfrac{x_1-1}{2} + \tfrac{x_3-7}{4} \quad \text{for Ⓒ}, \quad x_1 \geq 1, \quad x_2 = 0, \quad x_3 = 7. \tag{3.48}$$

$$\mathscr{D}_{3,2}^{\mathrm{III}}: \quad \tfrac{x_1 + \frac{1}{3}x_3 - \frac{10}{3}}{2} \quad \text{for Ⓒ}, \quad x_1 \geq 1, \quad x_2 = 0, \quad x_3 \leq 7. \tag{3.49}$$

The previously discussed subdivision in parts of $\mathscr{D}^{\mathrm{III}}$ is based on the buffer content that defines the time spent in the start-up mode. A correction might be necessary if the work in the system is increasing when either of the served buffers reaches its optimal value. Figure 3.16 illustrates that the work in the system is decreasing when $x_2 = 0$ and $x_3 > 0$. In (3.50) the derivative of the work in the system is presented for $x_2 > 0$ and $x_3 = 7$. This shows the work in the system is decreasing when $x_2 > 0$ and $x_3 = 7$, so the content of buffer 3 never defines the end of the start-up mode. The corrected candidate Lyapunov function in $\mathscr{D}_3^{\mathrm{III}}$ and $\mathscr{D}_{3,2}^{\mathrm{III}}$ is respectively given in (3.51) and (3.52).
Directly switching to setup never minimizes the mean extra work in the system, as the work in the system is decreasing in slow mode of buffer 2 and in slow mode of buffer 3. Hence, the domain parts $\mathscr{D}^{\mathrm{IV}}$ and $\mathscr{D}^{\mathrm{V}}$ do not exist in mode $\boldsymbol{C}$.

$$\dot{W} = \frac{\lambda_1}{\mu_1} + \frac{-(\mu_2 - \lambda_2)}{\mu_2} + \frac{-(\mu_3 - \lambda_3)}{\mu_3} = \frac{1}{2} + \frac{-1}{3} + \frac{-3}{4} < 0. \tag{3.50}$$

$$\mathscr{D}_3^{\mathrm{III}}: \quad \tfrac{x_1-1}{2} + \tfrac{x_2}{3} \quad \text{for Ⓒ}, \quad x_1 \geq 1, \quad x_2 \geq 0, \quad x_3 \leq 7. \tag{3.51}$$

$$\mathscr{D}_{3,2}^{\mathrm{III}}: \quad \tfrac{x_1-1}{2} \quad \text{for Ⓒ}, \quad x_1 \geq 1, \quad x_2 = 0, \quad x_3 \leq 7. \tag{3.52}$$

3.2.4 Candidate Lyapunov Function, $N = \{1, 2, 3\}$

The setup of all modes is the final part of the complete domain of the three flow example system. In case the server starts in setup, the definition equals the definition given in the example of an intersection of two directions. The value of the mean extra work in the system during periodic behaviour when the server starts in A, B or C, equals the value of $V(s, x_1, x_2, x_3)$ in respectively A, B, C x_0 time units later. The functions of x_1, x_2, x_3 at x_0 time units later, are listed in Table 3.2. The functions are more complicated than the functions of x_1, x_2 in the two flow system, because in the three direction system one buffer is served during setup, and the buffer contents are by definition non-negative.

This completes the derivation of the candidate Lyapunov function in the complete domain of the three flow example system. Combining the equations for different parts of the domain if possible yields a simplified expression of the candidate Lyapunov function $V(s, x_1, x_2, x_3)$, (3.53). The candidate Lyapunov function in setup of all modes is not denoted explicitly, to shorten the expression.
The number of equations and the complexity of the equations to derive (3.53), show the effect of adding one flow and one mode to the intersection of two directions. Comparing the candidate Lyapunov function derivation discussed in this section, to the example presented in the previous section, leaves to conclude that adding an extra mode to the system does not have a significant effect. Apart from needing to examine one more mode needs to derive a candidate Lyapunov function.
On the contrary, changing the composition of the mode, by for instance adding an extra direction that is served in a mode, makes the derivation more cumbersome. Furthermore, it changes the amount of subdivisions in the domain significantly, which results in a more complex candidate Lyapunov function. This observation stresses the convenience of establishing a general policy of which stability is implied without the need of explicitly deriving a candidate Lyapunov function of the system.

Table 3.2: Buffer contents after setup.

	A	B	C
x_1	$x_1 + \lambda_1 x_0$	$\max(0, x_1 - (\mu_1 - \lambda_1) x_0)$	$x_1 + \lambda_1 x_0$
x_2	$x_2 + \lambda_2 x_0$	$x_2 + \lambda_2 x_0$	$\max(0, x_2 - (\mu_2 - \lambda_2) x_0)$
x_3	$\max(0, x_3 - (\mu_3 - \lambda_3) x_0)$	$x_3 + \lambda_3 x_0$	$x_3 + \lambda_3 x_0$

Based on the results presented in this chapter, it is expected that a candidate Lyapunov function can be found for every system of which an optimal fixed time schedule is known and modes can be defined. Although it is an assumption that an explicit candidate Lyapunov function can be derived for a system of any size, actually performing the derivation is undesired. In the upcoming chapter the candidate Lyapunov functions of this chapter, are used to derive the control actions for each of the example systems.

$$
\begin{cases}
\frac{x_1-3}{2}+\max\left(0,\frac{x_2-2}{3}\right)+\frac{x_3-1}{4} & \text{for } \textcircled{A}, x_1\geq 3, x_3\geq 1,\\
\min\left(\frac{x_2-2}{3}+\frac{x_3-1}{4},\max\left(0,\frac{x_1-2}{2}\right)+\frac{x_2-4}{3}+\frac{x_3}{4}\right) & \text{for } \textcircled{A}, x_1\leq 3, x_2\geq 8-2x_1\geq 4, x_3\geq 1,\\
\min\left(\frac{x_2-2}{3}+\frac{x_3-1}{4},\frac{x_1-2}{2}+\frac{x_3}{4}\right) & \text{for } \textcircled{A}, 2\leq x_1\leq 3, x_2\leq 4, x_3\geq 1,\\
\frac{x_1-3}{2}+\frac{x_2-\frac{8}{3}+\frac{2}{3}x_3}{3} & \text{for } \textcircled{A}, x_1\geq 3, x_2\geq \frac{8}{3}-\frac{2}{3}x_3, x_3\leq 1,\\
\min\left(\frac{x_2-\frac{8}{3}+\frac{2}{3}x_3}{3},\frac{x_1-2}{2}+\frac{x_2-4}{3}+\frac{x_3}{4}\right) & \text{for } \textcircled{A}, x_1\leq 3, x_2\geq 4, 3x_1-8\leq x_3\leq 1,\\
\min\left(\frac{x_2-\frac{8}{3}+\frac{2}{3}x_3}{3},\frac{x_1-2}{2}+\frac{x_3}{4}\right) & \text{for } \textcircled{A}, x_1\leq 3, 8-2x_1\leq x_2\leq 4, 3x_1-8\leq x_3\leq 1,\\
\min\left(\frac{x_1-\frac{8}{3}-\frac{1}{3}x_3}{2}+\frac{x_2-\frac{8}{3}+\frac{2}{3}x_3}{3},\frac{x_1-2}{2}+\frac{x_2-4}{3}+\frac{x_3}{4}\right) & \text{for } \textcircled{A}, \frac{8}{3}+\frac{1}{3}x_3\leq x_1\leq 3, x_2\geq 4, x_3\leq 1,\\
\min\left(\frac{x_1-\frac{8}{3}-\frac{1}{3}x_3}{2}+\frac{x_2-\frac{8}{3}+\frac{2}{3}x_3}{3},\frac{x_1-2}{2}+\frac{x_3}{4}\right) & \text{for } \textcircled{A}, \frac{8}{3}+\frac{1}{3}x_3\leq x_1\leq 3, x_2\leq 4, x_3\leq 1,\\
0 & \text{for } \textcircled{A}, x_1\leq 3, x_2\leq 8-2x_1, x_3\leq 3x_1-8\leq 1,\\
0 & \text{for } \textcircled{A}, x_1\leq 3, x_2\leq \frac{8}{3}-\frac{2}{3}x_3, 3x_1-8\leq x_3\leq 1\\
V\left(\textcircled{A}, x_1+x_0, x_2+2x_0, \max\left(0,x_3-3x_0\right)\right) & \text{for 🅐},\\
\frac{x_1-1}{2}+\frac{x_2-6}{3}+\max\left(0,\frac{x_3-1}{4}\right) & \text{for } \textcircled{B}, x_1\geq 1, x_2\geq 6,\\
\frac{x_2-6}{3}+\frac{x_3+x_1-2}{4} & \text{for } \textcircled{B}, x_1\leq 1, x_2\geq 6, x_3\geq 2-x_1,\\
\frac{x_1-1}{2}+\frac{x_3-1}{4} & \text{for } \textcircled{B}, x_1\geq 1, x_2\leq 6, x_3\geq 7-x_2,\\
\max\left(0,\frac{x_2-x_1-5}{3}\right)+\frac{x_3+x_1-2}{4} & \text{for } \textcircled{B}, x_1\leq 1, x_2\leq 6, x_3\geq 2-x_1,\\
0 & \text{for } \textcircled{B}, x_1\leq 1, x_2\leq x_1+5\leq 6, x_3\leq 2-x_1,\\
0 & \text{for } \textcircled{B}, x_1\leq 1, x_1+5\leq x_2\leq 6, x_3\leq 7-x_2,\\
V\left(\textcircled{B}, \max\left(0,x_1-x_0\right), x_2+2x_0, x_3+x_0\right) & \text{for 🅑},\\
\max\left(0,\frac{x_1-1}{2}\right)+\frac{x_2}{3}+\max\left(0,\frac{x_3-7}{4}\right) & \text{for } \textcircled{C},\\
V\left(\textcircled{C}, x_1+x_0, \max\left(0,x_2-x_0\right), x_3+x_0\right) & \text{for 🅒}.
\end{cases}
\tag{3.53}
$$

Chapter 4. Control Actions

Traffic signals at an intersection are controlled by a server that performs control actions, as explained in detail in Section 1.3. A control action states if the server is in setup of a mode or processing in a mode. Additionally, the control actions list the rates at which the buffers are served.
In this chapter it is explained how control actions are designed. The control actions are derived from the result of Chapter 3, and the theory of Lyapunov-like functions, Theorem 4.1 and Theorem 4.2. To clearly explain the methodology of control action design, control actions of the example systems in Chapter 3 are designed in this chapter. However, the derivation of control actions concerns a general approach that is applicable to any derived candidate Lyapunov function.

The switched nature of the system provides multiple possibilities of control actions to perform. The first possible control action, is the option in which the server continues in the current mode. The second possibility is that the server performs setup, to start processing in the successive mode, referred to as: the server switches modes. The server should move the system states in the direction of steepest descent. Thus the control action to be chosen, corresponds to the option that minimizes the candidate Lyapunov function derivative.

In Section 4.1 the candidate Lyapunov function derivative of the intersection with two directions is designed. This design is based on the candidate Lyapunov function of the two flow system, (3.23). Based on the derivative of (3.23) it is decided if the server should continue in its current mode, or if the server should switch modes.
Section 4.2 explains the control action design of the example intersection of three directions. The results are simply posed, as the approach is already explained in Section 4.1.

4.1 Control Actions Design, $N = \{1, 2\}$

The candidate Lyapunov function of the two direction system is used to design the server actions (u_0, u_1, u_2) in each part of the domain. The control actions are designed such that the value of the Lyapunov function candidate is non-increasing over time. Furthermore, time-efficient convergence to optimal periodic behaviour is desired, this is realized when the candidate Lyapunov function derivative $\dot{V}(s, x_1, x_2)$ is minimized. If the derivative is minimized by switching to the subsequent mode, the control action becomes to switch to the subsequent mode. Otherwise, the server should continue in its current mode, and the control action is to continue in its current mode.

The derivative of the candidate Lyapunov function, $\dot{V}(s, x_1, x_2)$, is discussed Subsection 4.1.1. The control actions are derived in Subsection 4.1.2. Finally, a simplification of the control actions is discussed in Subsection 4.1.3. If the reader is exclusively interested in the final definition of the control actions, it is suggested to proceed to Subsection 4.1.3.

4.1.1 Candidate Lyapunov Function Derivative, $N = \{1, 2\}$

The candidate Lyapunov function derivative is determined based on the general definition of the derivative given in (4.1). By definition, the candidate Lyapunov function value is constant during setup of a mode, $\dot{V}(s, x_1, x_2) = 0$ for $s \in \{❶, ❷\}$, in all parts of the domain.
The candidate Lyapunov function derivative of the two direction system is explicitly defined by

computing the result of (4.1). The buffer content values at an infinitesimal small time increment later than t, the values at $t+\varepsilon$, are listed in Table 4.1. These values are a function of the arrival rates and the process rates. The buffer contents at $t+\varepsilon$, are implemented in (4.1) for each part of the domain, which yields the candidate Lyapunov function derivative $\dot{V}(s, x_1, x_2)$. For example, when the system is in Ⓐ with $x_1 \geq 15$ and $x_2 \geq 1$, the candidate Lyapunov function derivative becomes (4.2).

The complete candidate Lyapunov function derivative for the option in which the server continues in its current mode, is listed in (4.3). Mark the difference between the boundaries in (3.23) and (4.3). The candidate Lyapunov function derived in Chapter 3 is continuous: there are no jumps in values at the domain boundaries. The only jumps in the candidate Lyapunov function value occur when the system switches modes.

Although the candidate Lyapunov function is continuous at the domain boundary, the function in each of the domain parts is different. This difference leads to a different value of the candidate Lyapunov function derivative. Only one of the candidate Lyapunov function derivative values is used to design the control action, which results in the modified domain boundaries. The definition of the candidate Lyapunov function derivative at the boundary, is straightforward. If the server continues in its current mode, and the system values equal the boundary of a domain, continuing in its current mode results in entering a different domain part. Therefore, the candidate Lyapunov function derivative used equals the value the system enters when the server continues in its current mode.

An example is given to clarify this statement. If the system is in Ⓐ with $x_1 \leq 15$, $4 - \frac{1}{5}x_1 \leq x_2 \leq \frac{37}{40}x_1 + 4 \leq 5$, the domain part shares a boundary with Ⓐ, $x_1 \leq 15$, $\frac{37}{40}x_1 + 4 \leq x_2 \leq 5$. If $x_1 = 0$, the next time instant the system values enter the latter mentioned domain part. Hence, the candidate Lyapunov function derivative in Ⓐ with $x_1 = 0$ and $\frac{37}{40}x_1 + 4 \leq x_2 \leq 5$, is the candidate Lyapunov function derivative in Ⓐ, $x_1 = 0$, $\frac{37}{40}x_1 + 4 \leq x_2 \leq 5$. This method is used in the complete domain, which yields the modified boundaries in (4.3).

The candidate Lyapunov function derivative is different if the system starts in a mode and switches to setup of the successive mode. When the server switches to setup, the complete setup time needs to be performed. For instance, if the server is in Ⓐ and switches to 🅑, the value of the remaining setup time equals $x_0 = 3$. The candidate Lyapunov function derivative in case the server switches to setup of the successive mode, are the solution of (4.4) and the solution of (4.5) combined. This candidate Lyapunov function is referred to as the Lyapunov function at switching instants.

Due to the different domain part boundaries in both modes, additional expressions arise and the boundaries change. This significantly increases the length of the derivation. Hence, the derivation of $\dot{V}(s, x_1, x_2)$ at switching instants is listed in Appendix F. This appendix also includes a derivation of the candidate Lyapunov function derivative in case the system starts in setup, and switches to setup of the successive mode.

Table 4.1: Buffer contents x_1, x_2 at $t+\varepsilon$.

	$x_1(t+\varepsilon)$		$x_2(t+\varepsilon)$	
Ⓐ	$x_1(t) - (\mu_1 - \lambda_1)\varepsilon$	$\forall x_1 > 0$	$x_2(t) + \lambda_2\varepsilon$	
	$x_1(t)$	$\forall x_1 = 0$		
🅐,🅑	$x_1(t) + \lambda_1\varepsilon$		$x_2(t) + \lambda_2\varepsilon$	
Ⓑ	$x_1(t) + \lambda_1\varepsilon$		$x_2(t) - (\mu_2 - \lambda_2)\varepsilon$	$\forall x_2 > 0$
			$x_2(t)$	$\forall x_2 = 0$

$$\lim_{\varepsilon \to 0} \frac{V\left(s, x_1\left(t+\varepsilon\right), x_2\left(t+\varepsilon\right)\right) - V\left(s, x_1\left(t\right), x_2\left(t\right)\right)}{\varepsilon}. \tag{4.1}$$

$$\lim_{\varepsilon \to 0} \frac{x_1 - \left(\mu_1 - \lambda_1\right)\varepsilon - 15}{8\varepsilon} + \frac{x_2 + \lambda_2\varepsilon - 1}{9\varepsilon} - \frac{x_1 - 15}{8\varepsilon} - \frac{x_2 - 1}{9\varepsilon} = -\frac{37}{9} < 0. \tag{4.2}$$

$$\begin{cases}
\frac{x_1-(\mu_1-\lambda_1)\varepsilon-15}{\mu_1\varepsilon} + \frac{x_2+\lambda_2\varepsilon-1}{\mu_2\varepsilon} - \frac{x_1-15}{\mu_1\varepsilon} - \frac{x_2-1}{\mu_2\varepsilon} = -\frac{37}{72} & \text{for Ⓐ}, x_1 > 15, x_2 \geq 1, \\
\frac{x_1-(\mu_1-\lambda_1)\varepsilon-15}{\mu_1\varepsilon} - \frac{x_1-15}{\mu_1\varepsilon} = -\frac{5}{8} & \text{for Ⓐ}, x_1 > 15, x_2 \leq 1, \\
\frac{x_2+\lambda_2\varepsilon+\frac{1}{5}(x_1-(\mu_1-\lambda_1)\varepsilon)-4}{\mu_2\varepsilon} - \frac{x_2+\frac{1}{5}x_1-4}{\mu_2\varepsilon} = 0 & \text{for Ⓐ}, \frac{40}{37} < x_1 \leq 15, x_2 \geq 5, \\
\frac{x_1-(\mu_1-\lambda_1)\varepsilon}{\mu_1\varepsilon} + \frac{x_2+\lambda_2\varepsilon-5}{\mu_2\varepsilon} - \frac{x_1}{\mu_1\varepsilon} - \frac{x_2-5}{\mu_2\varepsilon} = -\frac{37}{72} & \text{for Ⓐ}, 0 < x_1 \leq \frac{40}{37}, x_2 \geq 5, \\
\frac{x_2+\lambda_2\varepsilon-5}{\mu_2\varepsilon} - \frac{x_2-5}{\mu_2\varepsilon} = \frac{1}{9} & \text{for Ⓐ}, x_1 = 0, x_2 \geq 5, \\
\frac{x_1-(\mu_1-\lambda_1)\varepsilon}{\mu_1\varepsilon} - \frac{x_1}{\mu_1\varepsilon} = -\frac{5}{8} & \text{for Ⓐ}, 0 < x_1 \leq 15, \frac{37}{40}x_1 + 4 < x_2 < 5, \\
\frac{x_1}{\mu_1\varepsilon} - \frac{x_1}{\mu_1\varepsilon} = 0 & \text{for Ⓐ}, x_1 = 0, 4 \leq x_2 < 5, \\
\frac{x_2+\lambda_2\varepsilon+\frac{1}{5}(x_1-(\mu_1-\lambda_1)\varepsilon)-4}{\mu_2\varepsilon} - \frac{x_2+\frac{1}{5}x_1-4}{\mu_2\varepsilon} = 0 & \text{for Ⓐ}, 0 < x_1 \leq 15, 4 - \frac{1}{5}x_1 < x_2 < \frac{37}{40}x_1 + 4 < 5, \\
0 - 0 = 0 & \text{for Ⓐ}, x_1 \leq 15, x_2 \leq 4 - \frac{1}{5}x_1, \\
0 & \text{for 🅐}, \\
\frac{x_1+\lambda_1\varepsilon-9}{\mu_1\varepsilon} + \frac{x_2-(\mu_2-\lambda_2)\varepsilon-8}{\mu_2\varepsilon} - \frac{x_1-9}{\mu_1\varepsilon} - \frac{x_2-8}{\mu_2\varepsilon} = -\frac{37}{72} & \text{for Ⓑ}, x_1 \geq 9, x_2 > 8, \\
\frac{x_2-(\mu_2-\lambda_2)\varepsilon-8}{\mu_2\varepsilon} - \frac{x_2-8}{\mu_2\varepsilon} = -\frac{8}{9} & \text{for Ⓑ}, x_1 \leq 9, x_2 > 8, \\
\frac{x_1+\lambda_1\varepsilon+\frac{3}{8}(x_2-(\mu_2-\lambda_2)\varepsilon)-12}{\mu_1\varepsilon} - \frac{x_1+\frac{3}{8}x_2-12}{\mu_1\varepsilon} = 0 & \text{for Ⓑ}, x_1 \geq 12 - \frac{3}{8}x_2, 0 < x_2 \leq 8, \\
\frac{x_1+\lambda_1\varepsilon-12}{\mu_1\varepsilon} - \frac{x_1-12}{\mu_1\varepsilon} = \frac{3}{8} & \text{for Ⓑ}, x_1 \geq 12 - \frac{3}{8}x_2, x_2 = 0, \\
0 - 0 = 0 & \text{for Ⓑ}, x_1 \leq 12 - \frac{3}{8}x_2, x_2 \leq 8, \\
0 & \text{for 🅑}.
\end{cases} \tag{4.3}$$

$$\lim_{\varepsilon \to 0} \frac{V\left(\text{🅑}, 3 - \varepsilon, x_1\left(t+\varepsilon\right), x_2\left(t+\varepsilon\right)\right) - V\left(\text{Ⓐ}, 0, x_1\left(t\right), x_2\left(t\right)\right)}{\varepsilon}. \tag{4.4}$$

$$\lim_{\varepsilon \to 0} \frac{V\left(\text{🅐}, 1 - \varepsilon, x_1\left(t+\varepsilon\right), x_2\left(t+\varepsilon\right)\right) - V\left(\text{Ⓑ}, 0, x_1\left(t\right), x_2\left(t\right)\right)}{\varepsilon}. \tag{4.5}$$

4.1.2 Control Actions, $N = \{1, 2\}$

In the previous subsections the candidate Lyapunov function derivative is defined for the entire state-space. $\dot{V}\left(s, x_1, x_2\right)$ is minimized if either the server continues processing or if the server switches to setup of the successive mode. The option that minimizes $\dot{V}\left(s, x_1, x_2\right)$ becomes the control action. In the vast majority of the domain the previously suggested method provides the control actions as function of the system states. In the domain parts where $\dot{V}\left(s, x_0, x_1, x_2\right) = 0$ regardless if the server switches or continues in its current, the control actions are undetermined. The control actions to perform in these specific undetermined domain parts, are addressed in Appendix G. The control action that results from Appendix G, is based on the most time efficient solution to obtain optimal periodic behaviour. The result of Appendix G and the other control actions completes the definition of the control actions in the entire domain, which is a long expression. To improve the readability, the control actions are first defined in case the server is in processing of a mode, so in Ⓐ or Ⓑ. After which, the control actions when the server is in setup of a mode are listed.

Control Actions when Processing

The derivative listed in (4.3) holds in case the server is in Ⓐ or Ⓑ and continues processing in this mode. This equation is compared to the derivative corresponding to the case of the server switches to setup of the successive mode, which is given in (F.59). This yields (4.6), the expression of control actions (u_0, u_1, u_2), in Ⓐ, Ⓑ. In setup of a mode no buffers are served, thus $u_1 = 0$ and $u_2 = 0$, irrespective of the specific domain part or the value of u_0.

$$
(u_0, u_1, u_2) = \begin{cases}
(\text{Ⓐ}, \mu_1, 0) & \text{if } \text{Ⓐ}, x_1 \geq 15, \\
(\text{Ⓐ}, \mu_1, 0) & \text{if } \text{Ⓐ}, 0 < x_1 \leq 15, x_2 \geq 5, \\
(\text{🅑}, 0, 0) & \text{if } \text{Ⓐ}, x_1 = 0, x_2 \geq 5, \\
(\text{🅑}, 0, 0) & \text{if } \text{Ⓐ}, 0 < x_1 \leq 15, \frac{37}{40}x_1 + 4 < x_2 < 5, \\
(\text{Ⓐ}, \lambda_1, 0) & \text{if } \text{Ⓐ}, x_1 = 0, 4 \leq x_2 < 5, \\
(\text{🅑}, 0, 0) & \text{if } \text{Ⓐ}, 0 < x_1 < \frac{15}{8} - \frac{3}{8}x_2 < 15, 4 - \frac{1}{5}x_1 < x_2 < \frac{37}{40}x_1 < 5, \\
(\text{🅑}, 0, 0) & \text{if } \text{Ⓐ}, 0 < \frac{15}{8} - \frac{3}{8}x_2 < x_1 < \frac{5}{8}x_2 - \frac{605}{296} < 15, 4 - \frac{1}{5}x_1 < x_2 < \frac{37}{40}x_1 < 5, \\
(\text{Ⓐ}, \mu_1, 0) & \text{if } \text{Ⓐ}, 0 < \frac{15}{8} - \frac{3}{8}x_2 < \frac{5}{8}x_2 - \frac{605}{296} < x_1 < 15, 4 - \frac{1}{5}x_1 < x_2 < \frac{37}{40}x_1 < 5, \\
(\text{Ⓐ}, \mu_1, 0) & \text{if } \text{Ⓐ}, \frac{15}{8} - \frac{3}{8}x_2 \leq x_1 < 15, x_2 \leq 4 - \frac{1}{5}x_1, \\
(\text{Ⓐ}, \mu_1, 0) & \text{if } \text{Ⓐ}, 0 < x_1 < 3x_2 < \frac{15}{8} - \frac{3}{8}x_2 < 15, x_2 < 4 - \frac{1}{5}x_1, \\
(\text{Ⓐ}, \lambda_1, 0) & \text{if } \text{Ⓐ}, 0 < x_1 \leq 3x_2 < \frac{15}{8} - \frac{3}{8}x_2 < 15, x_2 < 4 - \frac{1}{5}x_1, \\
(\text{🅑}, 0, 0) & \text{if } \text{Ⓐ}, 3x_2 < x_1 < \frac{15}{8} - \frac{3}{8}x_2 < 15, x_2 < 4 - \frac{1}{5}x_1, \\
(\text{Ⓑ}, 0, \mu_2) & \text{if } \text{Ⓑ}, x_2 \geq 8, \\
(\text{Ⓑ}, 0, \mu_2) & \text{if } \text{Ⓑ}, x_1 < 12 - \frac{3}{8}x_2, 0 < \frac{12}{5} - \frac{1}{5}x_1 \leq x_2 < 8, \\
(\text{🅐}, 0, 0) & \text{if } \text{Ⓑ}, x_1 < 12 - \frac{3}{8}x_2, 0 < x_2 < \frac{12}{5} - \frac{1}{5}x_1 < 8, \\
(\text{🅐}, 0, 0) & \text{if } \text{Ⓑ}, x_1 \geq 12 - \frac{3}{8}x_2, x_2 = 0, \\
(\text{Ⓑ}, 0, \mu_2) & \text{if } \text{Ⓑ}, 12 - \frac{3}{8}x_2 \leq x_1 < 12 + \frac{5}{8}x_2, 4 \leq x_2 < 8, \\
(\text{🅐}, 0, 0) & \text{if } \text{Ⓑ}, 12 - \frac{3}{8}x_2 \leq 12 + \frac{5}{8}x_2 \leq x_1, 4 \leq x_2 < 8, \\
(\text{🅐}, 0, 0) & \text{if } \text{Ⓑ}, 12 - \frac{3}{8}x_2 \leq x_1, 0 < x_2 < \frac{12}{5} - \frac{1}{5}x_1 \leq x_2 < 8, \\
(\text{Ⓑ}, 0, \mu_2) & \text{if } \text{Ⓑ}, x_1 < 12 - \frac{3}{8}x_2, 0 < \frac{12}{5} - \frac{1}{5}x_1 < x_2 < 8, \\
(\text{Ⓑ}, 0, \lambda_2) & \text{if } \text{Ⓑ}, x_1 < 12 - \frac{3}{8}x_2, \frac{12}{5} - \frac{1}{5}x_1 < 0 = x_2 < 8, \\
(\text{🅐}, 0, 0) & \text{if } \text{Ⓑ}, x_1 < 12 - \frac{3}{8}x_2, \frac{1}{5} < x_2 = \frac{12}{5} - \frac{1}{5}x_1 < 8, \\
(\text{Ⓑ}, 0, \mu_2) & \text{if } \text{Ⓑ}, x_1 < 12 - \frac{3}{8}x_2, 0 < x_2 = \frac{12}{5} - \frac{1}{5}x_1 < \frac{1}{5} < 8, \\
(\text{🅐}, 0, 0) & \text{if } \text{Ⓑ}, x_1 < 12 - \frac{3}{8}x_2, \frac{1}{3}x_1 < x_2 < \frac{12}{5} - \frac{1}{5}x_1 < 8, \\
(\text{Ⓐ}, 0, \mu_2) & \text{if } \text{Ⓑ}, x_1 < 12 - \frac{3}{8}x_2, 0 < x_2 < \frac{12}{5} - \frac{1}{5}x_1 \leq \frac{1}{3}x_1 < 8, \\
(\text{Ⓐ}, 0, \lambda_2) & \text{if } \text{Ⓑ}, x_1 < 12 - \frac{3}{8}x_2, 0 = x_2 < \frac{12}{5} - \frac{1}{5}x_1 \leq \frac{1}{3}x_1 < 8.
\end{cases}
\tag{4.6}
$$

Control Actions when in Setup

If the server is in setup, the comparison between the derivative if the server continues in its current mode or switches to setup of the successive mode is less complex. If the server continues in setup the candidate Lyapunov function derivative value equals zero, so by studying (F.60) the control actions are immediately found. The server should continue in its current setup for all domain parts where (F.60) equals ∞. If (F.60) equals $-\infty$, the server should switch to setup of the successive mode. In parts of the domain where (F.60) equals 0, the control action is undefined and the result from Appendix G is added. All these results combined give the control actions if the server starts in setup, (4.7).

$$
(u_0,u_1,u_2) = \begin{cases}
(\textcircled{A},0,0) & \text{if } \textcircled{A}, x_1 \geq \frac{104}{37}, x_2 \geq 5, \\
(\textcircled{B},0,0) & \text{if } \textcircled{A}, x_1 < \frac{104}{37}, x_2 \geq 5, \\
(\textcircled{A},0,0) & \text{if } \textcircled{A}, \frac{15}{8} - \frac{3}{8}x_2 \leq \frac{5}{8}x_2 - \frac{93}{296} \leq x_1 < 12, \frac{12}{5} - \frac{1}{5}x_1 \leq x_2 < 5, \\
(\textcircled{B},0,0) & \text{if } \textcircled{A}, \frac{15}{8} - \frac{3}{8}x_2 \leq x_1 < \frac{5}{8}x_2 - \frac{93}{296} < 12, \frac{12}{5} - \frac{1}{5}x_1 \leq x_2 < 5, \\
(\textcircled{A},0,0) & \text{if } \textcircled{A}, x_1 < \frac{15}{8} - \frac{3}{8}x_2 < 12, \frac{12}{5} - \frac{1}{5}x_1 \leq x_2 < 5, \\
(\textcircled{A},0,0) & \text{if } \textcircled{A}, \frac{15}{8} - \frac{3}{8}x_2 \leq x_1 < 12, x_2 < \frac{12}{5} - \frac{1}{5}x_1, \\
(\textcircled{A},0,0) & \text{if } \textcircled{A}, x_1 < \frac{15}{8} - \frac{3}{8}x_2 < 12, x_2 < \frac{12}{5} - \frac{1}{5}x_1, \\
(\textcircled{A},0,0) & \text{if } \textcircled{A}, x_1 \leq 3x_2 < \frac{15}{8} - \frac{3}{8}x_2 < 12, x_2 \leq \frac{12}{5} - \frac{1}{5}x_1, \\
(\textcircled{A},0,0) & \text{if } \textcircled{A}, 3x_2 < x_1 < \frac{15}{8} - \frac{3}{8}x_2 < 12, x_2 < \frac{12}{5} - \frac{1}{5}x_1, \\
(\textcircled{A},0,0) & \text{if } \textcircled{B}, x_1 \geq \frac{104}{37}, x_2 \geq 5, \\
(\textcircled{B},0,0) & \text{if } \textcircled{B}, x_1 < \frac{104}{37}, x_2 \geq 5, \\
(\textcircled{A},0,0) & \text{if } \textcircled{B}, x_1 \geq 12, x_2 < 5, \\
(\textcircled{B},0,0) & \text{if } \textcircled{B}, \frac{15}{8} - \frac{3}{8}x_2 \leq x_1 < 12, \frac{12}{5} - \frac{1}{5}x_1 \leq \frac{93}{185} + \frac{8}{5}x_1 \leq x_2 < 5, \\
(\textcircled{A},0,0) & \text{if } \textcircled{B}, \frac{15}{8} - \frac{3}{8}x_2 \leq x_1 < 12, \frac{12}{5} - \frac{1}{5}x_1 \leq x_2 < \frac{93}{185} + \frac{8}{5}x_1 < 5, \\
(\textcircled{B},0,0) & \text{if } \textcircled{B}, x_1 < \frac{15}{8} - \frac{3}{8}x_2 < 12, \frac{12}{5} - \frac{1}{5}x_1 \leq x_2 < 5, \\
(\textcircled{A},0,0) & \text{if } \textcircled{B}, \frac{15}{8} - \frac{3}{8}x_2 \leq x_1 < 12, x_2 < \frac{12}{5} - \frac{1}{5}x_1, \\
(\textcircled{B},0,0) & \text{if } \textcircled{B}, x_1 < \frac{15}{8} - \frac{3}{8}x_2 < 12, x_2 < \frac{12}{5} - \frac{1}{5}x_1, \\
(\textcircled{B},0,0) & \text{if } \textcircled{B}, 3x_2 < x_1 \leq \frac{15}{8} - \frac{3}{8}x_2 < 12, x_2 < \frac{12}{5} - \frac{1}{5}x_1, \\
(\textcircled{B},0,0) & \text{if } \textcircled{B}, x_1 < \frac{15}{8} - \frac{3}{8}x_2 < 3x_2 < 12, x_2 < \frac{12}{5} - \frac{1}{5}x_1,
\end{cases} \tag{4.7}
$$

The control actions derived in this subsection are more complex than the controller presented in [1], even when the parts of the domain that are not taken into account in [1] are omitted. The complexity shows in the parts of the domain referred to as $\mathscr{D}^{\mathrm{III_b}}$ and $\mathscr{D}^{\mathrm{V}}$. The extension made in this study was including the domain part referred to as $\mathscr{D}^{\mathrm{VI}}$, which should not lead to differences in the controller outside this domain part. An additional difference between the candidate Lyapunov function proposed in [1] and the Lyapunov function candidate derived in this thesis, is the definition of V in setup. Because of that it is assumed the complexity is increased due to the definition of the candidate Lyapunov function in setup. In Appendix H this assumption is examined.

4.1.3 Simplified Control Actions, $N = \{1, 2\}$

Defining the control actions as shown in the previous section, is cumbersome. Furthermore, Appendix H shows that the complexity can not entirely be dedicated to the difference in definition of the candidate Lyapunov functions. The goal is to derive a control policy that is easy to implement. If control actions are derived analogously to the two direction system for a larger system, a significant increase in the complexity of the control action expression arises. The objective to prove stability of the controlled system only restricts on the candidate Lyapunov function derivative value from increasing. If the time it takes to obtain periodic behaviour is of less importance, the server could simply continue in its current mode when V is non-increasing. So if the constraint of finding the most efficient solution is relaxed, a simpler expression of the control actions is found. Additionally, switching between setups is only of interest during the first time instant the server is started. The server switches modes if necessary after the setup is finished. Finally, if $\dot{V} = 0$ holds for the candidate Lyapunov function derivative, continuing in the current mode does not increase the value of V.

Although the simplified control actions might not converge the system as quickly as possible, the derivation of control actions significantly simplifies. The above mentioned adjustments results in the simplified expression of the control actions, (4.8).

$$(u_0, u_1, u_2) = \begin{cases} (\text{Ⓐ}, \mu_1, 0) & \text{if Ⓐ}, x_1 > 15, \\ (\text{Ⓐ}, \mu_1, 0) & \text{if Ⓐ}, 0 < x_1 \leq 15, x_2 \geq 5, \\ (\text{🅑}, 0, 0) & \text{if Ⓐ}, x_1 = 0, x_2 \geq 5, \\ (\text{Ⓐ}, \mu_1, 0) & \text{if Ⓐ}, x_0 = 0, 0 < x_1 \leq 15, 4 - \frac{1}{5}x_1 \leq x_2 < 5, \\ (\text{Ⓐ}, \lambda_1, 0) & \text{if Ⓐ}, x_0 = 0, x_1 = 0, 4 \leq x_2 < 5, \\ (\text{Ⓐ}, \mu_1, 0) & \text{if Ⓐ}, x_0 = 0, 0 < x_1 \leq 15, x_2 < 4 - \frac{1}{5}x_1, \\ (\text{Ⓐ}, \lambda_1, 0) & \text{if Ⓐ}, x_0 = 0, x_1 = 0, x_2 < 4 - \frac{1}{5}x_1, \\ (\text{🅐}, 0, 0) & \text{if 🅐}, \\ (\text{Ⓑ}, 0, \mu_2) & \text{if Ⓑ}, x_0 = 0, x_2 > 8, \\ (\text{Ⓑ}, 0, \lambda_2) & \text{if Ⓑ}, x_0 = 0, 0 < x_2 \leq 8, \\ (\text{🅐}, 0, 0) & \text{if Ⓑ}, x_1 \geq 12, x_2 = 0, \\ (\text{Ⓑ}, 0, \lambda_2) & \text{if Ⓑ}, x_1 < 12 - \frac{3}{8}, x_2 = 0, \\ (\text{🅑}, 0, 0) & \text{if 🅑}. \end{cases} \tag{4.8}$$

The control actions in (4.8) are defined based on the candidate Lyapunov function derivative. The result can straightforwardly be transformed in, (4.9). The control actions in (4.9) are equal to those presented in [1]. In the upcoming section the simplification of the design of control actions is immediately applied to the example intersection of three directions. This derivation clearly shows the simplification of the control action design.

$$(u_0, u_1, u_2) = \begin{cases} (\text{Ⓐ}, \mu_1, 0) & \text{if Ⓐ}, x_1 > 0, \\ (\text{Ⓐ}, \lambda_1, 0) & \text{if Ⓐ}, x_1 = 0, x_2 < 5, \\ (\text{🅑}, 0, 0) & \text{if Ⓐ}, x_1 = 0, x_2 \geq 5, \\ (\text{🅐}, 0, 0) & \text{if 🅐}, \\ (\text{Ⓑ}, 0, \mu_2) & \text{if Ⓑ}, x_2 > 0, \\ (\text{Ⓑ}, 0, \lambda_2) & \text{if Ⓑ}, x_1 < 12, x_2 = 0, \\ (\text{🅐}, 0, 0) & \text{if Ⓑ}, x_1 \geq 12, x_2 = 0, \\ (\text{🅑}, 0, 0) & \text{if 🅑}, x_0 > 0. \end{cases} \tag{4.9}$$

4.2 Design of Control Actions, $N = \{1, 2, 3\}$

In this section the conclusions of the previous section are used in the design of the control actions. Based on the simplifications listed in the previous section, the control action is to continue in the current mode if possible. In which if possible refers to the fact that the Lyapunov function value should be non-increasing.

First the candidate Lyapunov function derivative if the server continues in its current mode is derived. Then the candidate Lyapunov function in case the server switches to setup is derived. But this derivation is only required for the parts of the domain where V increases, if the server continues in its current mode. If the candidate Lyapunov function derivative equals zero, the control action is to continue in the current mode. Because of that, a three direction system equivalent of Appendix G is excluded.

The curse of dimensionality becomes obvious when the derivative is defined for (3.53). The final expressions of the candidate Lyapunov function derivative elongate significantly and the derivation is cumbersome. The method to derive the candidate Lyapunov function derivative is equivalent to the derivation in the two direction system. Hence, the derivation of the candidate Lyapunov function derivative is simply attached in Appendix E.

4.2.1 Control Actions, $N = \{1, 2, 3\}$

The control actions are based on the results of the derivation presented in Appendix E. While the candidate Lyapunov function is non-increasing, the server continues in its current mode. When the candidate Lyapunov function derivative becomes positive the server switches to the subsequent mode. The subsequent mode is the successive mode denoted in the fixed time schedule. This restriction endorses the desired steady-state behaviour is obtained. The resulting control actions (u_0, u_1, u_2, u_3) equal (4.10). The base for this equation is found in Appendix E. The expression of control actions for the three flow system illustrates the effect of an increase in system dimension. The number of options increases, which yields the extensive expression in (4.10).

$$(u_0, u_1, u_2, u_3) = \begin{cases} (\text{Ⓐ}, \mu_1, 0, \mu_3) & \text{if } \text{Ⓐ}, x_1 > 0, x_3 > 0, \\ (\text{Ⓐ}, \lambda_1, 0, \mu_3) & \text{if } \text{Ⓐ}, x_1 = 0, x_3 > 0, \\ (\text{Ⓐ}, \mu_1, 0, \lambda_3) & \text{if } \text{Ⓐ}, x_1 > 2, x_3 = 0, \\ (\text{🅑}, \mu_1, 0, 0) & \text{if } \text{Ⓐ}, 0 < x_1 \leq 2, x_3 = 0, \\ (\text{🅑}, \lambda_1, 0, 0) & \text{if } \text{Ⓐ}, x_1 = 0, x_3 = 0, \\ (\text{🅐}, 0, 0, \mu_3) & \text{if } \text{🅐}, x_3 > 0. \\ (\text{🅐}, 0, 0, \lambda_3) & \text{if } \text{🅐}, x_3 = 0. \\ (\text{Ⓑ}, \mu_1, \mu_2, 0) & \text{if } \text{Ⓑ}, x_1 > 0, x_2 > 0, \\ (\text{Ⓑ}, \mu_1, \lambda_2,) & \text{if } \text{Ⓑ}, x_1 > 0, x_2 = 0, \\ (\text{Ⓑ}, \lambda_1, \mu_2, 0) & \text{if } \text{Ⓐ}, x_1 = 0, x_2 > 1 \\ (\text{🅒}, 0, \mu_2, 0) & \text{if } \text{Ⓐ}, x_1 = 0, 0 < x_2 \leq 1 \\ (\text{🅒}, 0, \lambda_2, 0) & \text{if } \text{Ⓐ}, x_1 = 0, x_2 = 0 \\ (\text{🅑}, \mu_1, 0, 0) & \text{if } \text{🅑}, x_1 > 0. \\ (\text{🅑}, \lambda_1, 0, 0) & \text{if } \text{🅑}, x_1 = 0. \\ (\text{Ⓒ}, 0, \mu_2, \mu_3) & \text{if } \text{Ⓒ}, x_2 > 0, x_3 > 0, \\ (\text{Ⓒ}, 0, \mu_2, \lambda_3) & \text{if } \text{Ⓒ}, x_2 > 0, x_3 = 0, \\ (\text{Ⓒ}, 0, \lambda_2, \mu_3) & \text{if } \text{Ⓒ}, x_2 = 0, x_3 > 4, \\ (\text{🅐}, 0, 0, \mu_3) & \text{if } \text{Ⓒ}, x_2 = 0, 0 < x_3 \leq 4, \\ (\text{🅐}, 0, 0, \lambda_3) & \text{if } \text{Ⓒ}, x_2 = 0, x_3 = 0, \\ (\text{🅒}, 0, \mu_2, 0) & \text{if } \text{🅒}, x_2 > 0. \\ (\text{🅒}, 0, \lambda_2, 0) & \text{if } \text{🅒}, x_2 = 0. \end{cases} \tag{4.10}$$

4.3 Stability of the Controlled System

The final requirement on the control actions to address, is the fact that the control actions are required to stabilize the system. Preferably, the systems steady-state behaviour equals the optimal periodic behaviour. The control actions are derived based on the idea of proving stability with the Lyapunov-like function, Theorem 4.1, for which stability is defined in Theorem 4.2. This proves the stability of the controlled system in the feasible domain of [1], so the entire domain with the exception of the so called $\mathscr{D}^{\mathrm{VI}}$ domain part. However, in $\mathscr{D}^{\mathrm{VI}}$ the candidate Lyapunov function value equals zero, which triggers the idea of a LaSalle like extension.

In continuous systems LaSalle's theorem, an invariance principle, proves stability of the system if the value of the Lyapunov function is only negative semidefinite in specific cases, [43]. This implies an extension to hybrid systems of this invariance principle, could prove the stability of the controlled system.

Theorem 4.1. *A family of Lyapunov-like functions such that each vector field has its own*

Lyapunov function is used. The particularity of Lyapunov-like functions is that the decay of the function is required only when the system is active. A multiple Lyapunov-like function satisfies the following conditions, [34]:

- $V_j(\boldsymbol{x}) > 0 \qquad \forall \boldsymbol{x} \neq \boldsymbol{x}^*, j \in s$
- $V_j(\boldsymbol{x}) = 0 \qquad \forall \boldsymbol{x} = \boldsymbol{x}^*, j \in s$
- *The derivative of each $V_j(\boldsymbol{x})$ satisfies the relation*

$$\dot{V}_j(\boldsymbol{x}) \leq 0 \qquad \forall j \in s \tag{4.11}$$

when the j-th subsystem is active.

Theorem 4.2. *This theorem states the requirement for a switched system in the context of multiple Lyapunov-like functions, Theorem 4.7 in [34]. Consider a family of Lyapunov-like functions V_j, each associated with a vector field $A_j\boldsymbol{x}$. For $k < m$, let $t_k < t_m$ be the switching times for which $j(t_k) = j(t_m)$. If there exists a $\gamma > 0$ such that*

$$V_j(t_m)(\boldsymbol{x}(t_{m+1})) - V_j(t_k)(\boldsymbol{x}(t_{k+1})) \leq -\gamma \|\mathbf{x}(t_{k+1})\|^2, \tag{4.12}$$

then the switched system is stable.

The candidate Lyapunov function equals zero for buffer contents in periodic behaviour. However, buffer contents in the domain parts referred to as $\mathscr{D}^{\mathrm{VI}}$, the candidate Lyapunov function also equals zero. Invariance principles are derived for switched systems and for hybrid systems, for instance Theorem 8.2 of [44], and [45, 46, 47]. In [45] LaSalle's invariance principle is extended to certain classes of hybrid systems. It is shown how asymptotic stability is proven with multiple Lyapunov functions whose Lie derivatives are only negative semi-definite. The Lie derivatives of the different Lyapunov function parts are all negative semi-definite, which means that by the strategy suggested in [45] asymptotic stability of the controlled system is proven. This proves the system approaches the desired optimal periodic behaviour for all starting values in the state-space. However, the existence of a nonzero slow mode in a system is an additional requirement for the system to converge in optimal periodic behaviour for all starting values.

The control actions designed in this chapter, form the foundation to establish a control policy in Chapter 5. Comparing the control actions for the two and three flow example systems, leaves to conclude that the control actions of the three flow system are more extensive than the expression for the control actions of the two flow system. However, this is only due to the extra mode and flow, the derived control actions are straightforward. Furthermore, the control actions stabilize the system in optimal periodic behaviour, as required. The convergence might not be as time-efficient as possible, as some concessions were made to create control actions that are easy to implement. In Chapter 5 the expressions of the control actions are transformed into a control policy. This simplifies the expression of the controller, especially for large dimension systems.

Chapter 5. Control Policy

The curse of dimensionality shows in Chapter 3 and Chapter 4. When the system dimension increases, the derivation of control actions becomes cumbersome which leads to extensive expressions for the control actions. Control actions can be listed in words as a set of decision rules. In this chapter control policies are defined for system of various sizes, the derivation of a control policy is explained in this chapter. The process of transforming control actions to a set of decision rules is referred to as transforming the control actions into a control policy.

In Section 5.1 the control actions discussed in Chapter 4, are transformed into control policies. A strategy to determine a control policy for systems of all sizes is proposed in Section 5.2, provided that specific system requirements are met. The strategy is demonstrated by an illustrative example, which is designed such that defining modes of the system appears to be straightforward. Finally, case studies are presented to illustrate that defining modes in an actual fixed time schedule to obtain a control policy is possible.

5.1 Transform Control Actions

The expression of the control actions of a system can be transformed into a set of decision rules. The decision depends on the current state values of the system. An additional term is introduced to shorten the expression of the control policy, namely processing at maximum rate. Processing at maximum is defined as processing x_i at rate μ_i if $x_i > 0$, and processing x_i at rate λ_i if $x_i = 0$. This definition shortens the expression of the control policy, whilst all the information is preserved.

The transformation of the control actions listed in (4.9), into a control policy is straightforward as is illustrated below. Part of the expressions of (4.9) are repeated in this chapter to illustrate the process of obtaining a control policy. The control actions in A of the two flow system are listed in (5.1). Studying the first two lines of the control actions in A, it is concluded that x_1 is processed as long as $x_2 < 5$. This can be transformed in words: When in A with $x_2 < 5$ process x_1 at maximum rate. The next line of the control action in (5.1) can be explained in words as: When in A with $x_2 \geq 5$ and $x_1 = 0$ switch to B.

$$(u_0, u_1, u_2) = \begin{cases} (\text{A}, \mu_1, 0) & \text{if } \text{A}, x_1 > 0, \\ (\text{A}, \lambda_1, 0) & \text{if } \text{A}, x_1 = 0, x_2 < 5, \\ (\text{B}, 0, 0) & \text{if } \text{A}, x_1 = 0, x_2 \geq 5. \end{cases} \tag{5.1}$$

Similar transformations are performed for the remainder of (4.9), which yields the control policy of the two flow system:

- *When in A:*
  - continue processing at x_1 at maximum rate
  - switch to B if $x_1 = 0$, $x_2 \geq 5$.
- *When in B:*
  - continue processing x_2 at maximum rate
  - switch to A if $x_2 = 0$, $x_1 \geq 12$.

- *When in* A *or* B*:*
  - continue for x_0 time units, then start processing in the successive mode.

The complexity of the control policy presented above, is similar to the complexity of the control actions. However, the advantage of transforming the control actions into a control policy shows in systems of increased dimension, which is illustrated by transforming the control actions of the three flow example. The control policy is created by performing similar transformations as in the two flow example. The resulting control policy, corresponding to the control actions (4.10), equals:

- *When in* A*:*
  - continue processing x_1 and x_3 at their maximum rate.
  - switch to B if both $x_1 \leq 0$ and $x_3 = 0$.
- *When in* B*:*
  - continue processing x_1 and x_2 at their maximum rate.
  - switch to B if both $x_1 = 0$ and $x_2 \leq 1$.
- *When in* C*:*
  - continue processing x_2 and x_3 at their maximum rate.
  - switch to B if both $x_2 = 0$ and $x_3 \leq 4$.
- *When in* A*:*
  - continue processing x_3 at maximum rate for x_0 time units, then switch to A.
- *When in* B*:*
  - continue processing x_1 at maximum rate for x_0 time units, then switch to B.
- *When in* C*:*
  - continue processing x_2 at maximum rate for x_0 time units, then switch to C.

The control policy of the three flow system presented above, significantly shortens the expression of a controller for the three flow system. Similarities between the control policy of the two and three flow system appear when comparing the previously derived control policies. Furthermore, the control policy of the intersection of three directions is repetitive, which is of convenience in deriving a general control policy. The derivation of a general control policy is explained in the upcoming section.

5.2 Control Policy

In the previous section a control policy is derived for the example systems discussed in Chapter 3 and Chapter 4. Studying the control policies the repetitive nature of the policies is striking. However, the two policies also show differences. This difference is interesting, even though the control actions are derived from a similarly defined candidate Lyapunov function, the resulting control policy differs. In the two flow policy the switch between modes occurs when the served buffer is emptied and the unserved buffer reaches its threshold value. In the three flow policy however, a switch occurs when the primarily served buffer is emptied and the secondarily served buffer does not exceed its threshold value. The unserved buffer is of no importance in the control policy of the three flow system.
This difference is assumed to be the effect of the required slow modes. In the two flow system no other buffer is served, so the threshold on the unserved buffer value ensures the slow mode is performed in optimal periodic behaviour. In the three flow system another buffer is served which needs to be empty enough to progress to the next mode. This buffer simultaneously defines that the slow mode is performed in optimal periodic behaviour. Therefore, it is not necessary to take into account the value of the unserved buffers.

A general control policy of a system in mode $\boldsymbol{m}$ is determined, based on the previously discussed conclusions and the control policy of the three flow system. This general control policy holds for systems with modes that serve more than one direction at the time in a mode. The general

control policy is given in Lemma 5.3.
Lemma 5.3 enables to derive a control policy for each system of which a fixed time schedule is known. More specifically, a control policy can directly be derived from the optimal fixed time schedule, if modes can be defined, and the optimal periodic behaviour is computed. This assumption is examined in the case studies of the next section.

Lemma 5.3. *For a system that meets the requirements of Appendix A with a given optimal fixed time schedule in which more than one buffer is served in a mode, the control policy in mode* $\boldsymbol{m}$ *can directly be derived from the fixed time schedule.*

Let $\boldsymbol{m}$ *be the mode the system is currently in, with* $\boldsymbol{n}$ *its successive mode. The buffer contents at the mode switch in optimal periodic behaviour are defined as* $x_i^{n^*}$. *The set of buffers served during setup of mode* $\boldsymbol{n}$ *is called* b. *Let* s *equal the set of all served buffers in mode* $\boldsymbol{m}$, *except for the primarily served buffers. Let* p *represent the set of primarily served buffers and* x_0 *the remaining setup time, in that case the control policy equals:*

- *When in* m:
  - *continue processing* x_i *at its maximum rate,* $\forall i \in s$.
  - *switch to* n *if* $x_i \leq x_i^{n^*}$ $\forall i \in s$ *and* $x_j = 0$ $\forall j \in p$.
- *When in* n:
  - *continue processing* x_i $\forall i \in b$ *at maximum rate for* x_0 *time units, then switch to* n.

5.4 Case Studies

Two cases are studied to examine if the conclusion stated in the previous section is correct. Is it possible to define a control policy for a system of any size with Lemma 5.3, provided that the requirements are met, the system parameters are known, and an optimal fixed time schedule is known. To examine this statement modes are required to be defined in the fixed time schedule. The definition of mode is ambiguous. Therefore, first the process of defining modes used in this thesis is explained, after which the optimal periodic behaviour is computed. The derived modes, the fixed time schedule and the optimal behaviour can be combined with Lemma 5.3 which provides a control policy. This control policy is expected to be stabilizing, based on the results of the two and three flow example.

5.4.1 Intersection A2N279

Defining modes in a fixed time schedule is crucial in the determination process of a control policy. The modes can not be determined unambiguously based on the fixed time schedule itself, additional intersection data is needed. As mentioned in Section 1.3, the setup time σ of a mode depends on the clearance time needed. The time that is needed to switch from processing one flow, to processing the subsequent flow. In case of a larger system however, multiple flows might switch simultaneously. The question arises which clearance time determines the duration of the setup of a mode.

To explain how modes are defined first some terminology is introduced. Section 1.3 mentions clearance time, the time needed to switch between two conflicting flows. All these clearance times can be denoted in a so called conflict matrix, Σ. The matrix rows and columns represent all flows in the system, thus the matrix contains the squared number of elements in N. The row number, p of the matrix represents the flow given as p^{th} element in N, the column number, q of the matrix corresponds to the q^{th} element in N. An element of the conflict matrix is empty if there is no conflict between the flow represented by the row number and the flow represented by the column number. If there exists a conflict between the flows corresponding to the row and column number of the matrix, the element value equals the clearance time duration. A conflict is called active if the time between the end of a green period of one flow, and the start of the green period of a conflicting flow, exactly equals the clearance time denoted in the conflict matrix. These active conflicts are listed in the so called active conflict matrix, logically having equally located non-empty elements as the conflict matrix. The active conflict matrix elements equal 1 if the conflict corresponding to the element is active, and a zero if the conflict is not active.

The modes in a fixed time schedule can be derived with the definitions of conflict matrix and active conflict matrix. To illustrate the definition of modes a fixed time schedule of an existing intersection in the Netherlands is studied. The optimal fixed time schedule corresponding to a specific time-of-day of this intersection is shown in Figure 5.1 and the conflict matrix is given in (5.2). First all active conflicts in the fixed time schedule are determined, the flows in the system are $N = \{1, 2, 8, 9, 10, 12\}$. As the conflict matrix states, switching from flow 1 to flow 9 takes a setup time of zero. From the fixed time schedule in Figure 5.1, it can be concluded that the time between the end of green period of flow 1 and the start of green period of flow 9 does not exceed the clearance time, which means the conflict is active. Performing this review for each flow, yields the active conflict matrix, (5.3).

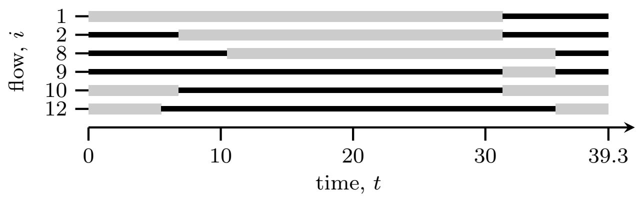

Figure 5.1: Fixed time schedule, A2N279

$$\Sigma = \left\{ \begin{array}{cccccc} & & & 0 & & \\ & & & 0 & 0 & 0 \\ & & & & & 0 \\ 4 & 3 & & & 0 & \\ & 0 & & & & \\ & 1 & 5 & 3 & & \end{array} \right\} \tag{5.2}$$

$$\Sigma_a = \left\{ \begin{array}{cccccc} & & & \mathbf{1} & & \\ & & & \mathbf{1} & \mathbf{1} & 0 \\ & & & & & \mathbf{1} \\ \mathbf{1} & 0 & & & 1 & \\ & \mathbf{1} & & & & \\ & 1 & \mathbf{1} & 0 & & \end{array} \right\} \tag{5.3}$$

Studying the active conflict matrix shows that when the green period of for instance flow 2 ends, multiple conflicts are active. The control policy needs to ensure that all clearance times are performed. Hence, the setup time is set to the active constraint with the largest clearance time. To clarify this definition, the constraints that determine the setup time, referred to as defining active conflicts, are emphasized in the active constraint matrix by bold numbers. Two bold numbers mean that there is no distinction between the active conflicts, both active conflicts require equal setup time.

Table 5.1 lists the time at the end of setup, based on the defining conflicts in the system. This table shows the setup of direction 10 and direction 12 overlap. The setup times are grouped which yields the duration of B. By grouping the following is meant: the minimum end of the green period of flow 10 and 12 provides the start of B, and the maximum end of setup of flow10, 12 represents the finish of setup. This method is used to define all setup zones in the fixed time schedule of the A2N279 intersection, which results in Figure 5.2.

Table 5.1: Computation setup zones, A2N279.

i	$t_{@\text{end green}}$	setup time	$t_{@\text{end setup}}$	group	$g_{T,i}$	$r_{T,i}$
1	**31.3**	0	**31.3**	C	31.3	8
2	**31.3**	0	**31.3**	C	24.5	14.8
8	**35.3**	0	35.3	A	24.8	14.5
9	**35.3**	4	**39.3**	A	4	35.3
10	6.8	0	6.8	B	14.8	24.5
12	**5.5**	5	**10.5**	B	9.5	29.5

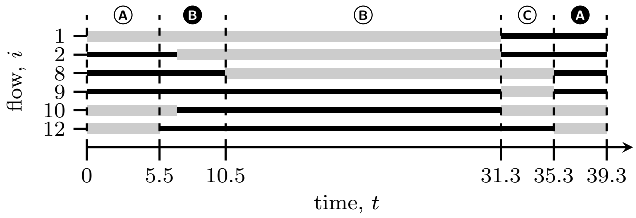

Figure 5.2: Fixed time schedule, A2N279

Control Policy

With the modes of the system defined, the control policy can be determined. As implied in Section 3.1, the specific values found in the control policy can easily be derived from the fixed time schedule and the intersection data. This final step to define a control policy for the A2N279 intersection is discussed below. The intersection data, the arrival and process rates of each flow, are used to compute the optimal buffer contents as function of time with the assumption that each buffer is emptied once during a cycle. The intersection data is given in Appendix I.
The duration of the green period of flow i, $g_{T,i}$, can be read in the fixed time schedule. The duration of the red period of flow i, $r_{T,i}$ is then easily computed with (5.4). The optimal content of buffer i is computed as function of time, based on the red and green periods and the arrival and service rates of flow i. For instance, $g_{T,1} = 31.3$ yields $r_{T,1} = 39.3 - 31.3 = 8$, this means during the red period $\lambda_1 * 8$ vehicles arrive. These vehicles are then cleared in $\frac{8\lambda_1}{\mu_1 - \lambda_1}$ time units. This computation for each of the flows at the A2N279 intersection, results in Figure 5.3, the optimal buffer contents of the A2N279 intersection. It is noticed that the actual buffer contents are integers, since this is an approximation the computed buffer contents in real numbers are used as it is assumed that this will provide a more accurate control policy.

Based on the knowledge gained in the analytically derived control policies a straightforward control policy for this case study is expected. The server should serve all buffers in a mode that need to be served in that mode, at maximum rate. The server continues to process in a mode until the threshold values are reached for all served buffers. When this occurs the server should switches to setup of the successive mode, before starting to serve in that mode. The previously suggested control policy is exactly the result of Lemma 5.3.

$$r_{T,i} = T - g_{T,i}. \tag{5.4}$$

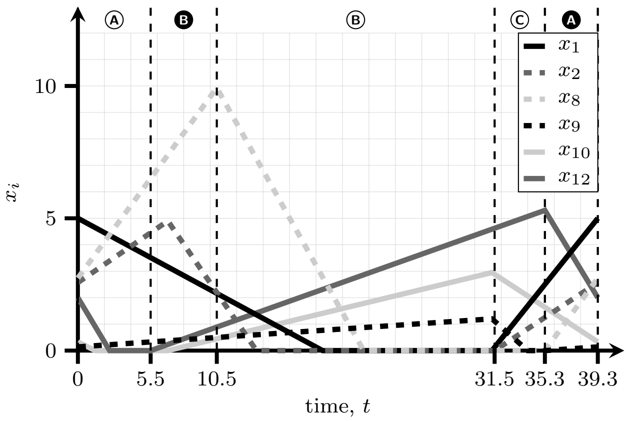

Figure 5.3: Buffer content in optimal periodic cycle.

Table 5.2: Buffer contents

	t	x_1	x_2	x_8	x_9	x_{10}	x_{12}
Ⓐ*	0	5.01	2.57	2.74	0.14	0.33	1.79
🅑*	5.5	3.52	4.46	6.51	0.34	0	0
Ⓑ*	10.5	2.16	2.20	9.93	0.51	0.45	0.98
Ⓒ	31.3	0	0	0	1.25	2.95	4.61
🅐*	35.3	2.50	1.32	0	0	1.64	5.27

5.4.2 Intersection 's Gravendijkwal

Deriving a control policy for this intersection is important, it is considered a relatively large intersection. If a stabilizing control policy can be derived for this intersection, it implies a control policy can be defined for an intersection of any size. With the definitions introduced in the previous section, modes of the 's Gravendijkwal intersection system are defined. Due to the increase in system size, there is an extension needed to the previously mentioned algorithm, this is explained in detail in the mode definition of this example. The fixed time schedule of the intersection studied in this section is shown in Figure 5.4. The corresponding conflict matrix is given in condensed form in Table I.3, due to the system size this table is only attached in the appendix.

The number of flows in the system increased significantly, compared to the previously studied intersection. The same method as used in the previous section is used to determine the modes in this fixed time schedule. First the conflict matrix is studied, in condensed form given in Table I.3. The active conflicts are derived and given in Table I.4. Determining the largest clearance times per direction as setup time, the setup time to perform when the corresponding buffer is emptied, yields the setup times given in Table 5.3. Applying all the active conflicts as setup times show overlap in grouped setup times, as shows in Table 5.3. To cope with all these setup times, the setup times are merged in case they overlap. The group containing the largest setup time values become the setup of the first mode, the smallest of the second mode and the other setup times become setup of the final mode. The minimum and maximum value of the setup times in a group, emphasized by bold numbers in Table 5.3, determine respectively the start and end of setup. This method results in three setup zones, and thus three modes, as shown in Figure 5.5.
Now in this system a problem occurs, as can be seen in Figure 5.5 some flows are only served during setup. This means that if the content of these buffers is exceptionally much, the green time can not be extended to clear these buffers. Therefore, an extension is suggested to the methodology of mode definition. If a direction is only served during setup, the end of the

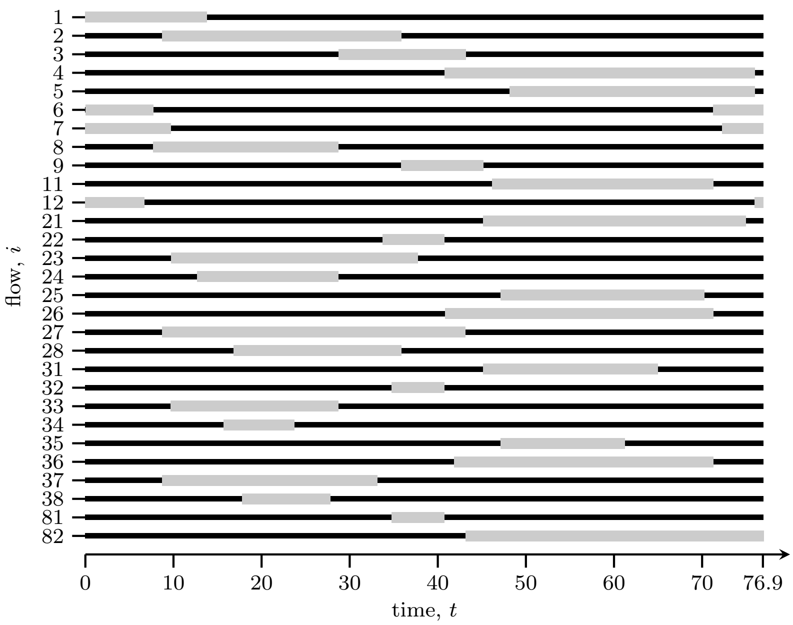

Figure 5.4: Fixed time schedule, 's Gravendijkwal

green time of these buffers create an additional mode. The duration of that mode equals zero, so effectively the server subsequently performs two setups if the system is in optimal periodic behaviour. This solves the unlikely problem that the buffer contents are exceptionally much and can not be processed.

To explicitly define the control policy of the 's Gravendijkwal intersection, the optimal buffer contents are derived. It concerns a large intersection, so the computation of the optimal buffer contents is exhaustive. The computation contains no new theories and is therefore only attached in Appendix I. The resulting buffer contents at the end of each mode are listed in Table 5.4. From the table and Lemma 5.3 a control policy is found for this system.

The method presented in this chapter results in a control policy for a system of various sizes. The resulting control policies are plain and easy to implement. The convergence to the desired periodic behaviour is based on the results of the two and three flow example, with no mathematical proof being provided. The absence of rigorous proof is intended because avoiding cumbersome computations is the goal of creating a general control policy. Although the definition of the modes in the system requires some derivation, it contains far less expressions than the analytical derivation of a control policy for large system sizes.
With the definition of a general control policy this study is finished. The upcoming chapter concludes on the results of this thesis. Furthermore, some recommendations are made to be considered in further research.

Table 5.3: Computation setup zones, 's Gravendijkwal.

i	$t_{\text{@end green}}$	setup time	$t_{\text{@end setup}}$	group	$g_{T,i}$	$r_{T,i}$
1	13.8	4	**17.8**	🅑	13.8	63.1
2	35.8	6	41.8	🅒	27.1	49.8
3	43.1	3	46.1	🅒	28.7	62.5
4	75.9	0	75.9	🅐	40.7	35.2
5	75.9	1	**76.9**	🅐	27.8	49.1
6	7.7	2	9.7	🅑	13.4	63.5
7	9.7	6	15.7	🅑	14.4	62.5
8	28.7	6	34.7	🅒	21	55.9
9	45.1	3	**48.1**	🅒	9.3	67.6
11	71.2	1	72.2	🅐	25.1	51.8
12	**6.7**	2	8.7	🅑	7.7	69.2
21	74.9	2	**76.9**	🅐	29.8	47.1
22	40.7	0	40.7	🅒	7	69.9
23	37.7	3	40.7	🅒	28	48.9
24	28.7	0	28.7	🅒	16	60.9
25	70.2	2	72.2	🅐	23	53.9
26	71.2	0	71.2	🅐	30.4	46.5
27	43.1	3	46.1	🅒	34.4	42.5
28	35.8	0	35.8	🅒	19	57.9
31	64.9	12	**76.9**	🅐	19.8	57.1
32	40.7	0	40.7	🅒	6	70.9
33	28.7	12	40.7	🅒	19	57.9
34	**23.7**	5	28.7	🅒	8	68.9
35	**61.2**	11	72.2	🅐	14	62.9
36	71.2	0	71.2	🅐	29.4	47.5
37	33.1	13	46.1	🅒	24.4	52.5
38	27.8	8	35.8	🅒	10	66.9
81	40.7	0	40.7	🅒	6	70.9
82	76.9	0	**76.9**	🅐	33.8	43.1

Table 5.4: Buffer contents in optimal periodic behaviour.

	t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{11}	x_{12}	x_{21}	x_{22}
Ⓐ*	0	4.38	4.56	4.69	0.08	0.33	1.8	3.03	5.35	1.77	1.89	1.46	0.01	0.14
🅑*	6.7	1.94	5.31	5.62	0.64	2.56	0	0	6.1	2.14	4.13	0	0.03	0.17
Ⓑ*	17.8	0.28	1.74	7.16	1.57	6.26	0.7	0.9	1.72	2.75	7.83	0.31	0.08	0.21
🅒*	23.7	0.69	0	7.98	2.06	8.23	1.11	1.56	0	3.08	9.79	0.47	0.1	0.23
Ⓒ*	48.1	2.38	1.37	0.69	0.6	16.37	2.81	4.27	2.16	0.17	14.96	1.15	0	0.03
🅐*	61.2	3.29	2.82	2.51	0	6.91	3.72	5.72	3.61	0.89	0	1.51	0	0.08

	x_{23}	x_{24}	x_{25}	x_{26}	x_{27}	x_{28}	x_{31}	x_{32}	x_{33}	x_{34}	x_{35}	x_{36}	x_{37}	x_{38}	x_{81}	x_{82}
Ⓐ*	0.09	0.11	0.02	0.02	0.09	0.11	0.17	0.5	0.67	0.74	0.22	0.08	0.61	0.68	0.4	0
🅑*	0.1	0.12	0.04	0.03	0.11	0.13	0.26	0.6	0.76	0.83	0.31	0.17	0.7	0.77	0.48	0.07
Ⓑ*	0	0	0.07	0.07	0	0	0.41	0.75	0	0	0.47	0.33	0	0.93	0.6	0.2
🅒*	0	0	0.08	0.08	0	0	0.5	0.83	0	0	0.55	0.41	0	0	0.67	0.26
Ⓒ*	0.02	0.04	0	0	0.01	0.03	0	0.1	0.27	0.34	0	0	0.21	0.28	0.08	0
🅐*	0.05	0.07	0	0	0.05	0.07	0	0.28	0.45	0.52	0	0	0.39	0.46	0.23	0

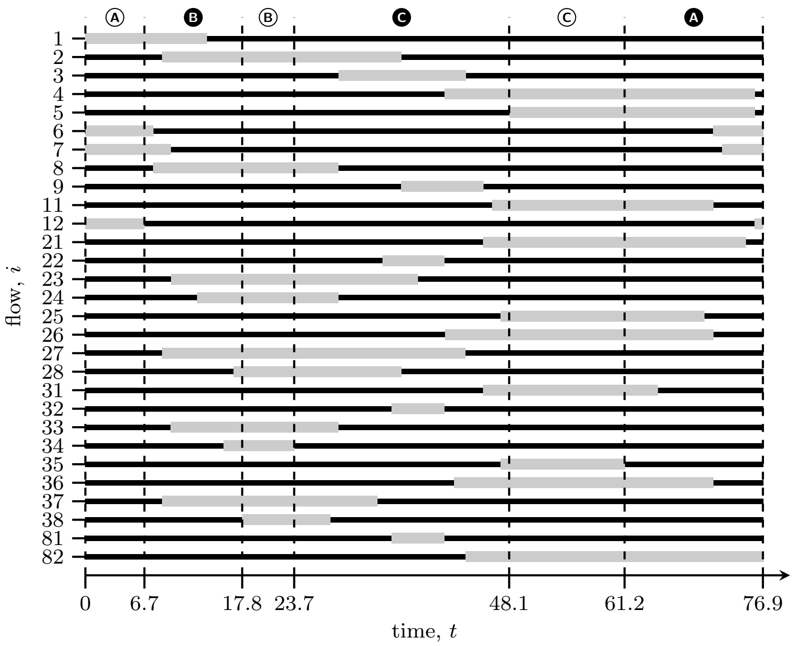

Figure 5.5: Fixed time schedule with modes, 's Gravendijkwal

Chapter 6. Conclusion

Thanks to beneficial application of control methods any improvements of traffic congestion might be countered by an increased capacity demand, as is also noted in [3]. Although the results presented in this thesis are theoretical and do not directly reduce traffic congestion, some interesting achievements are made.

A candidate Lyapunov function is derived for example systems of two sizes based on the approach proposed in [1]. The candidate Lyapunov function of the two and three flow system is defined in the entire domain, instead of in some feasible part of the domain. The resulting candidate Lyapunov function on the feasible part of the domain presented in [1] is equivalent to the expression of the candidate Lyapunov function given in [1]. Hence, a generalisation is established of the Lyapunov function in [1].
Furthermore, the derivation of a candidate Lyapunov function is structured, as illustrated in Chapter 3. Because of this structure, it is assumed based on the derived functions that a candidate Lyapunov function can be found for systems of various sizes. This assumption holds provided that the system meets the requirements in Appendix A.
It is concluded that the complexity of the candidate Lyapunov function derivation significantly increases when the system dimensions are increased, thus explicitly denoting the Lyapunov function for large system dimensions is infeasible.

The candidate Lyapunov functions described in Chapter 3 are the basis of the control actions. The control actions are derived based on the derivative of the candidate Lyapunov functions. The Lyapunov function should be non-increasing over time to possibly achieve stable system behaviour. The control actions chosen to be performed are the control actions that move the system states in the direction of steepest descent. The control actions minimize the candidate Lyapunov function derivative in each part of the domain, thus move the system in the direction of steepest descent.
The expression of the control actions resulting from the derivation are rather complex. If the requirement of quick convergence is relaxed, the expression of the control actions significantly simplifies. The definition of the control actions provides a non-increasing Lyapunov function, the Lyapunov function derivative is negative or zero. For the infeasible part of the domain, the Lyapunov function value equals zero. Therefore the stability proof of the controlled system is extended with an invariance principle, which proves the stability of the example systems.

The control actions of Chapter 4 are transformed into a control policy for the example systems. The resulting control policies can straightforwardly be implemented. The two flow system control policy and three flow system control policy are compared. From the repetitive nature of the policies a general control policy is derived. With this definition of a general control policy, a control policy can directly be obtained from the given fixed time schedule of an intersection.
A methodology to define modes in the system is derived so combined with the general control policy and the fixed time schedule, a control policy can be found for an intersection of any size. This conclusion is verified by performing two case studies. The mode definition requires some derivations but these are insignificant compared to the derivations needed to derive a control policy from the candidate Lyapunov function. With the measured traffic data of the case studies intersections, the fixed time schedules of these intersections and the defined modes a control policy can be found for both case studies.

Concluding, the research objective is met for the two flow system and for the three flow system. An easy to implement control policy is found based on derived candidate Lyapunov functions. Furthermore, it is proven that the derived control policies stabilize the two and three flow system in their optimal periodic behaviour.
For systems of increased dimension a method is developed to define modes in a fixed time schedule, and a general control policy is derived. With these tools a control policy for an intersection is established based on the data of the specific intersection.

Currently the control policy forces switching through modes in the predefined order. Due to the setup times this means that a signal might switch green when its corresponding queue is empty. The duration of transient behaviour could be shortened by allowing the server to switch to an arbitrary mode. For instance, in a situation where the buffer contents of each of the flows are highly unequally distributed, it is likely that switching in a non predetermined order is beneficiary for the convergence speed of the controlled system.

Apart from the fact that the results are promising and the research objective is met for specific example systems, the number of intersections at which flows are not affected by neighbouring intersections might not be significant. After all, the Netherlands is a relatively small country with compact infrastructure. If flows at intersections are affected by neighbouring intersections, controlling these intersections as if they are isolated might result in an increase of overall average waiting time. Communication between intersections to prevent intersections from counteracting could be a solution, as in network setting arriving vehicles often arrive in platoons. However, a decentralized controller is desired for the speed of computation and convergence. On that account an extension of a vehicle actuated control policy is suggested as a topic of future research, such that the policy is applicable for a network of intersections.

References

[1] E. Lefeber and J.E. Rooda. Controller design for switched linear systems with setups. *Physica A: Statistical Mechanics and its Applications*, 363:48–61, 2006.

[2] European Commission. Roadmap to a single european transport area — towards a competitive and resource-efficient transport system. *White Paper on Transport*, 1:4–28, 2011.

[3] M. Papageorgiou, C. Diakaki, V. Dinopoulou, A. Kotsialos, and Y. Wang. Review of road traffic control strategies. *Proceedings of the IEEE*, 91:2043–2067, 2003.

[4] European Commission. The paris protocol — a blueprint for tackling global climate change beyond 2020. *COM(2015)*, 81 final/2, 2015.

[5] F. Soylemezgiller, M. Kuscu, and D. Kilinc. A traffic congestion avoidance algorithm with dynamic road pricing for smart cities.

[6] Q.-Y. Luo, Z.-C. Juan, and L.-J. Gao. Urban road congestion pricing based on game theory. *International Conference on Electronic Commerce and Business Intelligence (ECBI)*, pages 139–142, 2009.

[7] M. Song and Y. Li. Feasibility study of urban road congestion pricing. *International Conference on Intelligent Computation Technology and Automation (ICICTA)*, 5:564–567, 2012.

[8] Y. Zhong, Y. Li, and S. Nie. Road congestion pricing model based on p system. *International Conference on Intelligent Computation Technology and Automation (ICICTA)*, 5:295–298, 2012.

[9] K.T. Bui, V.A. Huynh, and E. Frazzoli. Dynamic traffic congestion pricing mechanism with user-centric considerations. *International IEEE Conference on Intelligent Transportation Systems*, 15:147–154, 2012.

[10] Dunn engineering associates and Siemens Intelligent Transportation Systems. *Traffic Control Systems Handbook*. Department of Transportation, 1200 New Jersey Avenue, SE, Washington, DC 20590, 2005.

[11] M.S. van den Broek. Traffic signals, optimizing and analyzing traffic control systems. Master's thesis, Eindhoven University of Technology, The Netherlands, 2004.

[12] K.S. Duisters. Formulating an testing an algorithm for fixed time control of traffic intersections. Bachelor's Thesis, Eindhoven University of Technology, The Netherlands, 2013.

[13] D.A.J. van Zwieten. *Fluid Flow Switching Servers, Control and Observer Design*. PhD thesis, Eindhoven University of Technology, The Netherlands, 2014.

[14] S.T.G. Fleuren. Optimizing traffic light control. Master's thesis, Eindhoven University of Technology, The Netherlands, 2012.

[15] J.W.C.M. van der Vleuten. Networks of signalized traffic intersections, optimization and implementation. Master's thesis, Eindhoven University of Technology, The Netherlands, 2014.

[16] J.J.H. Karouta. Fixed-time schedule based vehicle actuated controller design for networks of intersections. Master's thesis, Eindhoven University of Technology, The Netherlands, 2015.

[17] J.A.W.M. van Eekelen. *Modelling and Control of Discrete Event Manufacturing Flow Lines.* PhD thesis, Eindhoven University of Technology, The Netherlands, 2008.

[18] H.J. H.-J. Chang and G.-T. Park. A study on traffic signal control at signalized intersections in vehicular ad hoc networks. *Ad Hoc Networks*, 11:2115–2124, 2013.

[19] M. Artimy. Local density estimation and dynamic transmission-range assignment in vehicular *Ad Hoc* networks. *IEEE Transactions on Intelligent Transportation Systems*, 8:400–412, 2007.

[20] A.J. Miller. A computer control system for traffic networks. *Proceedings 2nd International Symposium Traffic Theory*, pages 200–220, 1963.

[21] S. Sen and K.L. Head. Controlled optimization of phases at an intersection. *Transportation Science*, 31:5–17, 1997.

[22] G. Abu-Lebdeh and R.F. Benekohal. Design and evaluation of dynamic traffic management strategies for congested conditions. *Transportation Reasearch Part A*, 37:109–127, 2003.

[23] C. Wenjie, C. Lifeng, C. Zhanglong, and T. Shiliang. A realtime dynamic traffic control system based on wireless sensor network. In *International Conference Workshops on Parallel Processing, 2005. ICPP 2005 Workshops*, pages 258–264. IEEE, 2005.

[24] K.M. Yousef, J.N. Al-Karaki, and Shatnawi A.M. Intelligent traffic light flow control system using wireless sensors networks. *Journal of Information Science and Engineering*, 26:753–768, 2010.

[25] A. A.-K. Jeng, R.-H. Jan, C. Chen, and T.-L. Chang. Adaptive urban traffic signal control system with bus priority. In *Vehicular Technology Conference (VTC Spring), 2013 IEEE 77th*, pages 1–5. IEEE, 2013.

[26] P.S. Chakraborty, P. Nair, P.R. Sinha, and I.K. Behera. Real time optimized traffic management algorithm. *International Journal of Computer Science & Information Technology*, 6:119–135, 2014.

[27] F. Dion and B. Hellinga. A rule-based real-time traffic responsive signal control system with transit priority: application to an isolated intersection. *Transportation Research Part B*, 36:325–343, 2002.

[28] D.L. Borg and K. Scerri. Efficient traffic modelling and dynamic control of an urban region. *Transportation Research Procedia*, 6:224–238, 2015.

[29] X.-H. Yu and W.W. Recker. Stochastic adaptive control model for traffic signal systems. *Transportation Research Part C*, 14:263–282, 2006.

[30] A. Prati, I. Mikić, C. Grana, and M.M. Trivedi. Shadow detection algorithms for traffic flow analysis: a comparative study. *IEEE Intelligent Transportation Systems Proceedings*, pages 340–345, 2001.

[31] C. Singla, A. Kanungo, and A. Sharma. Smart traffic lights switching and traffic density calculation using video processing. *Recent Advances in Engineering and Computational Sciences (RAECS) Proceedings*, pages 340–345, 2014.

[32] K.-H. Chao and P.-Y. Chen. An intelligent traffic flow control system based on radio frequency identification and wireless sensor networks. *International Journal of Distributed Sensor Networks*, 2014:1–10, 2014.

[33] D. Helbing. Derivation of a fundamental diagram for urban traffic flow. *The European Physical Journal B*, 70:229–241, 2009.

[34] J. Lunze and F. Lamnabhi-Lagarrigue. *Handbook of Hybrid Systems Control, Theory, Tools, Applications.* Cambridge University Press, The Edinburgh Building, Cambridge CB2 8RU, UK, 2009.

[35] A.J. van der Schaft and J.M. Schumacher. *An Introduction to Hybrid Dynamical Systems.* Springer London, 236 Gray's inn Road, London WC1X 8HB, UK, 2000.

[36] A.S. Matveev and A.V. Savkin. *Qualitative Theory of Hybrid Dynamical Systems.* Birkhauser Boston, 175 Fifth Avenue, New York, 2000.

[37] V. Feoktistova, A. Matveev, E. Lefeber, and J.E. Rooda. Designs of optimal switching feedback decentralized control policies for fluid queueing networks. *Mathematics of Control, Signals, and Systems*, 24:477–503, 2012.

[38] F. Borrelli, M. Baotić, A. Bemporad, and M. Morari. Dynamic programming for constrained optimal control of discrete-time linear hybrid systems. *Automatica*, 41:1709–1721, 2005.

[39] G. Xie and L. Wang. Stabilization of switched linear systems with time-delay in detection of switching signal. *Journal of Mathematical Analysis and Applications*, 305:277–290, 2005.

[40] J. Selen. Dynamics of an abstract production system. Bachelor's Thesis, Eindhoven University of Technology, The Netherlands, 2010.

[41] Z. Sun. Stabilization and optimization of switched linear systems. *Automatica*, 42:783–788, 2006.

[42] H. Lin and P.J. Antsaklis. Stability and stabilizability of switched linear systems: A survey of recent results. *IEEE Transactions on Automatic Control*, 54:308–322, 2009.

[43] H.K. Khalil. *Nonlinear systems, third edition.* Prentice-Hall, Upper Saddle River, New Jersey, 2000.

[44] R. Goebel, R.G. Sanfelice, and A.R. Teel. *Hybrid Dynamical Systems, Modeling, Stability, and Robustness.* Princeton University Press, 41 William Street, Princeton, New Jersey, 2012.

[45] J.P. Hespanha. Uniform stability of switched linear systems: Extensions of lasalle's invariance principle. *IEEE Transactions on Automatic Control*, 49:470–482, 2004.

[46] R. Goebel, R.G. Sanfelice, and A.R. Teel. Invariance principles for switching systems via hybrid system techniques. *Systems & Control Letters*, 57:980–986, 2008.

[47] R. Goebel, R.G. Sanfelice, and A.R. Teel. Invariance principles for hybrid systems with connections to detectability and asymptotic stability. *IEEE Transactions on Automatic Control*, 52:2282–2297, 2007.

Appendix A. System Requirements

Any intersection or system considered in this thesis is assumed to meet the requirements given in this appendix. Requirements 1 and 2 are physical system requirements, they regard the system parameters and other limits. Requirements 3 to 7, are requirements that are assumed to be met by the fixed time schedule.
In this appendix n equals the minimum number of buffers served in processing of all modes. Let s_p be a set of the total number of buffers that is processed in each processing part of a mode, then $n = \min(s_p)$. For instance, in the three flow example system in A x_1 and x_3 are processed, in B x_1 and x_2 are processed and in C x_2 and x_3 are processed. This gives $s_p = \{2, 2, 2\}$ and $n = 2$. In this example the number of buffers served during processing is equal for all modes. However this is not a necessary condition, in example 's Gravendijkwal, Figure 5.5, $s_p = \{4, 10, 10\}$ and $n = 4$.

The following requirements are assumed to be met for a system:

1. The arrival rates, $\lambda_i \quad \forall i \in N$, are constant and known.
2. The process rates, $\mu_i \quad \forall i \in N$, are constant and known.
3. An optimal fixed time schedule is given that can be associated with the system.*
4. The amber time is modelled as partially red, and partially green time in the fixed time schedule.
5. All items that have arrived during one cycle of the fixed time schedule, are processed in that cycle. That is in optimal behaviour, each buffer is emptied at least once during the cycle of the fixed time schedule.
6. All buffers are served at least once in a cycle of the given fixed time schedule. (Follows indirectly from previously mentioned requirement.)
7. The machine utilization is smaller than one. Thus, if one buffer is served in each mode and all buffers are served once in a cycle,

$$\sum_{i \in N} \frac{\lambda_i}{\mu_i} = \rho_i < 1. \tag{A.1}$$

   Or in general (also applicable when multiple buffers are served simultaneously in a mode), if all buffers are served once in a cycle,

$$\sum_{i \in N} \frac{\lambda_i}{\mu_i} < n. \tag{A.2}$$

* "Can be associated with" refers to the fact that, the actual arrival and process rates of the system equal the arrival and process rates the fixed time schedule design is based on. Furthermore this fixed time schedule provides requirement 5.

Appendix B. Stability Fixed Time Control

In this appendix stability theorems for fixed time control are proven. The first section describes the situation when fixed time control is applied and all buffers are served at maximum rate, a situation comparable to a real applied fixed time schedule. Section B.2 presents a stability theorem in case fixed time control is executed such that at least the minimum slow mode is executed, this proof is important in the derivation of the candidate Lyapunov function of systems.
Some parameters of interest in the proof of the theorems are recapitulated or introduced. The parameter $t_{i,sm}$ represents the duration of the slow mode of buffer i. The minimum duration of slow mode for direction i, is the duration of slow mode for buffer i in optimal periodic behaviour. The symbol $\circledcirc$ refers to the mode subsequent to the start-up mode, where $\circledcirc$ can be setup or processing in a mode, $x_{i,\circledcirc^*}$ refers to the value of x_i at the start of $\circledcirc$ in optimal periodic behaviour. The time spent in the start-up mode, t_{su} meets $0 \leq t_{su} \leq t_{\textcircled{m}}$ if the system starts in setup of mode m and $0 \leq t_{su} \leq t_{\textcircled{m}}$ if the system starts in $x_0 = 0$ and mode $\boldsymbol{m}$.
The total red time of direction i in a cycle is represented by $t_{r,i}$, the total green time of a direction in one cycle is $t_{g,i}$.
With these parameters defined the proof of stable periodic behaviour when the fixed time schedule is executed repeatedly, is presented in the upcoming sections.

B.1 Stability, Serve Buffers at Maximal Rate

Theorem B.1. *Executing the fixed time schedule repeatedly, processing buffer contents at maximum rate if a buffer is served, stabilizes the system in a periodic orbit in a finite amount of time, provided that the system meets al requirements listed in Appendix A. More specifically all buffers i that have a slow mode of duration $t_{i,sm} > 0$ and all buffers for which $x_{i,as} \leq x_{i,\circledcirc^*}$, stabilize in optimal periodic behaviour.*

Proof. Let X_0 be the vector of buffer contents x_i at the time the fixed time control starts, $t = t_0$. After t_{su} time units the start-up mode is finished. When the start up mode is finished the buffer contents equal $X(t_{su})$, with $x_i(t_{su}) = x_{i,0} + \lambda_i t_{su}$ if buffer i is not served and $x_i(t_{su}) = \max(0, x_{i,0} - (\mu_i - \lambda_i)\, t_{su})$ if buffer i is served. From t_{su} on the time spent in each mode equals the time given in the fixed time schedule. Let X_a be the vector of the added buffer content during red time $x_{i,a}$, in one cycle of the fixed time schedule. The number of items arriving at buffer i during red time of a cycle equals, $x_{i,a} = t_{r,i}\lambda_i$. Let X_d be the vector of maximum content of all buffers that can be processed during green time of these buffers, in one cycle. The content of buffer i that can be processed during green time in one cycle equals $x_{i,d} = t_{g,i}\mu_i$. Requirement 5 in Appendix A states that $X_a \leq X_d$.

After the start-up mode is finished there are two options for each buffer content either $x_i(t_{su}) > x_{i,\circledcirc^*}$ or $x_i(t_{su}) \leq x_{i,\circledcirc^*}$. This gives four different cases of system behaviour.

1. If $x_i(t_{su}) \leq x_{i,\circledcirc^*}$, because of $X_a \leq X_d$ at least all items that arrive in the upcoming cycle are processed during that cycle. If $x_i(t_{su}) < x_{i,\circledcirc^*}$ the buffer is emptied earlier than in optimal periodic behaviour. This means the duration of the slow mode, t_{sm}, is increased. Hence $x_i(t_{su} + T) = x_i(t_{su})$, the arrival and process rates are constant so the buffer is in periodic

behaviour after $t_{su} + T$.

2. If $x_i(t_{su}) \leq x_{i,\circledcirc^*}$, because of $X_a \leq X_d$ at least all items that arrive in the upcoming cycle are processed during that cycle. If $x_i(t_{su}) = x_{i,\circledcirc^*}$ the buffer is emptied exactly at the time instant provided by the optimal periodic behaviour. Hence $x_i(t_{su} + T) = x_i(t_{su})$, given the arrival and process rates are constant, the buffer is in periodic behaviour after t_{su}.
3. If $x_i(t_{su}) > x_{i,\circledcirc^*}$ and if $x_{i,a} = x_{i,d}$, exactly all items that arrive in buffer i during the upcoming cycle are processed in that cycle. As $x_{i,a} = x_{i,d}$, the excess buffer content $x_i(t_{su}) - x_{i,\circledcirc^*}$, is never cleared. This means $x_i(t_{su}+T) = x_i(t_{su})$, given the arrival and process rates are constant, buffer i is in periodic behaviour after t_{su}.
4. If $x_i(t_{su}) > x_{i,\circledcirc^*}$ and if $x_{i,a} < x_{i,d}$, the number of items that arrive at buffer i during a cycle are less than the number of items that can be processed in a cycle. The extra content of buffer i decreases each cycle with $\min\left(x_i\left(t_s u + nT\right) - x_{i,\circledcirc^*}, x_{i,d} - x_{i,a}\right)$, in which n is the number of executed cycles. This means that after

$$\left\lceil \frac{x_i(t_{su}) - x_{i,\circledcirc^*}}{x_{i,d} - x_{i,a}} \right\rceil \tag{B.1}$$

cycles, x_i is in optimal periodic behaviour.

In case of option 1, 2 and 4 buffer i stabilizes in optimal periodic behaviour. If option 3 holds, the excess buffer content of buffer i is bounded to $x_i(t_{su}) - x_{i,\circledcirc^*}$. In the latter, the buffer content stabilizes in periodic behaviour but not optimal periodic behaviour.

Let R be the vector of cycles it takes to stabilize each buffer i in periodic behaviour, then

$$\max\left(R\right) * T \tag{B.2}$$

equals the maximum amount of time for which the fixed time schedule needs to be executed before all buffers stabilized in periodic behaviour. Since the buffer contents are finite, the maximum amount of time resulting from B.2 is finite.

Concluding, periodic behaviour is reached for all buffers in a finite amount of time. Furthermore, all buffers i that contain a slow mode of a duration exceeding zero, or buffers i for which $x_i(t_{su}) \leq x_{i,\circledcirc^*}$, stabilize in optimal periodic behaviour. □

B.2 Stability, Serve Buffers with Slow Mode

Theorem B.2. *Executing the fixed time schedule repeatedly, stabilizes the system in periodic behaviour in a finite amount of time. In this case executing the fixed time schedule repeatedly means processing buffer the content of buffer i at maximum rate, except for the minimum duration of slow mode of buffer i then buffer i is processed at rate λ_i.The slow mode is required to be executed, even if the buffer content is unequal to zero. More specifically executing the fixed time schedule as such, stabilizes the system in maximum duration of the start-up mode and one full cycle.*

Proof. Let X_0 be the vector of buffer contents x_i at the time the fixed time control starts, $t = t_0$. After t_{su} time units the start-up mode is finished. When the start up mode is finished the buffer contents equal $X(t_{su})$, with $x_{i,0} = x_i(t_0) + \lambda_i t_{su}$ if buffer i is not served and $x_i(t_{su}) = \max\left(0, x_{i,0} - (\mu_i - \lambda_i)\left(\max\left(0, t_{su} - t_{i,sm}\right)\right)\right)$ if buffer i is served. From t_{su} on the time spent in each mode equals the time given in the fixed time schedule. Let X_a be the vector of the added buffer content during red time $x_{i,a}$, in one cycle of the fixed time schedule. The number of items arriving at buffer i during red time of a cycle equals, $x_{i,a} = t_{r,i}\lambda_i$. Let X_p be the vector of content of all buffers that is processed during green time of these buffers, in one cycle. The content of buffer i that can be processed during green time in one cycle equals $x_{i,p} = \left(t_{g,i} - t_{sm}\right)\mu_i$. By definition, $X_a = X_p$ holds for the buffer content in optimal periodic behaviour.

After the start-up mode is finished there are two options for each buffer content either $x_i(t_{su}) > x_{i,\circledcirc^*}$ or $x_i(t_{su}) \leq x_{i,\circledcirc^*}$. This gives four different cases of system behaviour.

1. If $x_i(t_{su}) \leq x_{i,\circledcirc^*}$, because $X_a = X_p$ all items that arrive in the upcoming cycle, are processed during that cycle. If $x_i(t_{su}) < x_{i,\circledcirc^*}$ the buffer is emptied earlier than in optimal periodic behaviour. This means the duration of the slow mode, t_{sm}, is increased, which is allowed since the minimum slow mode is executed. Thus $x_i(t_{su} + T) = x_i(t_{su})$, the arrival and process rates are constant so the buffer is in periodic behaviour after $t_{su} + T$.
2. If $x_i(t_{su}) \leq x_{i,\circledcirc^*}$, because $X_a = X_p$ all items that arrive in the upcoming cycle, are processed during that cycle. If $x_i(t_{su}) = x_{i,\circledcirc^*}$ the buffer is emptied exactly at the time instant provided by the optimal periodic behaviour. Hence $x_i(t_{su}+T) = x_i(t_{su})$, given the arrival and process rates are constant, the buffer is in periodic behaviour after t_{su}.
3. If $x_i(t_{su}) > x_{i,\circledcirc^*}$, because $X_a = X_p$ all items that arrive in buffer i during the upcoming cycle, are processed in that cycle. As $x_{i,a} = x_{i,p}$, the excess buffer content at time t_{su}, $x_i(t_{su}) - x_{i,\circledcirc^*}$, is never cleared. This means $x_i(t_{su} + T) = x_i(t_{su})$, given the arrival and process rates are constant, buffer i is in periodic behaviour after t_{su}.

In case of option 1, 2 buffer i stabilizes in optimal periodic behaviour after respectively $t_{su} + T$, t_{su} time units. If option 3 holds, the excess buffer content of buffer i is bounded to $x_i(t_{su}) - x_{i,\circledcirc^*}$ and the buffer content is periodic after t_{su} time units. In case option 3 is true, the buffer content stabilizes in periodic behaviour but not optimal periodic behaviour.

Concluding, periodic behaviour is reached for all buffers in a finite amount of time, which is $t_{su}+T$ time units at most. Furthermore, buffers i for which $x_i(t_{su}) \leq x_{i,\circledcirc^*}$, stabilize in optimal periodic behaviour. $\square$

Appendix C. General Lyapunov Function Candidate, $N = \{1, 2\}$

This appendix presents a less intuitive derivation of the candidate Lyapunov function of an intersection of two directions. The candidate Lyapunov function in each mode is determined when the algorithm presented in this chapter is executed. Combining the results for all modes provides the candidate Lyapunov function for an intersection of two directions. The algorithm only provides a correct result if the system meets the requirements presented in Appendix A.
The general optimal fixed time schedule corresponding to an intersection of two directions is shown in Figure C.1.

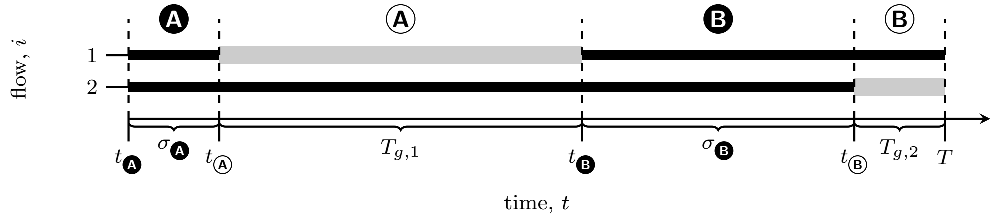

Figure C.1: General fixed time schedule, $N = \{1, 2\}$.

The candidate Lyapunov function value equals the mean extra work in the system in steady-state. This steady-state is obtained by repeatedly executing the fixed time schedule. The definition of repeatedly executing the fixed time schedule is explained in Section 3.1, and briefly recapitalized below.
In any derivation of a candidate Lyapunov function in this thesis the following is meant by repeatedly executing the fixed time schedule. The system starts in a given mode, it can either stay in this start-up mode or switch to the successive mode listed in the fixed time schedule. The time spent in each mode, subsequent to the start-up mode, is defined in the fixed time schedule and does not depend on the buffer contents. Except for the start-up mode, the server is obliged to stay in a mode for the duration of the mode registered in the fixed time schedule.
Furthermore, repeatedly executing the fixed time schedule is performed such that the slow mode is performed for at least the duration of the slow mode listed in optimal periodic behaviour. The duration of the slow mode equals the duration of $x_1 = 0$ in A or $x_2 = 0$ in B in Figure 3.4. The duration of the slow mode is extended in case the minimum duration of the slow mode has elapsed, whilst the time spent in the entire mode is still unequal to the duration of the mode denoted in the fixed time scheduleee. If the system is not in (extended) slow mode, the content of the served buffer x_i, is processed at rate μ_i. Thus the minimum duration of the slow mode equals the value given in the desired periodic behaviour. The maximum duration of the slow mode in $\boldsymbol{m}$ equals the duration of m registered in the fixed time schedule.
The work in the system in optimal periodic behaviour equals (C.1),

$$W(x_1, x_2) = \frac{x_1(t)}{\mu_1} + \frac{x_2(t)}{\mu_2}, \tag{C.1}$$

in which $x_1(t)$ and $x_2(t)$ equal the buffer contents corresponding to the value of x_1 and x_2 in optimal periodic behaviour at time t. The extra work in the system in steady-state is defined as

the difference between the work in the system in steady state and (C.1).
In this appendix mode m refers to the mode the candidate Lyapunov function in processing is derived for, mode n is its subsequent mode registered in the fixed time schedule. Let q refer to the unserved direction of mode $\boldsymbol{m}$ and p refer to the served buffer in mode $\boldsymbol{m}$. The arrival and process rates of type p, q are respectively λ_p, λ_q and μ_p, μ_q. A mode contains a slow mode if (C.2) holds. Section C.1 provides a description of the strategy to determine a candidate Lyapunov function if the mode does not contain a slow mode. Section C.2 describes the algorithm to find a candidate Lyapunov function if the mode contains a slow mode.
Note that all buffer contents are larger than or equal to zero by definition.

$$\frac{(T - T_{g,p})\,\lambda_p}{\mu_p - \lambda_p} < T_{g,p}. \tag{C.2}$$

C.1 Mode $\boldsymbol{m}$ without a Slow Mode

The buffer contents in optimal periodic behaviour are listed in Table C.1. These values are determined with the fixed time schedule in Figure C.1 and the system parameters. The buffer content at the start in each part of a mode in optimal periodic behaviour, are fundamental in the derivation of a candidate Lyapunov function. The subscript of the time t presents the start of which mode is considered and the star refers to the fact that it is the content in optimal periodic behaviour. This subscript is used as superscript for the buffer contents if it concerns the specific buffer contents listed in C.1. The buffer contents in Table C.1 are general, computing the value provides mode and system specific optimal buffer contents.

Table C.1: Buffer contents in optimal periodic behaviour.

$\boldsymbol{t}$	$(\boldsymbol{x_p},\, \boldsymbol{x_q})$
$t_{\textcircled{\textbf{m}}*}$	$\left(\left(T - T_{g,p} - \sigma_{\textcircled{\textbf{m}}}\right)\lambda_p, 0\right)$
$t_{\textcircled{m}*}$	$\left((T - T_{g,p})\,\lambda_p, \sigma_{\textcircled{\textbf{m}}}\lambda_q\right)$
$t_{\textcircled{\textbf{n}}*}$	$\left(0, \left(T_{g,p} + \sigma_{\textcircled{\textbf{m}}}\right)\lambda_q\right)$

Mode $\boldsymbol{m}$ does not contain a slow mode, which means that the domain is split in four parts. The different domain parts and corresponding boundaries are listed in (C.3).

$$\begin{array}{lll} \mathscr{D}^{\mathrm{I}} & \text{for} & x_p \geq x_p^{\textcircled{m}*}, \quad x_q \geq x_q^{\textcircled{m}*}, \\ \mathscr{D}^{\mathrm{II}} & \text{for} & x_p \geq x_p^{\textcircled{m}*}, \quad x_q \leq x_q^{\textcircled{m}*}, \\ \mathscr{D}^{\mathrm{III}} & \text{for} & x_p \leq x_p^{\textcircled{m}*}, \quad x_q^{\textcircled{\textbf{n}}*} - \frac{x_q^{\textcircled{\textbf{n}}*} - x_q^{\textcircled{m}*}}{x_p^{\textcircled{m}*}} x_p \leq x_q, \\ \mathscr{D}^{\mathrm{VI}} & \text{for} & x_p \leq x_p^{\textcircled{m}*}, \quad x_2 \leq x_q^{\textcircled{\textbf{n}}*} - \frac{x_q^{\textcircled{\textbf{n}}*} - x_q^{\textcircled{m}*}}{x_p^{\textcircled{m}*}} x_p. \end{array} \tag{C.3}$$

The candidate Lyapunov function is derived per part of the domain. In $\mathscr{D}^{\mathrm{I}}$ the mean extra work in the system is minimized if the server processes in the start-up mode for a time equal to the duration of $\textcircled{m}$ in the fixed time schedule. The candidate Lyapunov function in the system is (C.4).
The same duration of the start-up mode as used in $\mathscr{D}^{\mathrm{I}}$, minimizes the mean extra work in the system in $\mathscr{D}^{\mathrm{II}}$. In this case the unserved buffer shows optimal periodic behaviour in steady-state, the candidate Lyapunov function becomes (C.5).

$$\mathscr{D}^{\mathrm{I}}: \quad V(\textcircled{m}, x_p, x_q) = \frac{x_p - x_p^{\textcircled{m}*}}{\mu_p} + \frac{x_q - x_q^{\textcircled{m}*}}{\mu_q} \quad \text{for } x_p \geq x_p^{\textcircled{m}*}, \quad x_q \geq x_q^{\textcircled{m}*}. \tag{C.4}$$

$$\mathscr{D}^{\mathrm{II}}: \quad V(\textcircled{m}, x_p, x_q) = \frac{x_p - x_p^{\textcircled{m}*}}{\mu_p} \quad \text{for } x_p \geq x_p^{\textcircled{m}*}, \quad x_q \leq x_q^{\textcircled{m}*}. \tag{C.5}$$

The next part of the domain is $\mathscr{D}^{\mathrm{III}}$, in this part the value of x_q exceeds the value of x_q determined by the correlation between x_p and x_q. This correlation is the boundary between $\mathscr{D}^{\mathrm{III}}$ and $\mathscr{D}^{\mathrm{VI}}$. If $x_q^{\textcircled{\textbf{n}}*} - \frac{x_q^{\textcircled{\textbf{n}}*} - x_q^{\textcircled{m}*}}{x_p^{\textcircled{m}*}} x_p \leq x_q$ the mean extra work in the system is minimized when the time spent in the start-up mode equals the time to empty x_p, when empty the system should switch. This results in optimal periodic behaviour for x_p, the extra work is determined by the distance between x_q and $x_q^{\textcircled{\textbf{n}}*} - \frac{x_q^{\textcircled{\textbf{n}}*} - x_q^{\textcircled{m}*}}{x_p^{\textcircled{m}*}} x_p$. Which yields the candidate Lyapunov function in $\mathscr{D}^{\mathrm{III}}$, (C.6).

The final part of the domain, $\mathscr{D}^{\mathrm{VI}}$, is the part where $x_p \leq x_p^{\textcircled{m}*}$ and $x_2 \leq x_q^{\textcircled{\textbf{n}}*} - \frac{x_q^{\textcircled{\textbf{n}}*} - x_q^{\textcircled{m}*}}{x_p^{\textcircled{m}*}} x_p$. In this case the time spent in the start-up mode can be set such that the steady-state equals the optimal periodic behaviour, with corresponding extra work in the system of zero. Hence the candidate Lyapunov function in $\mathscr{D}^{\mathrm{VI}}$ equals (C.7).

$$\mathscr{D}^{\mathrm{III}}: \quad V(\textcircled{m}, x_p, x_q) = \frac{x_q - x_q^{\textcircled{\textbf{n}}*} + \frac{x_q^{\textcircled{\textbf{n}}*} - x_q^{\textcircled{m}*}}{x_p^{\textcircled{m}*}} x_p}{\mu_q} \quad \text{for } \ x_p \leq x_p^{\textcircled{m}*}, \quad x_q^{\textcircled{\textbf{n}}*} - \frac{x_q^{\textcircled{\textbf{n}}*} - x_q^{\textcircled{m}*}}{x_p^{\textcircled{m}*}} x_p \leq x_q. \tag{C.6}$$

$$\mathscr{D}^{\mathrm{VI}}: \quad V(\textcircled{m}, x_p, x_q) = 0 \quad \text{for } x_p \leq x_p^{\textcircled{m}*}, \quad x_2 \leq x_q^{\textcircled{\textbf{n}}*} - \frac{x_q^{\textcircled{\textbf{n}}*} - x_q^{\textcircled{m}*}}{x_p^{\textcircled{m}*}} x_p. \tag{C.7}$$

Setup is the final part of mode $\boldsymbol{m}$ to determine a candidate Lyapunov function in. The definition of the candidate Lyapunov function in setup is that it is equal to the candidate Lyapunov function in processing after setup is finished. In an intersection of two directions no flows are served during setup. Which yields the candidate Lyapunov function in $\textcircled{\textbf{m}}$,

$$V(\textcircled{\textbf{m}}, x_p, x_q) = V(\textcircled{m}, x_p + x_0\lambda_p, x_q + x_0\lambda_q). \tag{C.8}$$

C.2 Mode $\boldsymbol{m}$ with Slow Mode

If the work in the system decreases during slow mode, the extra domain parts $\mathscr{D}^{\mathrm{IV}}$, $\mathscr{D}^{\mathrm{V}}$ do not exist. The extra work in the system is minimized if the duration in the start up mode equals the registered in the fixed time schedule. Which means the strategy discussed in the previous section can simply be used, as the results of both strategies are equal.

The strategy discussed in this section becomes of interest when the work in the system increases during slow mode, but the mode contains a slow mode in optimal periodic behaviour. If the value of $x_p \leq x_p^{\textcircled{m}*}$ the candidate Lyapunov function could be minimized by immediately switching to setup of the subsequent mode. The buffer contents in optimal periodic behaviour are listed in Table C.2. The subscript of the time t presents the start of which mode is considered and the star refers to the fact that it is the content in optimal periodic behaviour. The extra addition to the subscript 0 when the time refers to the time in optimal periodic behaviour the slow mode starts, so the first occurrence of $x_p = 0$. The subscripts are used as superscript for the buffer contents if it concerns the specific buffer contents listed in Table C.2. The

Table C.2: Buffer contents in optimal periodic behaviour.

$\boldsymbol{t}$	$(\boldsymbol{x_p},\, \boldsymbol{x_q})$
$t_{\textcircled{\textbf{m}}^*}$	$\left(\left(T - T_{g,p} - \sigma_{\textcircled{\textbf{m}}}\right)\lambda_p, 0\right)$
$t_{\textcircled{m}^*}$	$\left(\left(T - T_{g,p}\right)\lambda_p, \sigma_{\textcircled{\textbf{m}}}\lambda_q\right)$
$t_{\textcircled{n}_0{}^*}$	$\left(0, \left(\sigma_{\textcircled{\textbf{m}}} + \frac{(T-T_{g,p})\lambda_p}{\mu_p-\lambda_p}\right)\lambda_q\right)$
$t_{\textcircled{\textbf{n}}^*}$	$\left(0, \left(T_{g,p} + \sigma_{\textcircled{\textbf{m}}}\right)\lambda_q\right)$

values are general, computing the value provides mode and system specific optimal buffer contents.

The derivation of the candidate Lyapunov function is started by presenting an overview of the resulting domain parts. The domain parts and their boundaries are explicitly denoted in (C.9).

$$
\begin{array}{lll}
\mathscr{D}^{\mathrm{I}} & \text{for} \quad x_p \geq x_p^{\textcircled{m}*}, & x_q \geq x_q^{\textcircled{m}*}, \\
\mathscr{D}^{\mathrm{II}} & \text{for} \quad x_p \geq x_p^{\textcircled{m}*}, & x_q \leq x_q^{\textcircled{m}*}, \\
\mathscr{D}^{\mathrm{III_a}} & \text{for} \quad \frac{40}{37} \leq x_p \leq x_p^{\textcircled{m}*}, & x_q \geq x_q^{\textcircled{\textbf{n}}*}, \\
\mathscr{D}^{\mathrm{III_b}} & \text{for} \quad x_p \leq x_p^{\textcircled{m}*}, & x_q^{\textcircled{n}_0*} - \frac{x_q^{\textcircled{n}_0*} - x_q^{\textcircled{m}*}}{x_p^{\textcircled{m}*}} x_p \leq x_2 \leq 4 + \frac{37}{40}x_1 \leq x_q^{\textcircled{\textbf{n}}*}, \\
\mathscr{D}^{\mathrm{IV}} & \text{for} \quad x_p \leq \frac{40}{37} \leq x_p^{\textcircled{m}*}, & x_2 \geq x_q^{\textcircled{\textbf{n}}*}, \\
\mathscr{D}^{\mathrm{V}} & \text{for} \quad x_p \leq \frac{40}{37} \leq x_p^{\textcircled{m}*}, & x_q^{\textcircled{n}_0*} - \frac{x_q^{\textcircled{n}_0*} - x_q^{\textcircled{m}*}}{x_p^{\textcircled{m}*}} x_p \leq 4 + \frac{37}{40}x_1 \leq x_2 \leq x_q^{\textcircled{\textbf{n}}*}, \\
\mathscr{D}^{\mathrm{VI}} & \text{for} \quad x_p \leq x_p^{\textcircled{m}*}, & x_2 \leq x_q^{\textcircled{n}_0*} - \frac{x_q^{\textcircled{n}_0*} - x_q^{\textcircled{m}*}}{x_p^{\textcircled{m}*}} x_p.
\end{array}
\tag{C.9}
$$

Next the candidate Lyapunov function is derived in each part of the domain listed in (C.9), starting in $\mathscr{D}^{\mathrm{I}}$. In $\mathscr{D}^{\mathrm{I}}$ the mean extra work in the system is minimized if the server processes in the start-up mode for a time equal to the duration of $\textcircled{m}$ in the fixed time schedule. The candidate Lyapunov function in the system is (C.10).
The same duration of the start-up mode as used in $\mathscr{D}^{\mathrm{I}}$, minimizes the mean extra work in the system in $\mathscr{D}^{\mathrm{II}}$. In this case the unserved buffer shows optimal periodic behaviour in steady-state, the candidate Lyapunov function becomes (C.11).

$$
\mathscr{D}^{\mathrm{I}}: \quad V(\textcircled{m}, x_p, x_q) = \frac{x_p - x_p^{\textcircled{m}*}}{\mu_p} + \frac{x_q - x_q^{\textcircled{m}*}}{\mu_q} \quad \text{for } x_p \geq x_p^{\textcircled{m}*}, \quad x_q \geq x_q^{\textcircled{m}*}. \tag{C.10}
$$

$$
\mathscr{D}^{\mathrm{II}}: \quad V(\textcircled{m}, x_p, x_q) = \frac{x_p - x_p^{\textcircled{m}*}}{\mu_p} \quad \text{for } x_p \geq x_p^{\textcircled{m}*}, \quad x_q \leq x_q^{\textcircled{m}*}. \tag{C.11}
$$

In $\mathscr{D}^{\mathrm{III}}$ the derivation becomes more complicated, yielding a split of $\mathscr{D}^{\mathrm{III}}$ in multiple parts. If the value of x_p becomes small enough, immediately progressing to $\textcircled{\textbf{n}}$ could result in minimum mean extra work in the stabilized system. By means of clarifying the origin of the domain split introduced before, first the domains $\mathscr{D}^{\mathrm{III_a}}$, $\mathscr{D}^{\mathrm{III_b}}$, $\mathscr{D}^{\mathrm{IV}}$ and $\mathscr{D}^{\mathrm{V}}$ are studied merged as $\mathscr{D}^{\mathrm{III}}$.
The part of $\mathscr{D}^{\mathrm{III}}$ where $x_q \geq x_q^{\textcircled{m}*}$ is examined. The optimal duration of the start-up mode equals the time it takes for x_p to become zero, plus the minimum duration of the slow mode. This results in the optimal periodic value of x_p, but buffer q preserves the extra amount of work it contained when the server started. The candidate Lyapunov function if the server remains in $\textcircled{m}$ instead of immediately switching to $\textcircled{\textbf{n}}$ in the start-up mode, is given in (C.12).

$$\mathscr{D}^{\mathrm{III_a}}: \quad V(m, x_p, x_q) = \frac{x_q - x_q^{n_0*} + \frac{x_q^{n_0*} - x_q^{m*}}{x_p^{m*}} x_p}{\mu_q} \quad \text{for} \;\; x_p \leq x_p^{m*}, \quad x_q \geq x_q^{m*}. \tag{C.12}$$

Additionally, the candidate Lyapunov function is derived for cases the the server immediately switches to n. If the server immediately starts setup, instead of continuing processing in m, the duration of the start-up mode equals zero. Because $x_q \geq x_q^{m*}$ and the time spent in the start-up mode equals zero, the system starts in its steady-state. The start value of x_p equals the extra content in buffer p. If $x_q = x_q^{m*}$ the content of buffer q does not contribute to the mean extra work in the system, whereas in all other cases x_q does contribute to the value of work in periodic behaviour. This results in the candidate Lyapunov function when the system starts in m and immediately switches to n, (C.13).

$$V(m, x_p, x_q) = \frac{x_p}{\mu_p} + \frac{x_q - x_q^{m*}}{\mu_q} \quad \text{for} \;\; x_p \leq x_p^{m*}, \quad x_q \geq x_q^{m*}. \tag{C.13}$$

$$x_p \leq \quad \text{for} \;\; x_p \leq x_p^{m*}, \quad x_q \geq x_q^{m*}. \tag{C.14}$$

Setup is the final part of mode $\boldsymbol{m}$ to determine a candidate Lyapunov function in. The definition of the candidate Lyapunov function in setup is that it is equal to the candidate Lyapunov function in processing after setup is finished. In an intersection of two directions no flows are served during setup. Which yields the candidate Lyapunov function in m,

$$V(m, x_p, x_q) = V(m, x_p + x_0\lambda_p, x_q + x_0\lambda_q). \tag{C.15}$$

Appendix D. Parametrization

The correlation between all buffer contents in optimal periodic behaviour can be determined based on the evolution of the buffer content in optimal periodic behaviour for all directions.
The evolution of the buffer content of all directions i that are served during mode m can be described by:

$$x_i(t) = (\lambda_i - \mu_i)\, t + x_i^{\textcircled{m}*}. \tag{D.1}$$

The evolution of the buffer content of all directions j that are not served during mode m can be described by:

$$x_j(t) = \lambda_j t + x_j^{\textcircled{m}*}. \tag{D.2}$$

The relation between all buffer contents can now be found by removing the time dependency of the evolution by using the time dependency of another buffer evolution. This gives that the correlation between buffers can be found using:

$$\lambda_j x_i = (\mu_i - \lambda_i)\, x_j = \lambda_j x_i^{\textcircled{m}*} + (\mu_i - \lambda_i)\, x_j^{\textcircled{m}*}. \tag{D.3}$$

The equation above describes the correlation between the buffer contents in mode m in optimal periodic behaviour and is indeed based on the evolution of the buffer contents in the periodic behaviour.

Appendix E. Intersection of Three Directions

This appendix is partially a recapitulation of Chapter 3 and Chapter 4. The recap is given to make this appendix self-contained. Section E.1, presents a complete explanation of candidate Lyapunov function for the three flow example system. In Section E.2 the base for the control action design is discussed, this base is the candidate Lyapunov function derivative.

E.1 Derivation of a Candidate Lyapunov Function

An intersection of three directions consists of three flows. Two directions are served simultaneously in all modes, except when the server is performing setup, then only one direction is served. The design of the fixed time schedule is such that all items arrived in one cycle, are cleared in that cycle. The fixed time schedule of this example system, Figure E.1, shows that all flows are conflict free. Therefore it would be possible to serve all directions simultaneously. Because of that the fixed time schedule in Figure E.1, will not occur in an actual traffic setting. For research objectives however, such a system is of interest. The results of the derivation of this example system are used to examine the changes in deriving a candidate Lyapunov function. It demonstrates the effect on the derivation of a candidate Lyapunov function when a flow and a mode are added to the system of two directions.

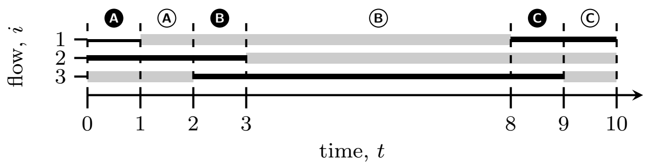

Figure E.1: Fixed time schedule, $N = \{1, 2, 3\}$.

In this section two new definitions are used, the primarily and secondary served buffers. The primarily served buffer of a mode, is the buffer that was already served in the preceding mode, and needs to be cleared during this mode. The secondary served buffer refers to the buffer that is served for the first time in the current mode, of which the service is continued in the subsequent mode. As can be concluded from Figure E.1, one buffer per mode is not defined yet. The buffer that is not served in a mode, is referred to as the unserved buffer of that mode. The items in the primarily and secondary served buffers, are processed during Ⓐ, Ⓑ, Ⓒ. In setup of a mode, $u_0 \in \{$🅐, 🅑, 🅒$\}$, the system is in setup to process respectively x_1, x_2, x_3. Processing the content of the primarily served buffer of a mode is continued in setup of the subsequent mode.
An extension is needed in the definition of slow mode, used in the intersection of two directions, to make the definition slow mode unambiguous in the intersection of three directions. In the two flow example system, slow mode was defined as the served buffer of the mode m being served at its arrival rate. In this example system multiple buffers are served during a mode. Thus, either the primarily served, the secondary served, or both buffers can be in slow mode. Therefore, in the three direction system, slow mode is specified as a slow mode of buffer i in mode $\boldsymbol{m}$. Herein i equals the flow served at its arrival rate, and $\boldsymbol{m}$ represents the mode the system is in.
To shorten the expressions of the candidate Lyapunov function, the equations omit

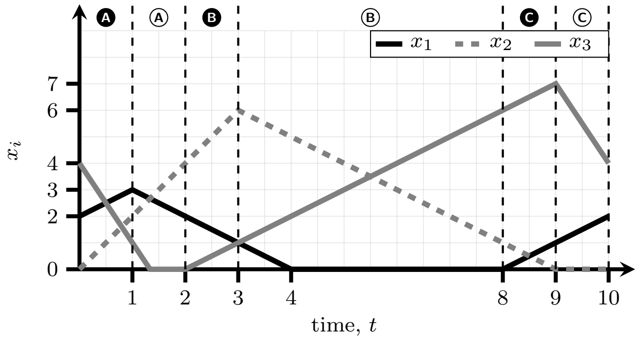

Figure E.2: Buffer contents as function of time.

Table E.1: Buffer content in optimal periodic behaviour.

	t	$(x_1\ x_2\ x_3)$
🅐*	0	(2,0,4)
Ⓐ*	1	(3,2,1)
Ⓐ	$\frac{4}{3}$	$(\frac{8}{3},\frac{8}{3},0)$
🅑*	2	(2,4,0)
Ⓑ*	3	(1,6,1)
Ⓑ	4	(0,5,2)
🅒*	8	(0,1,6)
Ⓒ*	9	(1,0,7)
Ⓒ	10	(2,0,4)

$V(s, x_1, x_2, x_3) =$, with $s \in \{$🅐, Ⓐ, 🅑, Ⓑ, C, Ⓒ$\}$. However, the symbols corresponding to the parts of the domain are listed prior to the equation of the candidate Lyapunov function in each part of the domain. This emphasizes the similarities within modes, and the resemblance between the two and three direction system. The candidate Lyapunov function described in this section is derived according the method explained in detail for the intersection of two directions, discussed in Section 3.1.

The system parameters and the fixed time schedule are designed such that each mode contains a slow mode of the primarily served buffer. Although it is not a necessary condition it is possible that the secondary served buffer is cleared during a mode.

The arrival rates are assumed to be constant and known, $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 1$, as are the process rates, $\mu_1 = 2$, $\mu_2 = 3$ and $\mu_3 = 4$. Figure E.2 shows the buffer contents as function of time when the system is in the desired periodic behaviour, the figure illustrates that all buffers are cleared once during a cycle. In Table E.1, the buffer contents are listed at each time instant of the fixed time schedule the service rate values change. These values are important in the candidate Lyapunov function derivation, the stars indicate that these values are the optimal buffer content at the start of setup, or start of processing in a mode.

Figure 3.16 is a graphical representation of the work in the system in optimal periodic behaviour as function of time. Although it is not the case in this example system, it is a possibility that the work in the system decreases during setup since the server continues processing one of the buffers in setup.

E.1.1 Mode *A*, $N = \{1, 2, 3\}$

Similar to the example of an intersection with two directions, the start values of the buffer contents can be in the complete domain. Based on the results of the previously discussed example it can be concluded that depending on the values of x_1, x_2 and x_3 the candidate Lyapunov function differs,

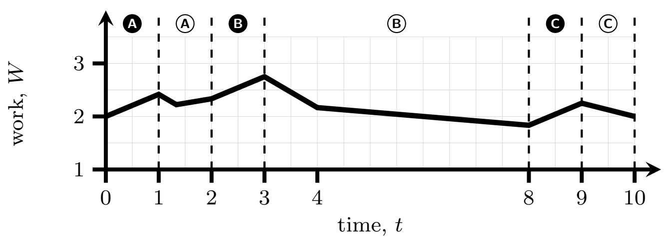

Figure E.3: Work in the system, $N = \{1, 2, 3\}$, during the optimal periodic cycle.

which results in subdivisions inside the complete domain.
In $\mathscr{D}^{\mathrm{I}}$, x_1, x_2 and x_3 all exceed their respective optimal values at $t = 1$ listed in Table E.1, $x_1 \geq 3$, $x_2 \geq 2$ and $x_3 \geq 1$. Studying Figure 3.16, the optimal time spent in the start-up mode is the maximum duration of Ⓐ. During periodic behaviour, achieved by executing the fixed time schedule repeatedly, yields candidate Lyapunov function in $\mathscr{D}^{\mathrm{I}}$, (E.1).
In the three flow system equivalent of $\mathscr{D}^{\mathrm{II}}$, $x_1 \geq 3$ and $x_3 \geq 1$, for the unserved buffer $x_2 \leq 2$ holds. This means the content of buffer 2 shows the desired periodic behaviour, the extra work in the system during periodic behaviour depends on the excess content of buffer 1 and buffer 3. The candidate Lyapunov function in $\mathscr{D}^{\mathrm{II}}$ equals (E.2).

$$\mathscr{D}^{\mathrm{I}}: \quad \tfrac{x_1-3}{2} + \tfrac{x_2-2}{3} + \tfrac{x_3-1}{4} \quad \text{for Ⓐ}, \quad x_1 \geq 3, \quad x_2 \geq 2, \quad x_3 \geq 1. \tag{E.1}$$

$$\mathscr{D}^{\mathrm{II}}: \quad \tfrac{x_1-3}{2} + \tfrac{x_3-1}{4} \quad \text{for Ⓐ}, \quad x_1 \geq 3, \quad x_2 \leq 2, \quad x_3 \geq 1. \tag{E.2}$$

The first part of the domain in which the analogy with the two direction system requires additional explanation is $\mathscr{D}^{\mathrm{III}}$. In the two flow example system the candidate Lyapunov function in $\mathscr{D}^{\mathrm{III}}$, $\mathscr{D}^{\mathrm{IV}}$, $\mathscr{D}^{\mathrm{V}}$, depended on the optimal time to spent in the start-up mode. Only one buffer was served in each mode in the two flow example system, so the optimal time spent in the start-up mode was the time needed to clear the served buffer. In the intersection of three direction, during each mode two buffers are served instead of one, this results in another subdivision of domain parts. For instance during Ⓐ, it is not only possible for x_1 to be less then the optimal value but $x_3 \leq 1$ is a possibility as well. The duration of the start-up mode depends on $x_1 - 1$ or x_3, based on which buffer content becomes its respective optimal value in the least amount of time. The buffer content that the time spent in the start-up mode is based on, is denoted in the subscript of the part of the domain.
In the intersection of two directions the extra content of the unserved buffer in $\mathscr{D}^{\mathrm{III}}$, $\mathscr{D}^{\mathrm{IV}}$ and $\mathscr{D}^{\mathrm{V}}$, was a function of the distance between the content in the unserved buffer and the boundary line that described the correlation between the served and unserved buffer. The general approach to derive a correlation between the buffer contents during Ⓜ, is found in Appendix D. Implementing Appendix D in Ⓐ results in the three flow equivalent of the boundary between the unserved and served buffers.
As in the two direction system, the correlation between the unserved and served buffers determine the boundary between created the boundary between $\mathscr{D}^{\mathrm{III}}$ and $\mathscr{D}^{\mathrm{VI}}$. In the latter part of the domain the candidate Lyapunov function value equals zero, however the domain boundaries depend on which buffer content the time spent in the start-up mode is based. If x_1 defines the duration of the start-up mode, Appendix D yields $3x_1 \geq x_3 + 8$, $2x_1 + x_2 \leq 8$ with $x_1 \leq 3$ and $x_3 \leq 1$. Hence, the candidate Lyapunov function in $\mathscr{D}_1^{\mathrm{VI}}$, (E.3). If x_3 determines the remaining time in the start-up mode, Appendix D gives $3x_1 \leq x_3 + 8$, $3x_2 + 2x_3 \leq 8$ should hold, with $x_1 \leq 3$, $x_3 \leq 1$, the candidate Lyapunov function in $\mathscr{D}_3^{\mathrm{VI}}$ equals (E.4).

$$\mathscr{D}_1^{\mathrm{VI}}: \quad 0 \quad \text{for Ⓐ}, \quad \tfrac{8}{3} \leq x_1 \leq 3, \quad x_2 \leq 8 - 2x_1, \quad x_3 \leq 3x_1 - 8 \leq 1. \tag{E.3}$$

$$\mathscr{D}_3^{\mathrm{VI}}: \quad 0 \quad \text{for Ⓐ}, \quad \tfrac{8}{3} \leq x_1 \leq 3, \quad x_2 \leq \tfrac{8}{3} - \tfrac{2}{3}x_3, \quad 3x_1 - 8 \leq x_3 \leq 1. \tag{E.4}$$

When $x_1 \leq 3$ and $x_3 \geq 1$ in the three flow equivalent of $\mathscr{D}^{\mathrm{III}}$ in the two direction system, the value of x_2 should exceed the boundary created by the correlation between x_1 and x_2, $2x_1 + x_2 \geq 8$. Since $x_1 \leq 3$, buffer 1 stabilizes in the desired optimal periodic behaviour, x_2 and x_3 contribute to the extra work in the system. The time spent in the start-up mode is based on the value of x_1, yielding the candidate Lyapunov function in $\mathscr{D}_1^{\mathrm{III}}$, (E.5).
When $x_1 \geq 3$ and $x_3 \leq 1$, the duration of the start-up mode is defined by the content of buffer 3. The boundary that determines the amount of extra work due to the excess content of buffer 2, becomes $3x_2 + 2x_3 \geq 8$. This part of the domain is referred to as $\mathscr{D}_3^{\mathrm{III}}$, in this part the candidate Lyapunov function equals (E.6).

It is a possibility that both x_1 and x_3 are less than their respective optimal values. The duration of the start up mode is determined by either $x_1 - \frac{8}{3}$ or x_3. When $x_1 - \frac{8}{3}$ defines the duration of the start-up mode, both $2x_1 + x_2 \geq 8$ and $3x_1 \leq 8 + x_3$ should hold. This gives the candidate Lyapunov function in $\mathscr{D}^{\mathrm{III}}_{1,3}$, (E.7). The subscript for this domain part refers to the fact that buffer 1 determines the duration of the start up mode and both buffer 1 and buffer 3 are less than their respective optimal values.
If not $x_1 - \frac{8}{3}$ but x_3 determines the duration of the start up mode, then $x_3 \leq 3x_1 - 8 \leq 1$ and $\frac{8}{3} + \frac{1}{3}x_3 \leq x_1 \leq 3$. The excess content of x_2 is defined by the correlation $3x_2 + 2x_3 \geq 8$, which gives the candidate Lyapunov function in $\mathscr{D}^{\mathrm{III}}_{3,1}$, (E.8).

$$\mathscr{D}^{\mathrm{III}}_{1}: \quad \tfrac{x_2-8+2x_1}{3} + \tfrac{x_3-1}{4} \quad \text{for (A)}, \quad \tfrac{8}{3} \leq x_1 \leq 3, \quad x_2 \geq 8-2x_1, \quad x_3 \geq 1. \tag{E.5}$$

$$\mathscr{D}^{\mathrm{III}}_{3}: \quad \tfrac{x_1-3}{2} + \tfrac{x_2-\frac{8}{3}+\frac{2}{3}x_3}{3} \quad \text{for (A)}, \quad x_1 \geq 3, \quad x_2 \geq \tfrac{8}{3}-\tfrac{2}{3}x_3, \quad x_3 \leq 1. \tag{E.6}$$

$$\mathscr{D}^{\mathrm{III}}_{1,3}: \quad \tfrac{x_2-8+2x_1}{3} + \tfrac{x_3+8-3x_1}{4} \quad \text{for (A)}, \quad \tfrac{8}{3} \leq x_1 \leq 3, \quad x_2 \geq 8-2x_1, \quad 3x_1-8 \leq x_3 \leq 1. \tag{E.7}$$

$$\mathscr{D}^{\mathrm{III}}_{3,1}: \quad \tfrac{x_1-\frac{8}{3}-\frac{1}{3}x_3}{2} + \tfrac{x_2-\frac{8}{3}+\frac{2}{3}x_3}{3} \quad \text{for (A)}, \quad \tfrac{8}{3}+\tfrac{1}{3}x_3 \leq x_1 \leq 3, \quad x_2 \geq \tfrac{8}{3}-\tfrac{2}{3}x_3, \quad x_3 \leq 1. \tag{E.8}$$

The split presented above is based on which buffer content defines the remaining time in the start-up mode, it structures the method to derive a candidate Lyapunov function but a review of the results is required. If $x_1 = 3$ and $x_3 > 0$, the work in the system decreases when the server continues in the start-up mode. The server switches if the time spent in the mode equals the maximum duration of (A). This gives a correction for the candidate Lyapunov function in $\mathscr{D}^{\mathrm{III}}_{1}$ and $\mathscr{D}^{\mathrm{III}}_{1,3}$. Because x_1 does not terminate the start-up mode, the candidate Lyapunov function for $\mathscr{D}^{\mathrm{III}}_{1}$ depends on the maximum duration of the fixed time schedule, which results in (E.9).
A similar motivation holds for the part of the domain where both x_1 and x_3 are less than their respective optimal values but the unserved buffer contains more than its optimal value. Again x_1 does not define the end of (A), the correct candidate Lyapunov function in $\mathscr{D}^{\mathrm{III}}_{1,3}$ is (E.10).

$$\mathscr{D}^{\mathrm{III}}_{1}: \quad \tfrac{x_2-2}{3} + \tfrac{x_3-1}{4} \quad \text{for (A)}, \quad \tfrac{8}{3} \leq x_1 \leq 3, \quad x_2 \geq 8-2x_1, \quad x_3 \geq 1. \tag{E.9}$$

$$\mathscr{D}^{\mathrm{III}}_{1,3}: \quad \tfrac{x_2-\frac{8}{3}+\frac{2}{3}x_3}{3} \quad \text{for (A)}, \quad \tfrac{8}{3} \leq x_1 \leq 3, \quad x_2 \geq \tfrac{8}{3}-\tfrac{2}{3}x_3, \quad 3x_1-8 \leq x_3 \leq 1. \tag{E.10}$$

The work in the system increases during the slow mode of buffer 3 in (A), see Figure 3.16. If the value of x_1 becomes small enough, immediate switching to (B) instead of starting in (A) could minimize the mean extra work in the system during periodic behaviour, as explained in the derivation of the candidate Lyapunov function of the intersection of two directions. Therefore an additional review of the results of the candidate Lyapunov function in $\mathscr{D}^{\mathrm{III}}$ is required, a similar review to the one performed in the derivation of the candidate Lyapunov function of the intersection of two directions that lead to the existence of $\mathscr{D}^{\mathrm{IV}}$.
In the domain parts $\mathscr{D}^{\mathrm{III}}_{1}$, $\mathscr{D}^{\mathrm{III}}_{3}$, $\mathscr{D}^{\mathrm{III}}_{1,3}$ and $\mathscr{D}^{\mathrm{III}}_{1,3}$ the candidate Lyapunov function value is determined if the server immediately switches to setup. In $\mathscr{D}^{\mathrm{III}}_{1}$, switching to setup of the subsequent mode, the content of x_3 contributes to the mean extra work of buffer 3. If $x_1 \leq 2$ and $x_2 \leq 4$, the buffer contents of buffer 1 and buffer 2 do not contribute to the mean extra work in the system during periodic behaviour. The candidate Lyapunov function in case the system immediately switches to (B) in $\mathscr{D}^{\mathrm{III}}_{1}$ is (E.11). To determine if immediately switching to setup minimizes the mean extra work in the system, (3.26) is compared to (3.32). This further subdivides the domain, the candidate Lyapunov function in $\mathscr{D}^{\mathrm{III}}_{1}$ is described by (E.12) to (E.16).

$$\mathscr{D}^{\mathrm{III}}_{1,\mathrm{setup}}: \quad \max\left(0, \tfrac{x_1-2}{2}\right) + \max\left(0, \tfrac{x_2-4}{3}\right) + \tfrac{x_3}{4} \quad \text{for Ⓐ}, \quad x_1 \leq 3, \quad x_2 \geq 8-2x_1, \quad 3x_1-8 \leq x_3 \geq 1. \tag{E.11}$$

$$\mathscr{D}^{\mathrm{III_a}}_{1}: \quad \tfrac{x_2-2}{3} + \tfrac{x_3-1}{4} \quad \text{for Ⓐ}, \quad \tfrac{17}{6} \leq x_1 \leq 3, \quad x_2 \geq 8-2x_1 \geq 4, \quad x_3 \geq 1. \tag{E.12}$$

$$\mathscr{D}^{\mathrm{III_b}}_{1}: \quad \tfrac{x_2-2}{3} + \tfrac{x_3-1}{4} \quad \text{for Ⓐ}, \quad 2 \leq x_1 \leq 3, \quad x_2 \leq \tfrac{3}{2}x_1 + \tfrac{1}{4}, \quad x_3 \geq 1. \tag{E.13}$$

$$\mathscr{D}^{\mathrm{IV_a}}_{1}: \quad \tfrac{x_1-2}{2} + \tfrac{x_2-4}{3} + \tfrac{x_3}{4} \quad \text{for Ⓐ}, \quad 2 \leq x_1 \leq \tfrac{17}{6}, \quad x_2 \geq 8-2x_1 \geq 4, \quad x_3 \geq 1. \tag{E.14}$$

$$\mathscr{D}^{\mathrm{IV_b}}_{1}: \quad \tfrac{x_2-4}{3} + \tfrac{x_3}{4} \quad \text{for Ⓐ}, \quad x_1 \leq 2, \quad x_2 \geq 8-2x_1 \geq 4, \quad x_3 \geq 1. \tag{E.15}$$

$$\mathscr{D}^{\mathrm{V}}_{1}: \quad \tfrac{x_1-2}{2} + \tfrac{x_3}{4} \quad \text{for Ⓐ}, \quad 2 \leq x_1 \leq 3, \quad \tfrac{3}{2}x_1 + \tfrac{1}{4} \leq x_2 \leq 4, \quad x_3 \geq 1. \tag{E.16}$$

In $\mathscr{D}^{\mathrm{III}}_{3}$ immediately switching to Ⓑ, results in mean extra work in the system in steady-state equal to (E.17). Comparing (3.27) and (E.17) concludes that immediately switching to setup of mode $\boldsymbol{B}$ never results in less mean extra work in the system in periodic behaviour. Thus the candidate Lyapunov function in (3.27) is correct.
In $\mathscr{D}^{\mathrm{III}}_{1,3}$ switching to mode $\boldsymbol{B}$ instead of processing in the start-up mode results in the candidate Lyapunov function given in (E.18). The comparison of (3.31) and (E.18), results in multiple parts of $\mathscr{D}^{\mathrm{III}}_{1,3}$, of which the candidate Lyapunov function is listed in (E.19) to (E.22).

$$\mathscr{D}^{\mathrm{III}}_{3,\mathrm{setup}}: \quad \tfrac{x_1-2}{2} + \max\left(0, \tfrac{x_2-4}{3}\right) + \tfrac{x_3}{4} \quad \text{for Ⓐ}, \quad x_1 \geq 3, \quad x_2 \geq \tfrac{8}{3} - \tfrac{2}{3}x_3, \quad x_3 \leq 1. \tag{E.17}$$

$$\mathscr{D}^{\mathrm{III}}_{1,3,\mathrm{setup}}: \quad \tfrac{x_1-2}{2} + \max\left(0, \tfrac{x_2-4}{3}\right) + \tfrac{x_3}{4} \quad \text{for Ⓐ}, \quad x_1 \leq 3, \quad x_2 \geq 8-2x_1, \quad 3x_1-8 \leq x_3 \leq 1. \tag{E.18}$$

$$\mathscr{D}^{\mathrm{III_a}}_{1,3}: \quad \tfrac{x_2-\frac{8}{3}+\frac{2}{3}x_3}{3} \quad \text{for Ⓐ}, \quad \tfrac{26}{9} - \tfrac{1}{18}x_3 \leq x_1 \leq 3, \quad x_2 \geq 4, \quad 3x_1-8 \leq x_3 \leq 1. \tag{E.19}$$

$$\mathscr{D}^{\mathrm{III_b}}_{1,3}: \quad \tfrac{x_2-\frac{8}{3}+\frac{2}{3}x_3}{3} \quad \text{for Ⓐ}, \quad \tfrac{2}{3}x_2 - \tfrac{1}{18}x_3 + \tfrac{2}{9} \leq x_1 \leq 3, \quad 8-2x_1 \leq x_2 \leq 4, \quad 3x_1-8 \leq x_3 \leq 1. \tag{E.20}$$

$$\mathscr{D}^{\mathrm{IV}}_{1,3}: \quad \tfrac{x_1-2}{2} + \tfrac{x_2-4}{3} + \tfrac{x_3}{4} \quad \text{for Ⓐ}, \quad x_1 \leq \tfrac{26}{9} - \tfrac{1}{18}x_3, \quad 8-2x_1 \leq x_2 \leq 4, \quad x_2 \geq 4, 3x_1-8 \leq x_3 \leq 1. \tag{E.21}$$

$$\mathscr{D}^{\mathrm{V}}_{1,3}: \quad \tfrac{x_1-2}{2} + \tfrac{x_3}{4} \quad \text{for Ⓐ}, \quad x_1 \leq \tfrac{2}{3}x_2 - \tfrac{1}{18}x_3 + \tfrac{2}{9}, \quad 8-2x_1 \leq x_2 \leq 4, \quad 3x_1-8 \leq x_3 \leq 1. \tag{E.22}$$

The final part of the domain in mode $\boldsymbol{A}$ to examine is $\mathscr{D}^{\mathrm{III}}_{3,1}$, in this part immediately switching to Ⓑ yields (E.23). Comparing (E.23) with (3.29), divides $\mathscr{D}^{\mathrm{III}}_{3,1}$ in several parts with corresponding candidate Lyapunov functions, (E.24) to (E.27).

$$\mathscr{D}^{\mathrm{III}}_{3,1,\mathrm{setup}}: \quad \tfrac{x_1-2}{2} + \max\left(0, \tfrac{x_2-4}{3}\right) + \tfrac{x_3}{4} \quad \text{for Ⓐ}, \quad \tfrac{8}{3} + \tfrac{1}{3}x_3 \leq x_1 \leq 3, \quad x_2 \geq \tfrac{8}{3} - \tfrac{2}{3}x_3, \quad x_3 \leq 1. \tag{E.23}$$

$$\mathscr{D}_{3,1}^{\mathrm{III_a}}: \quad \frac{x_1-\frac{8}{3}-\frac{1}{3}x_3}{2}+\frac{x_2-\frac{8}{3}+\frac{2}{3}x_3}{3} \quad \text{for Ⓐ}, \quad \tfrac{8}{3}+\tfrac{1}{3}x_3 \le x_1 \le 3, \quad x_2 \ge \tfrac{8}{3}-\tfrac{2}{3}x_3 \ge 4, \quad \tfrac{4}{7} \le x_3 \le 1. \tag{E.24}$$

$$\mathscr{D}_{3,1}^{\mathrm{III_b}}: \quad \frac{x_1-\frac{8}{3}-\frac{1}{3}x_3}{2}+\frac{x_2-\frac{8}{3}+\frac{2}{3}x_3}{3} \quad \text{for Ⓐ}, \quad \tfrac{8}{3}+\tfrac{1}{3}x_3 \le x_1 \le 3, \quad x_2 \le \tfrac{7}{12}x_3+\tfrac{11}{3} \le 4, \quad x_3 \le 1. \tag{E.25}$$

$$\mathscr{D}_{3,1}^{\mathrm{IV}}: \quad \frac{x_1-2}{2}+\frac{x_2-4}{3}+\frac{x_3}{4} \quad \text{for Ⓐ}, \quad \tfrac{8}{3}+\tfrac{1}{3}x_3 \le x_1 \le 3, \quad x_2 \ge 4, \quad x_3 \le \tfrac{4}{7}. \tag{E.26}$$

$$\mathscr{D}_{3,1}^{\mathrm{V}}: \quad \frac{x_1-2}{2}+\frac{x_3}{4} \quad \text{for Ⓐ}, \quad \tfrac{8}{3}+\tfrac{1}{3}x_3 \le x_1 \le 3, \quad \tfrac{7}{12}x_3+\tfrac{11}{3} \le x_2 \le 4, \quad x_3 \le 1. \tag{E.27}$$

E.1.2 Mode *B*, $N = \{1, 2, 3\}$

Although Ⓑ contains a slow mode, the work in the system decreases when buffer 1 is in slow mode, see Figure 3.16. When the server starts in mode ***B***, immediately switching in the start-up mode to mode ***C***, does not decrease the mean extra work in the system in periodic behaviour. Thus in mode ***B*** the three flow equivalent of $\mathscr{D}^{\mathrm{III}}$, is not divided in $\mathscr{D}^{\mathrm{III_a}}$, $\mathscr{D}^{\mathrm{III_b}}$, $\mathscr{D}^{\mathrm{IV}}$ and $\mathscr{D}^{\mathrm{V}}$.
In $\mathscr{D}^{\mathrm{I}}$, where $x_1 \ge 1$, $x_2 \ge 6$ and $x_3 \ge 1$, the candidate Lyapunov function is a function of all buffer contents, (E.28). Figure 3.16 shows that the work is continuously decreasing whilst the system is in Ⓑ, evidently the minimum mean extra work in the system is reached if the time spent in the start-up mode equals the maximum duration of Ⓑ.
$\mathscr{D}^{\mathrm{II}}$ gives $x_1 \ge 1$, $x_2 \ge 6$, whereas for the unserved buffer, $x_3 \le 1$ holds. The optimal time spent in the start-up mode is again the maximum duration of Ⓑ, as explained in more detail in the two flow example system. Since $x_3 \le 1$, buffer 3 does not contribute to the mean extra work in the system, so the candidate Lyapunov function in $\mathscr{D}^{\mathrm{II}}$ equals (E.29).

$$\mathscr{D}^{\mathrm{I}}: \quad \frac{x_1-1}{2}+\frac{x_2-6}{3}+\frac{x_3-1}{4} \quad \text{for Ⓑ}, \quad x_1 \ge 1, \quad x_2 \ge 6, \quad x_3 \ge 1. \tag{E.28}$$

$$\mathscr{D}^{\mathrm{II}}: \quad \frac{x_1-1}{2}+\frac{x_2-6}{3} \quad \text{for Ⓑ}, \quad x_1 \ge 1, \quad x_2 \ge 6, \quad x_3 \le 1. \tag{E.29}$$

In $\mathscr{D}^{\mathrm{VI}}$ both served buffers are less than the optimal value at the start of Ⓑ, the value of x_3 does not exceed the correlation that determines the boundary between $\mathscr{D}^{\mathrm{VI}}$ and $\mathscr{D}^{\mathrm{III}}$. In the three flow system two buffer contents can define the end of the start-up mode, either the remaining time in the start up mode depends on x_1, or it depends on x_2. The value of x_1 defines the time spent in the start-up mode if buffer 1 is cleared before $x_2 = 1$. With Appendix D it can be concluded that $x_1 \ge x_2 - 5$ should hold, otherwise x_1 would not determine the duration of the start-up mode. The correlation that determines the boundary equals $x_3 \le 2 - x_1$, if $x_1 \le 1$ and $x_2 \le 6$ the candidate Lyapunov function in $\mathscr{D}_1^{\mathrm{VI}}$ is (E.30).
If the content of buffer 2 defines the time spent in the start-up mode, by Appendix D it is concluded that $x_1 + 5 \le x_2 \le 6$ and $x_3 \le 7 - x_2$, if $x_1 \le 1$ and the candidate Lyapunov function equals (E.31). This completes the derivation of the candidate Lyapunov function in $\mathscr{D}^{VI}$ in Ⓑ, of the three direction example system.

$$\mathscr{D}_1^{\mathrm{VI}}: \quad 0 \quad \text{for Ⓑ}, \quad x_1 \le 1, \quad x_2 \le x_1 + 5 \le 6, \quad x_3 \le 2 - x_1. \tag{E.30}$$

$$\mathscr{D}_2^{\mathrm{VI}}: \quad 0 \quad \text{for Ⓑ}, \quad x_1 \le 1, \quad x_1 + 5 \le x_2 \le 6, \quad x_3 \le 7 - x_2. \tag{E.31}$$

In $\mathscr{D}^{\mathrm{III}}$ with $x_1 \le 1$ and $x_2 \ge 6$, x_3 needs to exceed the boundary created by the correlation between buffer 1 and buffer 3, particularly $x_3 \ge 2 - x_1$. Buffer 1 shows the desired behaviour when in steady-state since $x_1 \le 1$. The value of the excess content of x_2 and x_3, depends on the

time spent in the start-up mode. Thus the candidate Lyapunov function in $\mathscr{D}_1^{\mathrm{III}}$ equals (E.32).
If the value of x_2 defines the duration of the start-up mode, $x_1 \geq 1$ and $x_2 \leq 6$, the boundary that determines the extra content regarding the optimal periodic behaviour in buffer 3 becomes $x_3 \geq 7 - x_2$. This results in mean extra work in the system during periodic behaviour in $\mathscr{D}_2^{\mathrm{III}}$ equal to (E.33).
When buffer 1 and buffer 2 contain less than their respective optimal values and x_1 determines the duration of the start-up mode, both $x_3 \geq 2 - x_1$ and $x_1 \leq x_2 - 5$ hold. The extra content of buffer 2 and buffer 3 depend on the duration of the start-up mode and candidate Lyapunov function equals (E.34). However when the content of buffer 2 determines the duration of the start-up mode, with Appendix D it is concluded that $x_2 \leq x_1 + 5 \leq 6$ and $x_3 \geq 7 - x_2$. The candidate Lyapunov function in $\mathscr{D}_{2,1}^{\mathrm{III}}$ then equals (E.35).

$$\mathscr{D}_1^{\mathrm{III}}: \quad \tfrac{x_2-6}{3} + \tfrac{x_3+x_1-2}{4} \quad \text{for (B)} \quad x_1 \leq 1, \quad x_2 \geq 6, \quad x_3 \geq 2 - x_1. \tag{E.32}$$

$$\mathscr{D}_2^{\mathrm{III}}: \quad \tfrac{x_1-1}{2} + \tfrac{x_3+x_2-7}{4} \quad \text{for (B)} \; x_1 \geq 1, \quad x_2 \leq 6, \quad x_3 \geq 7 - x_2. \tag{E.33}$$

$$\mathscr{D}_{1,2}^{\mathrm{III}}: \quad \tfrac{x_2-x_1-5}{3} + \tfrac{x_3+x_1-2}{4} \quad \text{for (B)} \quad x_1 \leq 1, \quad x_1 + 5 \leq x_2 \leq 6, \quad x_3 \geq 2 - x_1. \tag{E.34}$$

$$\mathscr{D}_{2,1}^{\mathrm{III}}: \quad \tfrac{x_1-x_2+5}{2} + \tfrac{x_3+x_2-7}{4} \quad \text{for (B)} \quad x_1 \leq 1, \quad x_2 \leq x_1 + 5 \leq 6, \quad x_3 \geq 7 - x_2. \tag{E.35}$$

Starting the derivation of the candidate Lyapunov function in $\mathscr{D}^{\mathrm{III}}$ with subdividing the domain based on which buffer content defines the duration of the start-up mode, structures the derivation. Structuring is useful when the system size is extended. However the results need to be reviewed to determine if the derived function is complete and correct. From Figure 3.16 it can be concluded that if $x_1 = 0$ and $x_2 > 0$, the work is continuously decreasing, so the system should continue processing if the time spent in the start-up mode is less than the maximum duration of (B) stated in the fixed time schedule. If $x_2 = 6$ and $x_1 > 0$ and the system continues in (B), the derivative of work in the system equals (E.36) which proofs the work in the system is decreasing. The mean extra work in the system in periodic behaviour is minimized when the server continues in (B) as long as possible and the value of x_2 never defines the end of mode $\boldsymbol{B}$. A correction is needed in $\mathscr{D}_2^{\mathrm{III}}$ and $\mathscr{D}_{2,1}^{\mathrm{III}}$, the candidate Lyapunov function respectively becomes (E.37) and (E.38).

$$\dot{W} = \frac{\lambda_1 - \mu_1}{\mu_1} + \frac{\lambda_2 - \mu_2}{\mu_2} + \frac{\lambda_3}{\mu_3} = \frac{-1}{2} + \frac{-1}{3} + \frac{1}{4} < 0, \tag{E.36}$$

$$\mathscr{D}_{1,2}^{\mathrm{III}}: \quad \tfrac{x_1-1}{2} + \tfrac{x_3-1}{4} \quad \text{for (B)} \quad x_1 \geq 1, \quad x_2 \leq 6, \quad x_3 \geq 7 - x_2. \tag{E.37}$$

$$\mathscr{D}_{2,1}^{\mathrm{III}}: \quad \tfrac{x_3+x_1-2}{4} \quad \text{for (B)} \quad x_1 \leq 1, \quad x_2 \leq 6, \quad x_3 \geq 2 - x_1. \tag{E.38}$$

E.1.3 Mode $\boldsymbol{C}$, $N = \{1, 2, 3\}$

At $t = 9$ in the fixed time schedule the slow mode of buffer 2 in mode $\boldsymbol{C}$ starts. Figure 3.16 illustrates that the work in the system decreases when the system is in slow mode of buffer 2. Furthermore the properties of mode $\boldsymbol{C}$ are slightly different than the properties of mode $\boldsymbol{A}, \boldsymbol{B}$. In mode $\boldsymbol{C}$ the primarily served buffer is cleared during (C) instead of during (C).
Because the buffer contents are positive by definition, the boundaries of different domain parts are different compared to the boundaries in mode $\boldsymbol{A}$ and mode $\boldsymbol{B}$, $x_2 \leq 0$ becomes $x_2 = 0$. Even though mode $\boldsymbol{C}$ has different properties, the derivation of a candidate Lyapunov function is performed in a similar structure as used for mode $\boldsymbol{A}$ and $\boldsymbol{B}$. The successful derivation of the candidate Lyapunov function of mode $\boldsymbol{C}$, implies that the definition of mean extra work in the

system to derive a candidate Lyapunov function discussed in this chapter is applicable to systems with mode properties equivalent to the properties of mode $\boldsymbol{C}$. To emphasize the structure to derive a candidate Lyapunov function, the same methodology is used in mode $\boldsymbol{C}$ as for mode $\boldsymbol{A}$ and mode $\boldsymbol{B}$, even if this produces the same result multiple times.
In $\mathscr{D}^{\mathrm{I}}$ the candidate Lyapunov function is distinct, the values of the buffer contents equal $x_1 \geq 1$, $x_2 \geq 0$ and $x_3 \geq 7$. None of the buffer contents achieve the desired periodic behaviour, so the candidate Lyapunov function depends on all buffer contents, (E.39).
The next part of the domain is where the primarily and secondary served buffers content exceed their optimal value, $x_2 \geq 0$ and $x_3 \geq 7$, but $x_1 \leq 1$. The maximum duration of the start-up mode, minimizes the mean extra work in the system during periodic behaviour. Because $x_1 \leq 1$, x_1 does not contribute to the extra work, the candidate Lyapunov function in $\mathscr{D}^{\mathrm{II}}$ equals (E.40).

$$\mathscr{D}^{\mathrm{I}}: \quad \tfrac{x_1-1}{2} + \tfrac{x_2}{3} + \tfrac{x_3-7}{4} \quad \text{for } ©, \quad x_1 \geq 1, \quad x_2 \geq 0, \quad x_3 \geq 7. \tag{E.39}$$

$$\mathscr{D}^{\mathrm{II}}: \quad \tfrac{x_2}{3} + \tfrac{x_3-7}{4} \quad \text{for } ©, \quad x_1 \leq 1, \quad x_2 \geq 0, \quad x_3 \geq 7. \tag{E.40}$$

In $\mathscr{D}^{\mathrm{VI}}$ the content of both served buffers are $x_2 = 0$ and $x_3 \leq 7$, the value of x_1 is less than the correlation that determines the boundary between $\mathscr{D}^{\mathrm{VI}}$ and $\mathscr{D}^{\mathrm{III}}$. Mode $\boldsymbol{A}$ and mode $\boldsymbol{B}$ showed two options in this part, in mode $\boldsymbol{C}$ the same strategy is used. When x_2 defines the remaining time in ©, Appendix D gives $3x_2 \leq x_3 - 7$, otherwise the value of x_3 would determine the duration of the mode. If furthermore $x_1 \leq 1 - x_2$ and $x_2 = 0$, the candidate Lyapunov function of $\mathscr{D}_2^{\mathrm{VI}}$ can be written as (E.41). If the value of x_3 determines the duration of the start-up mode, then $x_3 \leq 3x_2 + 7 \leq 7$, $x_2 = 0$ and $3x_1 \leq 10 - x_3$. The first two statements yield $x_3 = 7$, which leaves to conclude that the candidate Lyapunov function of $\mathscr{D}_3^{\mathrm{VI}}$ equals (E.42). This function exactly equals (E.41), since $x_2 = 0$ is the only possible value.
There is no correction needed in this part of the domain, when $x_2 = 0$ the work is still decreasing, as mentioned at the start of this subsection.

$$\mathscr{D}_2^{\mathrm{VI}}: \quad 0 \quad \text{for } ©, \quad x_1 \leq 1, \quad x_2 = 0, \quad x_3 = 7. \tag{E.41}$$

$$\mathscr{D}_3^{\mathrm{VI}}: \quad 0 \quad \text{for } ©, \quad x_1 \leq 1, \quad x_2 = 0, \quad x_3 = 7. \tag{E.42}$$

Starting the derivation of $\mathscr{D}^{\mathrm{III}}$ for values of x_2 and x_3 that fulfil $x_2 \leq 0$ and $x_3 \geq 7$. In case $x_1 \geq 1 - x_2 \Rightarrow x_1 \geq 1$ the content of buffer 2 is optimal. The mean extra work in the system during periodic behaviour depends on x_1 and x_3, the candidate Lyapunov function in $\mathscr{D}_2^{\mathrm{III}}$ equals (E.43). When the content of buffer 3 determines the time spent in the start-up mode, $x_2 \geq 0$, $x_3 \leq 7$ and $3x_1 \geq 10 - x_3$, the candidate Lyapunov function equals E.44.
If both x_2 and x_3 are less than their respective optimal values, $x_2 \leq 0$ and $x_3 \leq 7$, the duration of the start-up mode can be defined by either x_2 or x_3. When the value of x_2 defines the duration of the start-up mode, both $3x_2 \leq x_3 - 7$ and $x_1 \geq 1$ should hold. Because $x_2 = 0$ the value of x_3 should fulfil $7 \leq x_3 \leq 7$ and becomes $x_3 = 7$ which yields the the candidate Lyapunov function in $\mathscr{D}_{2,3}^{\mathrm{III}}$, (E.45). If the content of buffer 3 determines the time spent in the start-up mode, $x_3 \leq 3x_2+7$ and $x_1 \geq \frac{10}{3} - \frac{1}{3}x_3$ should hold, the candidate Lyapunov function in $\mathscr{D}_{3,2}^{\mathrm{III}}$ equals (E.46).

$$\mathscr{D}_2^{\mathrm{III}}: \quad \tfrac{x_1-1}{2} + \tfrac{x_3-7}{4} \quad \text{for } ©, \quad x_1 \geq 1, \quad x_2 = 0, \quad x_3 \geq 7. \tag{E.43}$$

$$\mathscr{D}_3^{\mathrm{III}}: \quad \tfrac{x_1+\frac{1}{3}x_3-\frac{10}{3}}{2} + \tfrac{x_2}{4} \quad \text{for } ©, \quad x_1 \geq \tfrac{10}{3} - \tfrac{1}{3}x_3, \quad x_2 \geq 0, \quad x_3 \leq 7. \tag{E.44}$$

$$\mathscr{D}_{2,3}^{\mathrm{III}}: \quad \tfrac{x_1-1}{2} + \tfrac{x_3-7}{4} \quad \text{for } ©, \quad x_1 \geq 1, \quad x_2 = 0, \quad x_3 = 7. \tag{E.45}$$

$$\mathscr{D}^{\mathrm{III}}_{3,2}: \quad \frac{x_1+\frac{1}{3}x_3-\frac{10}{3}}{2} \quad \text{for Ⓒ}, \quad x_1 \geq 1, \quad x_2 = 0, \quad x_3 \leq 7. \tag{E.46}$$

The previously discussed subdivision in parts of $\mathscr{D}^{\mathrm{III}}$ is based on the buffer content that defines the time spent in the start-up mode. A correction might be necessary if the work in the system is increasing when either of the served buffers reaches its optimal value. Figure 3.16 illustrates that the work in the system is decreasing when $x_2 = 0$ and $x_3 > 0$. In (E.47) the derivative of the work in the system is presented for $x_2 > 0$ and $x_3 = 7$. This shows the work in the system is decreasing when $x_2 > 0$ and $x_3 = 7$, so the content of buffer 3 never defines the end of the start-up mode. The corrected candidate Lyapunov function in $\mathscr{D}^{\mathrm{III}}_3$ and $\mathscr{D}^{\mathrm{III}}_{3,2}$ is respectively given in (E.48) and (E.49).
Directly switching to setup never minimizes the mean extra work in the system, as the work in the system is decreasing in slow mode of buffer 2 and in slow mode of buffer 3. Hence there are the domain parts $\mathscr{D}^{\mathrm{IV}}$ and $\mathscr{D}^{\mathrm{V}}$ do not exist in mode $\boldsymbol{C}$.

$$\dot{W} = \frac{\lambda_1}{\mu_1} + \frac{-(\mu_2-\lambda_2)}{\mu_2} + \frac{-(\mu_3-\lambda_3)}{\mu_3} = \frac{1}{2} + \frac{-1}{3} + \frac{-3}{4} < 0. \tag{E.47}$$

$$\mathscr{D}^{\mathrm{III}}_{3}: \quad \frac{x_1-1}{2} + \frac{x_2}{3} \quad \text{for Ⓒ}, \quad x_1 \geq 1, \quad x_2 \geq 0, \quad x_3 \leq 7. \tag{E.48}$$

$$\mathscr{D}^{\mathrm{III}}_{3,2}: \quad \frac{x_1-1}{2} \quad \text{for Ⓒ}, \quad x_1 \geq 1, \quad x_2 = 0, \quad x_3 \leq 7. \tag{E.49}$$

E.1.4 Candidate Lyapunov function example system, $N = \{1, 2, 3\}$

The setup of all modes is the final part of the complete domain of the three flow example system. In case the server starts in setup, the definition equals the definition given in the example of an intersection of two directions. The value of the mean extra work in the system during periodic behaviour when the server starts in ❹, ❺, ❻, equals the value of $V(s, x_1, x_2, x_3)$ in respectively Ⓐ, Ⓑ, Ⓒ x_0 time units later. The functions of x_1, x_2, x_3 at x_0 time units later, are listed in Table E.2. The functions are more complicated than the functions of x_1, x_2 in the two flow system, because in the three direction system one buffer is served during setup and the buffer contents are by definition non-negative.

Table E.2: Buffer contents after setup.

	❹	❺	❻
x_1	$x_1 + \lambda_1 x_0$	$\max(0, x_1 - (\mu_1 - \lambda_1)x_0)$	$x_1 + \lambda_1 x_0$
x_2	$x_2 + \lambda_2 x_0$	$x_2 + \lambda_2 x_0$	$\max(0, x_2 - (\mu_2 - \lambda_2)x_0)$
x_3	$\max(0, x_3 - (\mu_3 - \lambda_3)x_0)$	$x_3 + \lambda_3 x_0$	$x_3 + \lambda_3 x_0$

The derivation of the candidate Lyapunov function is finished by determining the candidate Lyapunov function during ❹, ❺, ❻, respectively (E.50), (E.51) and (E.52).

$$V(\text{❹}, x_1, x_2, x_3) = V(\text{Ⓐ}, x_1 + x_0, x_2 + 2x_0, x_3 - 3x_0). \tag{E.50}$$

$$V(\text{❺}, x_1, x_2, x_3) = V(\text{Ⓑ}, \max(0, x_1 - x_0), x_2 + 2x_0, x_3 + x_0). \tag{E.51}$$

$$V(\text{❻}, x_1, x_2, x_3) = V(\text{Ⓒ}, x_1 + x_0, \max(0, x_2 - x_0), x_3 + x_0). \tag{E.52}$$

$$
\begin{cases}
\frac{x_1-3}{2} + \max\left(0, \frac{x_2-2}{3}\right) + \frac{x_3-1}{4} & \text{for Ⓐ}, x_1 \geq 3, x_3 \geq 1, \\
\min\left(\frac{x_2-2}{3} + \frac{x_3-1}{4}, \max\left(0, \frac{x_1-2}{2}\right) + \frac{x_2-4}{3} + \frac{x_3}{4}\right) & \text{for Ⓐ}, x_1 \leq 3, x_2 \geq 8-2x_1 \geq 4, x_3 \geq 1, \\
\min\left(\frac{x_2-2}{3} + \frac{x_3-1}{4}, \frac{x_1-2}{2} + \frac{x_3}{4}\right) & \text{for Ⓐ}, 2 \leq x_1 \leq 3, x_2 \leq 4, x_3 \geq 1, \\
\frac{x_1-3}{2} + \frac{x_2-\frac{8}{3}+\frac{2}{3}x_3}{3} & \text{for Ⓐ}, x_1 \geq 3, x_2 \geq \frac{8}{3} - \frac{2}{3}x_3, x_3 \leq 1, \\
\min\left(\frac{x_2-\frac{8}{3}+\frac{2}{3}x_3}{3}, \frac{x_1-2}{2} + \frac{x_2-4}{3} + \frac{x_3}{4}\right) & \text{for Ⓐ}, x_1 \leq 3, x_2 \geq 4, 3x_1-8 \leq x_3 \leq 1, \\
\min\left(\frac{x_2-\frac{8}{3}+\frac{2}{3}x_3}{3}, \frac{x_1-2}{2} + \frac{x_3}{4}\right) & \text{for Ⓐ}, x_1 \leq 3, 8-2x_1 \leq x_2 \leq 4, 3x_1-8 \leq x_3 \leq 1, \\
\min\left(\frac{x_1-\frac{8}{3}-\frac{1}{3}x_3}{2} + \frac{x_2-\frac{8}{3}+\frac{2}{3}x_3}{3}, \frac{x_1-2}{2} + \frac{x_2-4}{3} + \frac{x_3}{4}\right) & \text{for Ⓐ}, \frac{8}{3} + \frac{1}{3}x_3 \leq x_1 \leq 3, x_2 \geq 4, x_3 \leq 1, \\
\min\left(\frac{x_1-\frac{8}{3}-\frac{1}{3}x_3}{2} + \frac{x_2-\frac{8}{3}+\frac{2}{3}x_3}{3}, \frac{x_1-2}{2} + \frac{x_3}{4}\right) & \text{for Ⓐ}, \frac{8}{3} + \frac{1}{3}x_3 \leq x_1 \leq 3, x_2 \leq 4, x_3 \leq 1, \\
0 & \text{for Ⓐ}, x_1 \leq 3, x_2 \leq 8-2x_1, x_3 \leq 3x_1-8 \leq 1, \\
0 & \text{for Ⓐ}, x_1 \leq 3, x_2 \leq \frac{8}{3} - \frac{2}{3}x_3, 3x_1-8 \leq x_3 \leq 1 \\
V\left(\text{Ⓐ}, x_1+x_0, x_2+2x_0, \max\left(0, x_3-3x_0\right)\right) & \text{for 🅐}, \\
\frac{x_1-1}{2} + \frac{x_2-6}{3} + \max\left(0, \frac{x_3-1}{4}\right) & \text{for Ⓑ}, x_1 \geq 1, x_2 \geq 6, \\
\frac{x_2-6}{3} + \frac{x_3+x_1-2}{4} & \text{for Ⓑ}, x_1 \leq 1, x_2 \geq 6, x_3 \geq 2-x_1, \\
\frac{x_1-1}{2} + \frac{x_3-1}{4} & \text{for Ⓑ}, x_1 \geq 1, x_2 \leq 6, x_3 \geq 7-x_2, \\
\max\left(0, \frac{x_2-x_1-5}{3}\right) + \frac{x_3+x_1-2}{4} & \text{for Ⓑ}, x_1 \leq 1, x_2 \leq 6, x_3 \geq 2-x_1, \\
0 & \text{for Ⓑ}, x_1 \leq 1, x_2 \leq x_1+5 \leq 6, x_3 \leq 2-x_1, \\
0 & \text{for Ⓑ}, x_1 \leq 1, x_1+5 \leq x_2 \leq 6, x_3 \leq 7-x_2, \\
V\left(\text{Ⓑ}, \max\left(0, x_1-x_0\right), x_2+2x_0, x_3+x_0\right) & \text{for 🅑}, \\
\max\left(0, \frac{x_1-1}{2}\right) + \frac{x_2}{3} + \max\left(0, \frac{x_3-7}{4}\right) & \text{for Ⓒ}, \\
0 & \text{for Ⓒ}, x_1 \leq 1, x_2 = 0, x_3 = 7, \\
V\left(\text{Ⓒ}, x_1+x_0, \max\left(0, x_2-x_0\right), x_3+x_0\right) & \text{for 🅒}.
\end{cases}
\tag{E.53}
$$

This completes the derivation of the candidate Lyapunov function in the complete domain of the three flow example system. Combining the equations for different parts of the domain if possible yields a simplified expression of the candidate Lyapunov function $V(s, x_1, x_2, x_3)$, (E.53). The candidate Lyapunov function in setup of all modes is not denoted explicitly, to shorten the expression.

The number of equations and the complexity of the split in domains used to derive (E.53), shows the effect of adding one flow and one mode to the intersection of two directions. Comparing the candidate Lyapunov function derivation discussed in this section, to the example presented in the previous section, leaves to concludes that adding an extra mode to the system does not have an significant effect. Besides making the derivation of a candidate Lyapunov function more exhaustive because one more mode needs to be examined. On the contrary, changing the composition of the mode, by for instance adding an extra direction that is served in a mode, makes the derivation more cumbersome. Furthermore it changes the amount of subdivisions in the domain significantly, which results in a more complex candidate Lyapunov function. This observation stresses the convenience of establishing a general policy of which stability is implied without the need of explicitly deriving a candidate Lyapunov function of the system.

Based on the results presented in this chapter, it is expected that a candidate Lyapunov function can be found for every system of which an optimal fixed time schedule is known and modes can be defined. Although it is an educatedly guess that an explicit candidate Lyapunov function can be derived for a system of any size, actually performing the derivation is undesired.

E.2 Design of Control Actions, $N = \{1,2,3\}$

In Section 4.1 control actions are derived based on the candidate Lyapunov function derivative. Determining control actions in the three flow system is equivalent, but the number of expressions that combined define the candidate Lyapunov function derivative increases significantly. The candidate Lyapunov function derivative is derived explicitly, based on (4.1).
This section presents all candidate Lyapunov function derivative equations, which combined result in E.92. The order of derivation is equal to the order the candidate Lyapunov function expressions are presented in the previous section. Due to the definition of the candidate Lyapunov function the function value is constant during setup, this means the candidate Lyapunov function derivative equals zero in setup.

E.2.1 Derivative if Server in Processing

The candidate Lyapunov function value depends on the evolution of the buffer contents, when a buffer is emptied the buffer content is processed at the arrival rate which means the evolution of the buffer content changes. The buffer content at time instant $t+\varepsilon$ for all buffers in all modes is listed in Table E.3. It should be noted that the expressions given in this table are for the specific example system, as is the candidate Lyapunov function derivative.

Table E.3: Buffer contents at time instant $t+\varepsilon$, three flow intersection.

	In A	**In B**	**In C**
$x_1(t+\varepsilon)$	If $x_1 > 0$, $x_1(t) - (\mu_1 - \lambda_1)\varepsilon$. If $x_1 = 0$, $x_1(t)$	If $x_1 > 0$, $x_1(t) - (\mu_1 - \lambda_1)\varepsilon$. If $x_1 = 0$, $x_1(t)$	$x_1(t) + \lambda_1\varepsilon$.
$x_2(t+\varepsilon)$	$x_2(t) + \lambda_2\varepsilon$.	If $x_2 > 0$, $x_2(t) - (\mu_2 - \lambda_2)\varepsilon$. If $x_2 = 0$, $x_2(t)$	If $x_2 > 0$, $x_2(t) - (\mu_2 - \lambda_2)\varepsilon$. If $x_2 = 0$, $x_2(t)$
$x_3(t+\varepsilon)$	If $x_3 > 0$, $x_3(t) - (\mu_3 - \lambda_3)\varepsilon$. If $x_3 = 0$, $x_3(t)$.	$x_3(t) + \lambda_3\varepsilon$.	If $x_3 > 0$, $x_3(t) - (\mu_3 - \lambda_3)\varepsilon$. If $x_3 = 0$, $x_3(t)$.

The derivation of the candidate Lyapunov function derivative if the server is processing, is described per mode in the upcoming subsections. If the evolution of the buffer contents changes, the candidate Lyapunov function derivative changes as well, as was the case in the two direction example. This results in extra expressions and adjustments to the domain boundaries, when either one of the buffer contents equals zero.

Mode *A*

$$\mathscr{D}^{\mathrm{I}}: \quad \frac{x_1-(\mu_1-\lambda_1)\varepsilon-3}{2\varepsilon} + \frac{x_2+\lambda_2\varepsilon-2}{3\varepsilon} + \frac{x_3-(\mu_3-\lambda_3)\varepsilon-1}{4\varepsilon} - \frac{x_1-3}{2\varepsilon} - \frac{x_2-2}{3\varepsilon} - \frac{x_3-1}{4\varepsilon} = -\frac{7}{12} < 0, \quad \text{for A}, \quad x_1 \geq 3, \quad x_2 \geq 2, \quad x_3 \geq 1. \tag{E.54}$$

$$\mathscr{D}^{\mathrm{II}}: \quad \frac{x_1-(\mu_1-\lambda_1)\varepsilon-3}{2\varepsilon} + \frac{x_3-(\mu_3-\lambda_3)\varepsilon-1}{4\varepsilon} - \frac{x_1-3}{2\varepsilon} - \frac{x_3-1}{4\varepsilon} = -\frac{5}{4} < 0, \quad \text{for A}, \quad x_1 \geq 3, \quad x_2 \leq 2, \quad x_3 \geq 1. \tag{E.55}$$

$$\mathscr{D}_1^{\mathrm{VI}}: \quad = 0, \quad \text{for A}, \quad \frac{8}{3} \leq x_1 \leq 3, \quad x_2 \leq 8 - 2x_1, \quad x_3 \leq 3x_1 - 8 \leq 1. \tag{E.56}$$

$$\begin{aligned}\mathscr{D}_3^{\mathrm{VI}}: &\qquad = 0,\\ &\text{for } \text{Ⓐ}, \quad \tfrac{8}{3} \le x_1 \le 3, \quad x_2 \le \tfrac{8}{3} - \tfrac{2}{3}x_3, \quad 3x_1 - 8 \le x_3 \le 1.\end{aligned} \tag{E.57}$$

Because (E.55) is not a function of x_2 there is no additional expression needed if $x_2 = 0$. Similar motivation holds for (E.56) and (E.57), if either one of the buffer contents in the domain becomes zero the candidate Lyapunov function derivative does not change.

$$\begin{aligned}\mathscr{D}_1^{\mathrm{III_a}}: &\quad \tfrac{x_2+\lambda_2\varepsilon-2}{3\varepsilon} - \tfrac{x_3-(\mu_3-\lambda_3)-1}{4\varepsilon} - \tfrac{x_2-2}{3\varepsilon} - \tfrac{x_3-1}{4\varepsilon} = -\tfrac{1}{12} < 0,\\ &\text{for } \text{Ⓐ}, \quad \tfrac{17}{6} \le x_1 \le 3, \quad x_2 \ge 8 - 2x_1 \ge 4, \quad x_3 \ge 1.\end{aligned} \tag{E.58}$$

$$\begin{aligned}\mathscr{D}_1^{\mathrm{III_b}}: &\quad \tfrac{x_2+\lambda_2\varepsilon-2}{3\varepsilon} + \tfrac{x_3-(\mu_3-\lambda_3)\varepsilon-1}{4\varepsilon} - \tfrac{x_2-2}{3\varepsilon} - \tfrac{x_3-1}{4\varepsilon} = -\tfrac{1}{12} < 0,\\ &\text{for } \text{Ⓐ}, \quad 2 \le x_1 \le 3, \quad x_2 \le \tfrac{3}{2}x_1 + \tfrac{1}{4}, \quad x_3 \ge 1.\end{aligned} \tag{E.59}$$

$$\begin{aligned}\mathscr{D}_1^{\mathrm{IV_a}}: &\quad \tfrac{x_1-(\mu_1-\lambda_1)\varepsilon-2}{2\varepsilon} + \tfrac{x_2+\lambda_2\varepsilon-4}{3\varepsilon} + \tfrac{x_3-(\mu_3-\lambda_3)\varepsilon}{4\varepsilon} - \tfrac{x_1-2}{2\varepsilon} - \tfrac{x_2-4}{3\varepsilon} - \tfrac{x_3}{4\varepsilon} = -\tfrac{7}{12} < 0,\\ &\text{for } \text{Ⓐ}, \quad 2 \le x_1 \le \tfrac{17}{6}, \quad x_2 \ge 8 - 2x_1 \ge 4, \quad x_3 \ge 1.\end{aligned} \tag{E.60}$$

$$\begin{aligned}\mathscr{D}_1^{\mathrm{IV_b}}: &\quad \tfrac{x_2+\lambda_2\varepsilon-4}{3\varepsilon} + \tfrac{x_3-(\mu_3-\lambda_3)\varepsilon}{4\varepsilon} - \tfrac{x_2-4}{3\varepsilon} - \tfrac{x_3}{4\varepsilon} = -\tfrac{1}{12} < 0,\\ &\text{for } \text{Ⓐ}, \quad x_1 \le 2, \quad x_2 \ge 8 - 2x_1 \ge 4, \quad x_3 \ge 1.\end{aligned} \tag{E.61}$$

$$\begin{aligned}\mathscr{D}_1^{\mathrm{V}}: &\quad \tfrac{x_1-(\mu_1-\lambda_1)\varepsilon-2}{2\varepsilon} + \tfrac{x_3-(\mu_3-\lambda_3)\varepsilon}{4\varepsilon} - \tfrac{x_1-2}{2\varepsilon} - \tfrac{x_3}{4\varepsilon} = -\tfrac{5}{4} < 0,\\ &\text{for } \text{Ⓐ}, \quad 2 \le x_1 \le 3, \quad \tfrac{3}{2}x_1 + \tfrac{1}{4} \le x_2 \le 4, \quad x_3 \ge 1.\end{aligned} \tag{E.62}$$

$$\begin{aligned}\mathscr{D}_3^{\mathrm{III}}: &\quad \tfrac{x_1-(\mu_1-\lambda_1)\varepsilon-3}{2\varepsilon} + \tfrac{x_2+\lambda_2\varepsilon-\frac{8}{3}+\frac{2}{3}(x_3-(\mu_3-\lambda_3)\varepsilon)}{3\varepsilon} - \tfrac{x_1-3}{2\varepsilon} - \tfrac{x_2-\frac{8}{3}+\frac{2}{3}x_3}{3\varepsilon} = -\tfrac{1}{2} < 0,\\ &\text{for } \text{Ⓐ}, \quad x_1 \ge 3, \quad x_2 \ge \tfrac{8}{3} - \tfrac{2}{3}x_3, \quad 0 < x_3 \le 1.\end{aligned} \tag{E.63}$$

If in (E.63) the value of x_3 becomes zero than the buffer content evolution changes. In that case the candidate Lyapunov function derivative equals (E.64).

$$\begin{aligned}\mathscr{D}_3^{\mathrm{III}}: &\quad \tfrac{x_1-(\mu_1-\lambda_1)\varepsilon-3}{2\varepsilon} + \tfrac{x_2+\lambda_2\varepsilon-\frac{8}{3}+\frac{2}{3}x_3}{3\varepsilon} - \tfrac{x_1-3}{2\varepsilon} - \tfrac{x_2-\frac{8}{3}+\frac{2}{3}x_3}{3\varepsilon} = \tfrac{1}{6} > 0,\\ &\text{for } \text{Ⓐ}, \quad x_1 \ge 3, \quad x_2 \ge \tfrac{8}{3}, \quad x_3 = 0.\end{aligned} \tag{E.64}$$

$$\begin{aligned}\mathscr{D}_{1,3}^{\mathrm{III_a}}: &\quad \tfrac{x_2+\lambda_2\varepsilon-\frac{8}{3}+\frac{2}{3}(x_3-(\mu_3-\lambda_3)\varepsilon)}{3\varepsilon} - \tfrac{x_2-\frac{8}{3}+\frac{2}{3}x_3}{3\varepsilon} = 0,\\ &\text{for } \text{Ⓐ}, \quad \tfrac{26}{9} - \tfrac{1}{18}x_3 \le x_1 \le 3, \quad x_2 \ge 4, \quad 3x_1 - 8 \le x_3 \le 1.\end{aligned} \tag{E.65}$$

$$\begin{aligned}\mathscr{D}_{1,3}^{\mathrm{III_b}}: &\quad \tfrac{x_2+\lambda_2\varepsilon-\frac{8}{3}+\frac{2}{3}(x_3-(\mu_3-\lambda_3)\varepsilon)}{3\varepsilon} - \tfrac{x_2-\frac{8}{3}+\frac{2}{3}x_3}{3\varepsilon} = 0,\\ &\text{for } \text{Ⓐ}, \quad \tfrac{2}{3}x_2 - \tfrac{1}{18}x_3 + \tfrac{2}{9} \le x_1 \le 3, \quad 8 - 2x_1 \le x_2 \le 4, \quad 0 < 3x_1 - 8 \le x_3 \le 1.\end{aligned} \tag{E.66}$$

If in (E.66) and $x_3 = 0$ then the buffer content evolution changes. In that case the candidate Lyapunov function derivative equals (E.67).

$$\begin{aligned}\mathscr{D}_{1,3}^{\mathrm{III_b}}: &\quad \tfrac{x_2+\lambda_2\varepsilon-\frac{8}{3}+\frac{2}{3}x_3}{3\varepsilon} - \tfrac{x_2-\frac{8}{3}+\frac{2}{3}x_3}{3\varepsilon} = \tfrac{2}{3} > 0,\\ &\text{for } \text{Ⓐ}, \quad \tfrac{2}{3}x_2 - \tfrac{1}{18}x_3 + \tfrac{2}{9} \le x_1 \le 3, \quad 8 - 2x_1 \le x_2 \le 4, \quad 3x_1 - 8 \le x_3 = 0.\end{aligned} \tag{E.67}$$

$$\begin{aligned}\mathscr{D}_{1,3}^{\mathrm{IV}}: &\quad \tfrac{x_1-(\mu_1-\lambda_1)\varepsilon-2}{2\varepsilon} + \tfrac{x_2+\lambda_2\varepsilon-4}{3\varepsilon} + \tfrac{x_3-(\mu_3-\lambda_3)\varepsilon}{4\varepsilon} - \tfrac{x_1-2}{2\varepsilon} - \tfrac{x_2-4}{3\varepsilon} - \tfrac{x_3}{4\varepsilon} = -\tfrac{7}{12} < 0,\\ &\text{for } \text{Ⓐ}, \quad x_1 \le \tfrac{26}{9} - \tfrac{1}{18}x_3, \quad 8 - 2x_1 \le x_2 \le 4, \quad 3x_1 - 8 \le x_3 \le 1.\end{aligned} \tag{E.68}$$

$$\begin{aligned}\mathscr{D}_{1,3}^{\mathrm{V}}: &\quad \tfrac{x_1-(\mu_1-\lambda_1)\varepsilon-2}{2\varepsilon} + \tfrac{x_3-(\mu_3-\lambda_3)\varepsilon}{4\varepsilon} - \tfrac{x_1-2}{2\varepsilon} - \tfrac{x_3}{4\varepsilon} = -\tfrac{5}{4} < 0,\\ &\text{for } \text{Ⓐ}, \quad x_1 \le \tfrac{2}{3}x_2 - \tfrac{1}{18}x_3 + \tfrac{2}{9}, \quad 8 - 2x_1 \le x_2 \le 4, \quad 3x_1 - 8 \le x_3 \le 1.\end{aligned} \tag{E.69}$$

$$\mathscr{D}_{3,1}^{\mathrm{III_a}}: \quad \frac{x_1-(\mu_1-\lambda_1)\varepsilon-\frac{8}{3\varepsilon}-\frac{1}{3}(x_3-(\mu_3-\lambda_3)\varepsilon)}{2\varepsilon}+\frac{x_2+\lambda_2\varepsilon-\frac{8}{3}+\frac{2}{3}(x_3-(\mu_3-\lambda_3)\varepsilon)}{3\varepsilon}-\frac{x_1-\frac{8}{3}-\frac{1}{3}x_3}{2\varepsilon}-\frac{x_2-\frac{8}{3}+\frac{2}{3}x_3}{3\varepsilon}=0,$$
$$\text{for Ⓐ}, \quad \frac{8}{3}+\frac{1}{3}x_3 \le x_1 \le 3, \quad x_2 \ge \frac{8}{3}-\frac{2}{3}x_3 \ge 4, \quad \frac{4}{7} \le x_3 \le 1. \tag{E.70}$$

$$\mathscr{D}_{3,1}^{\mathrm{III_b}}: \quad \frac{x_1-(\mu_1-\lambda_1)\varepsilon-\frac{8}{3\varepsilon}-\frac{1}{3}(x_3-(\mu_3-\lambda_3)\varepsilon)}{2\varepsilon}+\frac{x_2+\lambda_2\varepsilon-\frac{8}{3}+\frac{2}{3}(x_3-(\mu_3-\lambda_3)\varepsilon)}{3\varepsilon}-\frac{x_1-\frac{8}{3}-\frac{1}{3}x_3}{2\varepsilon}-\frac{x_2-\frac{8}{3}+\frac{2}{3}x_3}{3\varepsilon}=0,$$
$$\text{for Ⓐ}, \quad \frac{8}{3}+\frac{1}{3}x_3 \le x_1 \le 3, \quad x_2 \le \frac{7}{12}x_3+\frac{11}{3} \le 4, \quad x_3 \le 1. \tag{E.71}$$

$$\mathscr{D}_{3,1}^{\mathrm{IV}}: \quad \frac{x_1-(\mu_1-\lambda_1)\varepsilon-2}{2\varepsilon}+\frac{x_2+\lambda_2\varepsilon-4}{3\varepsilon}+\frac{x_3-(\mu_3-\lambda_3)\varepsilon}{4}-\frac{x_1-2}{2\varepsilon}-\frac{x_2-4}{3\varepsilon}-\frac{x_3}{4\varepsilon}=-\frac{7}{12}<0,$$
$$\text{for Ⓐ}, \quad \frac{8}{3}+\frac{1}{3}x_3 \le x_1 \le 3, \quad x_2 \ge 4, \quad x_3 \le \frac{4}{7}. \tag{E.72}$$

$$\mathscr{D}_{3,1}^{\mathrm{V}}: \quad \frac{x_1-(\mu_1-\lambda_1)\varepsilon-2}{2\varepsilon}+\frac{x_3-(\mu_3-\lambda_3)\varepsilon}{4\varepsilon}-\frac{x_1-2}{2\varepsilon}-\frac{x_3}{4\varepsilon}=-\frac{5}{4}<0,$$
$$\text{for Ⓐ}, \quad \frac{8}{3}+\frac{1}{3}x_3 \le x_1 \le 3, \quad \frac{7}{12}x_3+\frac{11}{3} \le x_2 \le 4, \quad x_3 \le 1. \tag{E.73}$$

$$\dot{V}_{\mathrm{P2P}} = \begin{cases}
<0 & \text{for Ⓐ}, x_1 \ge 3, x_3 \ge 1. \\
<0 & \text{for Ⓐ}, \frac{17}{6} \le x_1 \le 3, x_2 \ge 8-2x_1 \ge 4, x_3 \ge 1, \\
<0 & \text{for Ⓐ}, 2 \le x_1 \le 3, x_2 \le \frac{3}{2}x_1+\frac{1}{4}, x_3 \ge 1, \\
<0 & \text{for Ⓐ}, 2 \le x_1 \le \frac{17}{6}, x_2 \ge 8-2x_1 \ge 4, x_3 \ge 1, \\
<0 & \text{for Ⓐ}, x_1 \le 2, x_2 \ge 8-2x_1 \ge 4, x_3 \ge 1, \\
<0 & \text{for Ⓐ}, 2 \le x_1 \le 3, \frac{3}{2}x_1+\frac{1}{4} \le x_2 \le 4, x_3 \ge 1, \\
<0 & \text{for Ⓐ}, x_1 \ge 3, x_2 \ge \frac{8}{3}-\frac{2}{3}x_3, 0<x_3 \le 1, \\
>0 & \text{for Ⓐ}, x_1 \ge 3, x_2 \ge \frac{8}{3}, x_3 = 0, \\
0 & \text{for Ⓐ}, \frac{26}{9}-\frac{1}{18}x_3 \le x_1 \le 3, x_2 \ge 4, 3x_1-8 \le x_3 \le 1, \\
0 & \text{for Ⓐ}, \frac{2}{3}x_2-\frac{1}{18}x_3+\frac{2}{9} \le x_1 \le 3, 8-2x_1 \le x_2 \le 4, 0<3x_1-8 \le x_3 \le 1, \\
>0 & \text{for Ⓐ}, \frac{2}{3}x_2-\frac{1}{18}x_3+\frac{2}{9} \le x_1 \le 3, 8-2x_1 \le x_2 \le 4, 3x_1-8 \le x_3 = 0, \\
<0 & \text{for Ⓐ}, x_1 \le \frac{26}{9}-\frac{1}{18}x_3, 8-2x_1 \le x_2 \le 4, 3x_1-8 \le x_3 \le 1, \\
<0 & \text{for Ⓐ}, x_1 \le \frac{2}{3}x_2-\frac{1}{18}x_3+\frac{2}{9}, 8-2x_1 \le x_2 \le 4, 3x_1-8 \le x_3 \le 1, \\
0 & \text{for Ⓐ}, \frac{8}{3}+\frac{1}{3}x_3 \le x_1 \le 3, x_2 \ge \frac{8}{3}-\frac{2}{3}x_3 \ge 4, \frac{4}{7} \le x_3 \le 1, \\
0 & \text{for Ⓐ}, \frac{8}{3}+\frac{1}{3}x_3 \le x_1 \le 3, x_2 \le \frac{7}{12}x_3+\frac{11}{3} \le 4, x_3 \le 1, \\
<0 & \text{for Ⓐ}, \frac{8}{3}+\frac{1}{3}x_3 \le x_1 \le 3, x_2 \ge 4, x_3 \le \frac{4}{7}, \\
<0 & \text{for Ⓐ}, \frac{8}{3}+\frac{1}{3}x_3 \le x_1 \le 3, \frac{7}{12}x_3+\frac{11}{3} \le x_2 \le 4, x_3 \le 1, \\
0 & \text{for Ⓐ}, \frac{8}{3} \le x_1 \le 3, x_2 \le 8-2x_1, x_3 \le 3x_1-8 \le 1, \\
0 & \text{for Ⓐ}, \frac{8}{3} \le x_1 \le 3, x_2 \le \frac{8}{3}-\frac{2}{3}x_3, 3x_1-8 \le x_3 \le 1.
\end{cases} \tag{E.74}$$

Mode *B*

$$\mathscr{D}^{\mathrm{I}}: \quad \frac{x_1-(\mu_1-\lambda_1)\varepsilon-1}{2\varepsilon}+\frac{x_2-(\mu_2-\lambda_2)\varepsilon-6}{3\varepsilon}+\frac{x_3+\lambda_3\varepsilon-1}{4\varepsilon}-\frac{x_1-1}{2\varepsilon}-\frac{x_2-6}{3\varepsilon}-\frac{x_3-1}{4\varepsilon}=-\frac{7}{12}<0,$$
$$\text{for Ⓑ}, \quad x_1 \ge 1, \quad x_2 \ge 6, \quad x_3 \ge 1. \tag{E.75}$$

$$\mathscr{D}^{\mathrm{II}}: \quad \frac{x_1-(\mu_1-\lambda_1)\varepsilon-1}{2\varepsilon}+\frac{x_2-(\mu_2-\lambda_2)\varepsilon-6}{3\varepsilon}-\frac{x_1-1}{2\varepsilon}-\frac{x_2-6}{3\varepsilon}=-\frac{5}{6}<0,$$
$$\text{for Ⓑ}, \quad x_1 \ge 1, \quad x_2 \ge 6, \quad x_3 \le 1. \tag{E.76}$$

$$\mathscr{D}_1^{\mathrm{VI}}: \quad =0,$$
$$\text{for Ⓑ}, \quad x_1 \le 1, \quad x_2 \le x_1+5 \le 6, \quad x_3 \le 2-x_1. \tag{E.77}$$

$$\mathscr{D}_2^{\mathrm{VI}}: \quad = 0, \quad \text{for } Ⓑ, \quad x_1 \le 1, \quad x_1 + 5 \le x_2 \le 6, \quad x_3 \le 7 - x_2. \tag{E.78}$$

$$\mathscr{D}_1^{\mathrm{III}}: \quad \tfrac{x_2-(\mu_2-\lambda_2)\varepsilon-6}{3\varepsilon} + \tfrac{x_3+\lambda_3\varepsilon+x_1-(\mu_1-\lambda_1)\varepsilon-2}{4\varepsilon} - \tfrac{x_2-6}{3\varepsilon} - \tfrac{x_3+x_1-2}{4\varepsilon} = -\tfrac{1}{3} < 0, \quad \text{for } Ⓑ, \quad x_1 \le 1, \quad x_2 \ge 6, \quad x_3 \ge 2 - x_1. \tag{E.79}$$

$$\mathscr{D}_{1,2}^{\mathrm{III}}: \quad \tfrac{x_1-(\mu_1-\lambda_1)\varepsilon-1}{2\varepsilon} + \tfrac{x_3+\lambda_3\varepsilon-1}{4\varepsilon} - \tfrac{x_1-1}{2\varepsilon} - \tfrac{x_3-1}{4\varepsilon} = -\tfrac{1}{4} < 0, \quad \text{for } Ⓑ, \quad x_1 \le 1, \quad x_2 \le 6, \quad x_3 \ge 7 - x_2. \tag{E.80}$$

$$\mathscr{D}_{1,2}^{\mathrm{III}}: \quad \tfrac{x_2-(\mu_2-\lambda_2)\varepsilon-(x_1-(\mu_1-\lambda_1)\varepsilon)-5}{3\varepsilon} + \tfrac{x_3+\lambda_3\varepsilon+x_1-(\mu_1-\lambda_1)\varepsilon-2}{4\varepsilon} - \tfrac{x_2-x_1-5}{3\varepsilon} - \tfrac{x_3+x_1-2}{4\varepsilon} = 0, \quad \text{for } Ⓑ, \quad x_1 \le 1, \quad x_1 + 5 \le x_2 \le 6, \quad x_3 \ge 2 - x_1. \tag{E.81}$$

$$\mathscr{D}_{2,1}^{\mathrm{III}}: \quad \tfrac{x_3+\lambda_3\varepsilon+x_1-(\mu_1-\lambda_1)\varepsilon-2}{4\varepsilon} - \tfrac{x_3+x_1-2}{4\varepsilon} = 0, \quad \text{for } Ⓑ, \quad x_1 \le 1, \quad x_2 \le 6, \quad x_3 \ge 2 - x_1. \tag{E.82}$$

$$\dot{V}_{\mathrm{P2P}} = \begin{cases} < 0 & \text{for } Ⓑ, x_1 \ge 1, x_2 \ge 6, x_3 \ge 1, \\ < 0 & \text{for } Ⓑ, x_1 \ge 1, x_2 \ge 6, x_3 \le 1, \\ < 0 & \text{for } Ⓑ, x_1 \le 1, x_2 \ge 6, x_3 \ge 2 - x_1, \\ < 0 & \text{for } Ⓑ, x_1 \ge 1, x_2 \le 6, x_3 \ge 7 - x_2, \\ 0 & \text{for } Ⓑ, x_1 \le 1, x_1 + 5 \le x_2 \le 6, x_3 \ge 2 - x_1, \\ 0 & \text{for } Ⓑ, x_1 \le 1, x_2 \le 6, x_3 \ge 2 - x_1, \\ 0 & \text{for } Ⓑ x_1 \le 1, x_2 \le x_1 + 5 \le 6, x_3 \le 2 - x_1, \\ 0 & \text{for } Ⓑ, x_1 \le 1, x_1 + 5 \le x_2 \le 6, x_3 \le 7 - x_2, \end{cases} \tag{E.83}$$

Mode *C*

$$\mathscr{D}^{\mathrm{I}}: \quad \tfrac{x_1\lambda_1\varepsilon-1}{2\varepsilon} + \tfrac{x_2-(\mu_2-\lambda_2)\varepsilon}{3\varepsilon} + \tfrac{x_3-(\mu_3-\lambda_3)\varepsilon-7}{4\varepsilon} - \tfrac{x_1-1}{2\varepsilon} - \tfrac{x_2}{3\varepsilon} - \tfrac{x_3-7}{4\varepsilon} = -\tfrac{7}{12} < 0, \quad \text{for } Ⓒ, \quad x_1 \ge 1, \quad x_2 \ge 0, \quad x_3 \ge 7. \tag{E.84}$$

$$\mathscr{D}^{\mathrm{II}}: \quad \tfrac{x_2-(\mu_2-\lambda_2)\varepsilon}{3\varepsilon} + \tfrac{x_3-(\mu_3-\lambda_3)\varepsilon-7}{4\varepsilon} - \tfrac{x_2}{3\varepsilon} - \tfrac{x_3-7}{4\varepsilon} = -\tfrac{13}{12} < 0, \quad \text{for } Ⓒ, \quad x_1 \le 1, \quad x_2 \ge 0, \quad x_3 \ge 7. \tag{E.85}$$

$$\mathscr{D}_2^{\mathrm{VI}}: \quad = 0, \quad \text{for } Ⓒ, \quad x_1 \le 1, \quad x_2 = 0, \quad x_3 = 7. \tag{E.86}$$

$$\mathscr{D}_2^{\mathrm{III}}: \quad \tfrac{x_1+\lambda_1\varepsilon-1}{2\varepsilon} + \tfrac{x_3-(\mu_3-\lambda_3)\varepsilon-7}{4\varepsilon} - \tfrac{x_1-1}{2\varepsilon} - \tfrac{x_3-7}{4\varepsilon} = -\tfrac{7}{12} < 0, \quad \text{for } Ⓒ, \quad x_1 \ge 1, \quad x_2 = 0, \quad x_3 \ge 7. \tag{E.87}$$

$$\mathscr{D}_3^{\mathrm{III}}: \quad \tfrac{x_1+\lambda_1\varepsilon-1}{2\varepsilon} + \tfrac{x_2-(\mu_2-\lambda_2)\varepsilon}{3\varepsilon} - \tfrac{x_1-1}{2\varepsilon} - \tfrac{x_2}{3\varepsilon} = \tfrac{1}{6} > 0, \quad \text{for } Ⓒ, \quad x_1 \ge 1, \quad x_2 \ge 0, \quad x_3 \le 7. \tag{E.88}$$

$$\mathscr{D}_{2,3}^{\mathrm{III}}: \quad \tfrac{x_1+\lambda_1\varepsilon-1}{2\varepsilon} + \tfrac{x_3-(\mu_3-\lambda_3)\varepsilon-7}{4\varepsilon} - \tfrac{x_1-1}{2\varepsilon} - \tfrac{x_3-7}{4\varepsilon} = -\tfrac{7}{12} < 0, \quad \text{for } Ⓒ, \quad x_1 \ge 1, \quad x_2 = 0, \quad x_3 = 7. \tag{E.89}$$

$$\mathscr{D}_{3,2}^{\mathrm{III}}: \quad \tfrac{x_1+\lambda_1\varepsilon-1}{2\varepsilon} - \tfrac{x_1-1}{2\varepsilon} = \tfrac{1}{2} > 0, \quad \text{for } Ⓒ, \quad x_1 \ge 1, \quad x_2 = 0, \quad x_3 \le 7. \tag{E.90}$$

$$\dot{V}_{\mathrm{P2P}} = \begin{cases} < 0 & \text{for } Ⓒ, x_1 \ge 1, x_2 \ge 0, x_3 \ge 7, \\ < 0 & \text{for } Ⓒ, x_1 \le 1, x_2 \ge 0, x_3 \ge 7, \\ < 0 & \text{for } Ⓒ, x_1 \ge 1, x_2 = 0, x_3 \ge 7, \\ > 0 & \text{for } Ⓒ, x_1 \ge 1, x_2 \ge 0, x_3 \le 7, \\ > 0 & \text{for } Ⓒ, x_1 \ge 1, x_2 = 0, x_3 \le 7, \\ 0 & \text{for } Ⓒ, x_1 \le 1, x_2 = 0, x_3 = 7. \end{cases} \tag{E.91}$$

Derivative in Process

Combining the derivatives in process of mode gives an expression for the candidate Lyapunov function derivative if the server is processing, namely (E.92)

$$
\dot{V}_{\mathrm{P2P}} = \begin{cases}
< 0 & \text{for } \text{A}, x_1 \geq 3, x_2 \geq 2, x_3 \geq 1. \\
< 0 & \text{for } \text{A}, x_1 \geq 3, x_2 \leq 2, x_3 \geq 1, \\
< 0 & \text{for } \text{A}, x_1 \leq 2, x_2 \geq 8 - 2x_1 \geq 4, x_3 \geq 1, \\
< 0 & \text{for } \text{A}, 2 \leq x_1 \leq 3, x_2 \leq 4, x_3 \geq 1, \\
< 0 & \text{for } \text{A}, x_1 \geq 3, x_2 \geq \frac{8}{3} - \frac{2}{3}x_3, 0 < x_3 \leq 1, \\
> 0 & \text{for } \text{A}, x_1 \geq 3, x_2 \geq \frac{8}{3}, x_3 = 0, \\
0 & \text{for } \text{A}, \frac{26}{9} - \frac{1}{18}x_3 \leq x_1 \leq 3, x_2 \geq 4, 3x_1 - 8 \leq x_3 \leq 1, \\
0 & \text{for } \text{A}, \frac{2}{3}x_2 - \frac{1}{18}x_3 + \frac{2}{9} \leq x_1 \leq 3, 8 - 2x_1 \leq x_2 \leq 4, 3x_1 - 8 \leq x_3 \leq 1, \\
> 0 & \text{for } \text{A}, \frac{2}{3}x_2 - \frac{1}{18}x_3 + \frac{2}{9} \leq x_1 \leq 3, 8 - 2x_1 \leq x_2 \leq 4, 3x_1 - 8 \leq x_3 = 0, \\
< 0 & \text{for } \text{A}, x_1 \leq \frac{26}{9} - \frac{1}{18}x_3, 8 - 2x_1 \leq x_2 \leq 4, 3x_1 - 8 \leq x_3 \leq 1, \\
< 0 & \text{for } \text{A}, x_1 \leq \frac{2}{3}x_2 - \frac{1}{18}x_3 + \frac{2}{9}, 8 - 2x_1 \leq x_2 \leq 4, 3x_1 - 8 \leq x_3 \leq 1, \\
0 & \text{for } \text{A}, \frac{8}{3} + \frac{1}{3}x_3 \leq x_1 \leq 3, x_2 \geq \frac{8}{3} - \frac{2}{3}x_3 \geq 4, \frac{4}{7} \leq x_3 \leq 1, \\
0 & \text{for } \text{A}, \frac{8}{3} + \frac{1}{3}x_3 \leq x_1 \leq 3, x_2 \leq \frac{7}{12}x_3 + \frac{11}{3} \leq 4, x_3 \leq 1, \\
< 0 & \text{for } \text{A}, \frac{8}{3} + \frac{1}{3}x_3 \leq x_1 \leq 3, x_2 \geq 4, x_3 \leq \frac{4}{7}, \\
< 0 & \text{for } \text{A}, \frac{8}{3} + \frac{1}{3}x_3 \leq x_1 \leq 3, \frac{7}{12}x_3 + \frac{11}{3} \leq x_2 \leq 4, x_3 \leq 1, \\
0 & \text{for } \text{A}, \frac{8}{3} \leq x_1 \leq 3, x_2 \leq 8 - 2x_1, x_3 \leq 3x_1 - 8 \leq 1, \\
0 & \text{for } \text{A}, \frac{8}{3} \leq x_1 \leq 3, x_2 \leq \frac{8}{3} - \frac{2}{3}x_3, 3x_1 - 8 \leq x_3 \leq 1, \\
< 0 & \text{for } \text{B}, x_1 \geq 1, x_2 \geq 6, x_3 \geq 1, \\
< 0 & \text{for } \text{B}, x_1 \geq 1, x_2 \geq 6, x_3 \leq 1, \\
< 0 & \text{for } \text{B}, x_1 \leq 1, x_2 \geq 6, x_3 \geq 2 - x_1, \\
< 0 & \text{for } \text{B}, x_1 \geq 1, x_2 \leq 6, x_3 \geq 7 - x_2, \\
0 & \text{for } \text{B}, x_1 \leq 1, x_1 + 5 \leq x_2 \leq 6, x_3 \geq 2 - x_1, \\
0 & \text{for } \text{B}, x_1 \leq 1, x_2 \leq 6, x_3 \geq 2 - x_1, \\
0 & \text{for } \text{B} x_1 \leq 1, x_2 \leq x_1 + 5 \leq 6, x_3 \leq 2 - x_1, \\
0 & \text{for } \text{B}, x_1 \leq 1, x_1 + 5 \leq x_2 \leq 6, x_3 \leq 7 - x_2, \\
< 0 & \text{for } \text{C}, x_1 \geq 1, x_2 \geq 0, x_3 \geq 7, \\
< 0 & \text{for } \text{C}, x_1 \leq 1, x_2 \geq 0, x_3 \geq 7, \\
< 0 & \text{for } \text{C}, x_1 \geq 1, x_2 = 0, x_3 \geq 7, \\
> 0 & \text{for } \text{C}, x_1 \geq 1, x_2 \geq 0, x_3 \leq 7, \\
> 0 & \text{for } \text{C}, x_1 \geq 1, x_2 = 0, x_3 \leq 7, \\
0 & \text{for } \text{C}, x_1 \leq 1, x_2 = 0, x_3 = 7.
\end{cases} \tag{E.92}
$$

Appendix F. Switching Instants

Due to the number of domain parts, the candidate Lyapunov function derivative in setup at switching instants is comprehensive. The interest of this derivation is debatable. The only time instant the resulting control action can occur is when the server is starts in setup. In all other situations the server would not have switched mode, if the candidate Lyapunov functions direction of steepest descent was the mode it switched out of. However, for the purpose of completeness, the candidate Lyapunov function derivative at switching instant in setup is explained in this appendix. The reader is referred to Section 4.1, if the interested is in the origin of the equations.

F.1 Derivative if Server Switches to Setup

The control action to perform depends on the candidate Lyapunov function derivative. If the derivative is minimized by switching to the subsequent mode, the control action should be to switch. Otherwise, the server should continue in its current mode.
The server is required to move the system states in the direction of steepest descent of the candidate Lyapunov function, thereby stabilizing the system as time efficiently as possible. The candidate Lyapunov function derivative is listed in (4.3), for cases the server continues in its current mode. The candidate Lyapunov function derivative derived in this section, is the derivative in case the server starts in a mode and switches to setup of the subsequent mode. Notice that, if the server switches to setup, the complete setup time needs to be performed. For instance, if the server is in Ⓐ and switches to ❸, the value of the remaining setup time becomes: $x_0 = 3$. The candidate Lyapunov function derivative is now defined via (F.1).

F.1.1 Mode A

Implementing the buffer content values at $t+\varepsilon$ in case the server switches to setup in (F.1), gives the candidate Lyapunov function derivative at switching instants in Ⓐ. The resulting candidate Lyapunov function derivative at switching instants, is presented for the entire domain in Ⓐ in (F.2) to (F.16). The candidate Lyapunov function derivative at switching instants in Ⓑ, is given by (F.19) to (F.30). It should be noted that additional expressions arise and the boundaries are shifted, due to the difference in domain parts of both modes.

$$\lim_{\varepsilon\to 0}\frac{V\left(❸, 3-\varepsilon, x_1\left(t+\varepsilon\right), x_2\left(t+\varepsilon\right)\right) - V\left(Ⓐ, 0, x_1\left(t\right), x_2\left(t\right)\right)}{\varepsilon}. \tag{F.1}$$

$$\mathscr{D}^{\mathrm{I}}: \quad \frac{x_1+\lambda_1\varepsilon+3(3-\varepsilon)-9}{\mu_1\varepsilon}+\frac{x_2+\lambda_2\varepsilon+3-\varepsilon-8}{\mu_2\varepsilon}-\frac{x_1-15}{\mu_1\varepsilon}-\frac{x_2-1}{\mu_2\varepsilon}=\frac{167}{72\varepsilon}\to\infty,\ \text{if } \varepsilon\to 0, \quad \text{for Ⓐ},\quad x_1\geq 15,\quad x_2\geq 5. \tag{F.2}$$

$$\mathscr{D}^{\mathrm{I}}: \quad \frac{x_1+\lambda_1\varepsilon+\frac{27}{8}(3-\varepsilon)+\frac{3}{8}(x_2+\lambda_2\varepsilon)-12}{\mu_1\varepsilon}-\frac{x_1-15}{\mu_1\varepsilon}-\frac{x_2-1}{\mu_2\varepsilon}=\frac{-37x_2+1009}{576\varepsilon}\to\infty,\ \text{if } \varepsilon\to 0, \quad \text{for Ⓐ},\quad x_1>\geq 15,\quad 1\leq x_2<5. \tag{F.3}$$

$$\mathscr{D}^{\mathrm{II}}: \quad \frac{x_1+\lambda_1\varepsilon+\frac{27}{8}(3-\varepsilon)+\frac{3}{8}(x_2+\lambda_2\varepsilon)-12}{\mu_1\varepsilon}-\frac{x_1-15}{\mu_1\varepsilon}=\frac{3x_2+105}{64\varepsilon}\to\infty,\ \text{if } \varepsilon\to 0, \quad \text{for Ⓐ},\quad x_1\geq 15,\quad x_2<1. \tag{F.4}$$

$$\mathscr{D}^{\mathrm{III_a}}: \quad \frac{x_1+\lambda_1\varepsilon+3(3-\varepsilon)-9}{\mu_1\varepsilon}+\frac{x_2+\lambda_2\varepsilon+(3-\varepsilon)-8}{\mu_2\varepsilon}-\frac{x_2+\frac{1}{5}x_1-4}{\mu_2\varepsilon}=\frac{37x_1-40}{360\varepsilon}\to\infty,\ \text{if } \varepsilon\to 0, \quad \text{for Ⓐ},\quad \frac{40}{37}\leq x_1<15,\quad x_2\geq 5. \tag{F.5}$$

$$\begin{aligned}\mathscr{D}^{\mathrm{IV}}:\quad & \tfrac{x_1+\lambda_1\varepsilon+3(3-\varepsilon)-9}{\mu_1\varepsilon}+\tfrac{x_2+\lambda_2\varepsilon+(3-\varepsilon)-8}{\mu_2\varepsilon}-\tfrac{x_1}{\mu_1\varepsilon}-\tfrac{x_2-5}{\mu_2\varepsilon}=0,\\ & \text{for A},\quad x_1<\tfrac{40}{37},\quad x_2\geq 5.\end{aligned} \tag{F.6}$$

$$\begin{aligned}\mathscr{D}^{\mathrm{III_b}}:\quad & \tfrac{x_1+\lambda_1\varepsilon+\frac{27}{8}(3-\varepsilon)+\frac{3}{8}(x_2+\lambda_2\varepsilon)-12}{\mu_1\varepsilon}-\tfrac{x_2+\frac{1}{5}-4}{\mu_2\varepsilon}=\tfrac{\frac{296}{5}x_1-37x_2+121}{576\varepsilon}\to\infty,\ \text{if } \varepsilon\to 0,\\ & \text{for A},\quad \tfrac{15}{8}-\tfrac{3}{8}x_2\leq\tfrac{5}{8}x_2-\tfrac{605}{296}\leq x_1<15,\quad 4-\tfrac{1}{5}x_1\leq x_2<\tfrac{37}{40}x_1+4<5.\end{aligned} \tag{F.7}$$

$$\begin{aligned}\mathscr{D}^{\mathrm{III_b}}:\quad & \tfrac{x_1+\lambda_1\varepsilon+\frac{27}{8}(3-\varepsilon)+\frac{3}{8}(x_2+\lambda_2\varepsilon)-12}{\mu_1\varepsilon}-\tfrac{x_2+\frac{1}{5}-4}{\mu_2\varepsilon}=\tfrac{\frac{296}{5}x_1-37x_2+121}{576\varepsilon}\to-\infty,\text{if } \varepsilon\to 0,\\ & \text{for A},\quad \tfrac{15}{8}-\tfrac{3}{8}x_2\leq x_1<\tfrac{5}{8}x_2-\tfrac{605}{296}<15,\quad 4-\tfrac{1}{5}x_1\leq x_2<\tfrac{37}{40}x_1+4<5.\end{aligned} \tag{F.8}$$

$$\begin{aligned}\mathscr{D}^{\mathrm{III_b}}:\quad & 0-\tfrac{x_2+\frac{1}{5}x_1-4}{\mu_2\varepsilon}=\tfrac{4-\frac{1}{5}x_1-x_2}{9\varepsilon}\to\infty,\ \text{if } \varepsilon\to 0,\\ & \text{for A},\quad x_1<\tfrac{15}{8}-\tfrac{3}{8}x_2<15,\quad 4-\tfrac{1}{5}x_1=x_2<\tfrac{37}{40}x_1+4<5.\end{aligned} \tag{F.9}$$

$$\begin{aligned}\mathscr{D}^{\mathrm{III_b}}:\quad & 0-\tfrac{x_2+\frac{1}{5}x_1-4}{\mu_2\varepsilon}=\tfrac{4-\frac{1}{5}x_1-x_2}{9\varepsilon}\to-\infty,\ \text{if } \varepsilon\to 0,\\ & \text{for A},\quad x_1<\tfrac{15}{8}-\tfrac{3}{8}x_2<15,\quad 4-\tfrac{1}{5}x_1\leq x_2<\tfrac{37}{40}x_1+4<5.\end{aligned} \tag{F.10}$$

$$\begin{aligned}\mathscr{D}^{\mathrm{V}}:\quad & \tfrac{x_1+\lambda_1\varepsilon+\frac{27}{8}(3-\varepsilon)+\frac{3}{8}(x_2+\lambda_2\varepsilon)-12}{\mu_1\varepsilon}-\tfrac{x_1}{\mu_1\varepsilon}=\tfrac{3x_2-15}{64\varepsilon}\to\infty,\ \text{if } \varepsilon\to 0,\\ & \text{for A},\quad \tfrac{15}{8}-\tfrac{3}{8}x_2\leq x_1<15,\quad \tfrac{37}{40}x_1+4\leq x_2=5.\end{aligned} \tag{F.11}$$

$$\begin{aligned}\mathscr{D}^{\mathrm{V}}:\quad & \tfrac{x_1+\lambda_1\varepsilon+\frac{27}{8}(3-\varepsilon)+\frac{3}{8}(x_2+\lambda_2\varepsilon)-12}{\mu_1\varepsilon}-\tfrac{x_1}{\mu_1\varepsilon}=\tfrac{3x_2-15}{64\varepsilon}\to-\infty,\ \text{if } \varepsilon\to 0,\\ & \text{for A},\quad \tfrac{15}{8}-\tfrac{3}{8}x_2\leq x_1\leq 15,\quad \tfrac{37}{40}x_1+4\leq x_2<5.\end{aligned} \tag{F.12}$$

$$\begin{aligned}\mathscr{D}^{\mathrm{V}}:\quad & 0-\tfrac{x_1}{\mu_1\varepsilon}=-\tfrac{x_1}{8\varepsilon}\to-\infty,\ \text{if } \varepsilon\to 0,\\ & \text{for A},\quad 0<x_1<\tfrac{15}{8}-\tfrac{3}{8}x_2,\quad \tfrac{37}{40}x_1+4\leq x_2<5.\end{aligned} \tag{F.13}$$

$$\begin{aligned}\mathscr{D}^{\mathrm{V}}:\quad & 0-\tfrac{x_1}{\mu_1\varepsilon}=\tfrac{0}{8\varepsilon}\to\infty,\ \text{if } \varepsilon\to 0,\\ & \text{for A},\quad x_1=0,\quad \tfrac{37}{40}x_1+4\leq x_2<5.\end{aligned} \tag{F.14}$$

$$\begin{aligned}\mathscr{D}^{\mathrm{VI}}:\quad & \tfrac{x_1+\lambda_1\varepsilon+\frac{27}{8}(3-\varepsilon)+\frac{3}{8}(x_2+\lambda_2\varepsilon)-12}{\mu_1\varepsilon}-0=\tfrac{8x_1+3x_2-15}{65\varepsilon}\to\infty,\ \text{if } \varepsilon\to 0,\\ & \text{for A},\quad \tfrac{15}{8}-\tfrac{3}{8}x_2\leq x_1\leq 15,\quad x_2\leq 4-\tfrac{1}{5}x_1.\end{aligned} \tag{F.15}$$

$$\begin{aligned}\mathscr{D}^{\mathrm{VI}}:\quad & 0-0=0,\\ & \text{for A},\quad x_1\leq\tfrac{15}{8}-\tfrac{3}{8}x_2,\quad x_2<4-\tfrac{1}{5}x_1.\end{aligned} \tag{F.16}$$

$$\dot{V}_{\mathrm{P2S}}=\begin{cases}\infty & \text{for A}, x_1\geq 15, x_2\geq 5,\\ \infty & \text{for A}, x_1\geq 15, 1\leq x_2<5,\\ \infty & \text{for A}, x_1\geq 15, x_2<1,\\ \infty & \text{for A}, \frac{40}{37}\leq x_1<15, x_2\geq 5,\\ 0 & \text{for A}, x_1<\frac{40}{37}, x_2\geq 5,\\ \infty & \text{for A}, \frac{15}{8}-\frac{3}{8}x_2\leq\frac{5}{8}x_2-\frac{605}{296}\leq x_1<15, 4-\frac{1}{5}x_1\leq x_2\leq\frac{37}{40}x_1+4\leq 5,\\ -\infty & \text{for A}, \frac{15}{8}-\frac{3}{8}x_2\leq x_1<\frac{5}{8}x_2-\frac{605}{296}<15, 4-\frac{1}{5}x_1\leq x_2\leq\frac{37}{40}x_1+4\leq 5,\\ \infty & \text{for A}, x_1<\frac{15}{8}-\frac{3}{8}x_2<15, 4-\frac{1}{5}x_1=x_2<\frac{37}{40}x_1+4<5,\\ -\infty & \text{for A}, x_1<\frac{15}{8}-\frac{3}{8}x_2<15, 4-\frac{1}{5}x_1\leq x_2<\frac{37}{40}x_1+4<5,\\ \infty & \text{for A}, \frac{15}{8}-\frac{3}{8}x_2\leq x_1<15, \frac{37}{40}x_1+4\leq x_2=5,\\ -\infty & \text{for A}, \frac{15}{8}-\frac{3}{8}x_2\leq x_1<15, \frac{37}{40}x_1+4\leq x_2<5,\\ -\infty & \text{for A}, 0<x_1<\frac{15}{8}-\frac{3}{8}x_2, \frac{37}{40}x_1+4\leq x_2<5,\\ \infty & \text{for A}, x_1=0, \frac{37}{40}x_1+4\leq x_2<5,\\ \infty & \text{for A}, \frac{15}{8}-\frac{3}{8}x_2\leq x_1<15, x_2<4-\frac{1}{5}x_1,\\ 0 & \text{for A}, x_1<\frac{15}{8}-\frac{3}{8}x_2, x_2<4-\frac{1}{5}x_1.\end{cases} \tag{F.17}$$

F.1.2 Mode B

Similar derivations are performed in case the system starts in Ⓑ. The switch to consider is switching to 🅐, with original setup time of $x_0 = 1$. The derivative is then equal to (F.18). It should be noted that specific parts of the domain in Ⓐ can not be entered via setup. Setup at least requires 1 time unit, in this one time unit $x_1 \geq 3$. Thus, $\mathscr{D}^{\mathrm{IV}}$ and $\mathscr{D}^{\mathrm{V}}$ are always skipped.

$$\lim_{\varepsilon \to 0} \frac{V\left(\text{🅐}, 1-\varepsilon, x_1(t+\varepsilon), x_2(t+\varepsilon)\right) - V\left(\text{Ⓑ}, 0, x_1(t), x_2(t)\right)}{\varepsilon}. \tag{F.18}$$

$$\begin{aligned} \mathscr{D}^{\mathrm{I}}: \quad & \tfrac{x_1+\lambda_1\varepsilon+\frac{27}{8}(1-\varepsilon)-15}{\mu_1\varepsilon} + \tfrac{x_2+\lambda_2\varepsilon+(1-\varepsilon)-1}{\mu_2\varepsilon} - \tfrac{x_2-8}{\mu_2\varepsilon} - \tfrac{x_1-9}{\mu_1\varepsilon} = \tfrac{45}{72\varepsilon} \to \infty, \text{ if } \varepsilon \to 0, \\ & \text{for Ⓑ}, \quad x_1 \geq 12, \quad x_2 \geq 8. \end{aligned} \tag{F.19}$$

$$\begin{aligned} \mathscr{D}^{\mathrm{I}}: \quad & \tfrac{x_2+\lambda_2\varepsilon+\frac{8}{5}(1-\varepsilon)+\frac{1}{5}(x_1+\lambda_1\varepsilon)-4}{\mu_2\varepsilon} - \tfrac{x_2-8}{\mu_2\varepsilon} - \tfrac{x_1-9}{\mu_1\varepsilon} = \tfrac{-37x_1+629}{360\varepsilon} \to \infty, \text{ if } \varepsilon \to 0, \\ & \text{for Ⓑ}, \quad 9 \leq x_1 < 12, \quad x_2 \geq 8. \end{aligned} \tag{F.20}$$

$$\begin{aligned} \mathscr{D}^{\mathrm{II}}: \quad & \tfrac{x_2+\lambda_2\varepsilon+\frac{8}{5}(1-\varepsilon)+\frac{1}{5}(x_1+\lambda_1\varepsilon)-4}{\mu_2\varepsilon} - \tfrac{x_2-8}{\mu_2\varepsilon} = \tfrac{x_1+28}{45\varepsilon} \to \infty, \text{ if } \varepsilon \to 0, \\ & \text{for Ⓑ}, \quad x_1 < 9, \quad x_2 \geq 8. \end{aligned} \tag{F.21}$$

$$\begin{aligned} \mathscr{D}^{\mathrm{III}}: \quad & \tfrac{x_2+\lambda_2\varepsilon+\frac{8}{5}(1-\varepsilon)+\frac{1}{5}(x_1+\lambda_1\varepsilon)-4}{\mu_2\varepsilon} - \tfrac{x_1+\frac{3}{8}x_2-12}{\mu_1\varepsilon} = \tfrac{-\frac{37}{8}x_1+\frac{185}{64}x_2+\frac{111}{2}}{45\varepsilon} \to \infty, \text{ if } \varepsilon \to 0, \\ & \text{for Ⓑ}, \quad 12-\tfrac{3}{8}x_2 \leq x_1 < 12+\tfrac{5}{8}x_2, \quad 4 \leq x_2 < 8. \end{aligned} \tag{F.22}$$

$$\begin{aligned} \mathscr{D}^{\mathrm{III}}: \quad & \tfrac{x_2+\lambda_2\varepsilon+\frac{8}{5}(1-\varepsilon)+\frac{1}{5}(x_1+\lambda_1\varepsilon)-4}{\mu_2\varepsilon} - \tfrac{x_1+\frac{3}{8}x_2-12}{\mu_1\varepsilon} = \tfrac{-\frac{37}{8}x_1+\frac{185}{64}x_2+\frac{111}{2}}{45\varepsilon} \to -\infty, \text{ if } \varepsilon \to 0, \\ & \text{for Ⓑ}, \quad 12-\tfrac{3}{8}x_2 \leq 12+\tfrac{5}{8}x_2 \leq x_1, \quad 4 \leq x_2 < 8. \end{aligned} \tag{F.23}$$

$$\begin{aligned} \mathscr{D}^{\mathrm{III}}: \quad & \tfrac{x_2+\lambda_2\varepsilon+\frac{8}{5}(1-\varepsilon)+\frac{1}{5}(x_1+\lambda_1\varepsilon)-4}{\mu_2\varepsilon} - 0 = \tfrac{5x_2+x_1-12}{45\varepsilon} \to \infty, \text{ if } \varepsilon \to 0, \\ & \text{for Ⓑ}, \quad x_1 < 12-\tfrac{3}{8}x_2, \quad \tfrac{12}{5}-\tfrac{1}{5}x_1 \leq x_2 < 8. \end{aligned} \tag{F.24}$$

$$\begin{aligned} \mathscr{D}^{\mathrm{III}}: \quad & 0 - \tfrac{x_1+\frac{3}{8}x_2-12}{\mu_1\varepsilon} = \tfrac{-x_1-\frac{3}{8}x_2+12}{8\varepsilon} \to -\infty, \text{ if } \varepsilon \to 0, \\ & \text{for Ⓑ}, \quad 12-\tfrac{3}{8}x_2 < x_1, \quad 0 < x_2 < \tfrac{12}{5}-\tfrac{1}{5}x_1 < 8. \end{aligned} \tag{F.25}$$

$$\begin{aligned} \mathscr{D}^{\mathrm{III}}: \quad & 0 - \tfrac{x_1+\frac{3}{8}x_2-12}{\mu_1\varepsilon} = \tfrac{-x_1-\frac{3}{8}x_2+12}{8\varepsilon} \to \infty, \text{ if } \varepsilon \to 0, \\ & \text{for Ⓑ}, \quad 12-\tfrac{3}{8}x_2 = x_1, \quad 0 < x_2 < \tfrac{12}{5}-\tfrac{1}{5}x_1 < 8. \end{aligned} \tag{F.26}$$

$$\begin{aligned} \mathscr{D}^{\mathrm{III}}: \quad & 0 - \tfrac{x_1-12}{\mu_1\varepsilon} = \tfrac{-x_1+12}{8\varepsilon} \to -\infty, \text{ if } \varepsilon \to 0, \\ & \text{for Ⓑ}, \quad 12 < x_1, \quad x_2 = 0. \end{aligned} \tag{F.27}$$

$$\begin{aligned} \mathscr{D}^{\mathrm{III}}: \quad & \tfrac{x_2+\lambda_2\varepsilon+\frac{8}{5}(1-\varepsilon)+\frac{1}{5}(x_1+\lambda_1\varepsilon)-4}{\mu_2\varepsilon} - 0 = \tfrac{5x_2+x_1-12}{45\varepsilon} \to \infty, \text{ if } \varepsilon \to 0, \\ & \text{for Ⓑ}, \quad x_1 < 12-\tfrac{3}{8}x_2, \quad 4 \leq x_2 < 8. \end{aligned} \tag{F.28}$$

$$\begin{aligned} \mathscr{D}^{\mathrm{VI}}: \quad & \tfrac{x_2+\lambda_2\varepsilon+\frac{8}{5}(1-\varepsilon)+\frac{1}{5}(x_1+\lambda_1\varepsilon)-4}{\mu_2\varepsilon} - 0 = \tfrac{5x_2+x_1-12}{45\varepsilon} \to \infty, \text{ if } \varepsilon \to 0, \\ & \text{for Ⓑ}, \quad x_1 < 12-\tfrac{3}{8}x_2, \quad \tfrac{12}{5}-\tfrac{1}{5}x_1 \leq x_2 < 4. \end{aligned} \tag{F.29}$$

$$\begin{aligned} \mathscr{D}^{\mathrm{VI}}: \quad & 0 - 0 = 0, \\ & \text{for Ⓑ}, \quad x_1 < 12-\tfrac{3}{8}x_2, \quad x_2 < \tfrac{12}{5}-\tfrac{1}{5}x_1 < 8. \end{aligned} \tag{F.30}$$

$$\dot{V}_{\mathrm{P2S}} = \begin{cases} \infty & \text{for B}, x_1 \geq 12, x_2 \geq 8, \\ \infty & \text{for B}, 9 \leq x_1 < 12, x_2 \geq 8, \\ \infty & \text{for B}, x_1 < 9, x_2 \geq 8, \\ \infty & \text{for B}, 12 - \frac{3}{8}x_2 \leq x_1 < 12 + \frac{5}{8}x_2, 4 \leq x_2 < 8, \\ -\infty & \text{for B}, 12 - \frac{3}{8}x_2 \leq 12 + \frac{5}{8}x_2 < x_1, 4 \leq x_2 < 8, \\ \infty & \text{for B}, x_1 < 12 - \frac{3}{8}x_2, \frac{12}{5} - \frac{1}{5}x_1 \leq x_2 < 8, \\ -\infty & \text{for B}, 12 - \frac{3}{8}x_2 \leq x_1, 0 < x_2 < \frac{12}{5} - \frac{1}{5}x_1 < 8, \\ \infty & \text{for B}, 12 - \frac{3}{8}x_2 = x_1, 0 < x_2 < \frac{12}{5} - \frac{1}{5}x_1 < 8, \\ -\infty & \text{for B}, 12 < x_1, x_2 = 0, \\ \infty & \text{for B}, x_1 < 12 - \frac{3}{8}x_2, 4 \leq x_2 < 8, \\ \infty & \text{for B}, x_1 < 12 - \frac{3}{8}x_2, \frac{12}{5} - \frac{1}{5}x_1 \leq x_2 < 4, \\ 0 & \text{for B}, x_1 < 12 - \frac{3}{8}x_2, x_2 < \frac{12}{5} - \frac{1}{5}x_1 < 8. \end{cases} \tag{F.31}$$

F.2 Derivative if Server Switches between Setups

If the server is in setup and continues setup of the same mode, the candidate Lyapunov function equals zero by definition. If the server is in setup of one mode and switches to another however, determining the the candidate Lyapunov function derivative is extensive. The derivative is only of interest at the first time instant the server is started, it should be ensured that the complete setup time is executed by the server. This leaves to conclude that $x_0 = 3$ for B and $x_0 = 1$ for A.

F.2.1 Mode *A*

Starting the derivation for cases the server started in A, the subsequent mode would be A if $x_0 = 0$ and the mode was finished, but the server can only switch to B. Implementing the buffer contents during setup in (F.32), the candidate Lyapunov function derivative if the server started in A is derived.

$$\lim_{\varepsilon \to 0} \frac{V\left(\text{B}, 3 - \varepsilon, x_1\left(t + \varepsilon\right), x_2\left(t + \varepsilon\right)\right) - V\left(\text{A}, 1, x_1\left(t\right), x_2\left(t\right)\right)}{\varepsilon}. \tag{F.32}$$

$$\begin{gathered} \frac{x_1+\lambda_1\varepsilon+3(3-\varepsilon)-9}{\mu_1\varepsilon} + \frac{x_2+\lambda_2\varepsilon+3-\varepsilon-8}{\mu_2\varepsilon} - \frac{x_1+3-15}{\mu_1\varepsilon} - \frac{x_2+1-1}{\mu_2\varepsilon} = \frac{68}{72\varepsilon} \to \infty, \text{ if } \varepsilon \to 0, \\ \text{for A}, \quad x_1 \geq 12, \quad x_2 \geq 5. \end{gathered} \tag{F.33}$$

$$\begin{gathered} \frac{x_1+\lambda_1\varepsilon+3(3-\varepsilon)-9}{\mu_1\varepsilon} + \frac{x_2+\lambda_2\varepsilon+3-\varepsilon-8}{\mu_2\varepsilon} - \frac{x_2+\frac{8}{5}+\frac{1}{5}x_1-4}{\mu_2\varepsilon} = \frac{\frac{37}{8}x_1-13}{45\varepsilon} \to \infty, \text{ if } \varepsilon \to 0, \\ \text{for A}, \quad \frac{104}{37} \leq x_1 < 12, \quad x_2 \geq 5. \end{gathered} \tag{F.34}$$

$$\begin{gathered} \frac{x_1+\lambda_1\varepsilon+3(3-\varepsilon)-9}{\mu_1\varepsilon} + \frac{x_2+\lambda_2\varepsilon+3-\varepsilon-8}{\mu_2\varepsilon} - \frac{x_2+\frac{8}{5}+\frac{1}{5}x_1-4}{\mu_2\varepsilon} = \frac{\frac{37}{8}x_1-13}{45\varepsilon} \to -\infty, \text{ if } \varepsilon \to 0, \\ \text{for A}, \quad x_1 < \frac{104}{37}, \quad x_2 \geq 5. \end{gathered} \tag{F.35}$$

$$\begin{gathered} \frac{x_1+\lambda_1\varepsilon+\frac{27}{8}(3-\varepsilon)+\frac{3}{8}(x_2+\lambda_2\varepsilon)-12}{\mu_1\varepsilon} - \frac{x_2+\frac{8}{5}+\frac{1}{5}x_1-4}{\mu_2\varepsilon} = \frac{\frac{37}{5}x_1+\frac{93}{40}-\frac{37}{8}x_2}{72\varepsilon} \to \infty, \text{ if } \varepsilon \to 0, \\ \text{for A}, \quad \frac{15}{8} - \frac{3}{8}x_2 \leq \frac{5}{8}x_2 - \frac{93}{296} \leq x_1 < 12, \quad 4 \leq x_2 < 5. \end{gathered} \tag{F.36}$$

$$\begin{gathered} \frac{x_1+\lambda_1\varepsilon+\frac{27}{8}(3-\varepsilon)+\frac{3}{8}(x_2+\lambda_2\varepsilon)-12}{\mu_1\varepsilon} - \frac{x_2+\frac{8}{5}+\frac{1}{5}x_1-4}{\mu_2\varepsilon} = \frac{\frac{37}{5}x_1+\frac{93}{40}-\frac{37}{8}x_2}{72\varepsilon} \to -\infty, \text{ if } \varepsilon \to 0, \\ \text{for A}, \quad \frac{15}{8} - \frac{3}{8}x_2 \leq x_1 < \frac{5}{8}x_2 - \frac{93}{296} < 12, \quad 4 \leq x_2 < 5. \end{gathered} \tag{F.37}$$

$$\begin{gathered} 0 - \frac{x_2+\frac{8}{5}+\frac{1}{5}x_1-4}{\mu_2\varepsilon} = \frac{-x_2-\frac{1}{5}x_1+\frac{12}{5}}{9\varepsilon} \to \infty, \text{ if } \varepsilon \to 0, \\ \text{for A}, \quad x_1 < \frac{15}{8} - \frac{3}{8}x_2 < 12, \quad 4 \leq x_2 < 5. \end{gathered} \tag{F.38}$$

$$\begin{gathered} \frac{x_1+\lambda_1\varepsilon+\frac{27}{8}(3-\varepsilon)+\frac{3}{8}(x_2+\lambda_2\varepsilon)-12}{\mu_1\varepsilon} - \frac{x_2+\frac{8}{5}+\frac{1}{5}x_1-4}{\mu_2\varepsilon} = \frac{\frac{37}{5}x_1+\frac{93}{40}-\frac{37}{8}x_2}{72\varepsilon} \to \infty, \text{ if } \varepsilon \to 0, \\ \text{for A}, \quad \frac{15}{8} - \frac{3}{8}x_2 \leq \frac{5}{8}x_2 - \frac{93}{296} \leq x_1 < 12, \quad \frac{12}{5} - \frac{1}{5}x_1 \leq x_2 < 4. \end{gathered} \tag{F.39}$$

$$\begin{gathered}\frac{x_1+\lambda_1\varepsilon+\frac{27}{8}(3-\varepsilon)+\frac{3}{8}(x_2+\lambda_2\varepsilon)-12}{\mu_1\varepsilon}-\frac{x_2+\frac{8}{5}+\frac{1}{5}x_1-4}{\mu_2\varepsilon}=\frac{\frac{37}{5}x_1+\frac{93}{40}-\frac{37}{8}x_2}{72\varepsilon}\to-\infty,\ \text{if } \varepsilon\to 0,\\ \text{for } \textcircled{A},\quad \tfrac{15}{8}-\tfrac{3}{8}x_2\le x_1<\tfrac{5}{8}x_2-\tfrac{93}{296}<12,\quad \tfrac{12}{5}-\tfrac{1}{5}x_1\le x_2<4.\end{gathered}\tag{F.40}$$

$$\begin{gathered}0-\frac{x_2+\frac{8}{5}+\frac{1}{5}x_1-4}{\mu_2\varepsilon}=\frac{-x_2-\frac{1}{5}x_1+\frac{12}{5}}{9\varepsilon}\to\infty,\ \text{if } \varepsilon\to 0,\\ \text{for } \textcircled{A},\quad x_1<\tfrac{15}{8}-\tfrac{3}{8}x_2<12,\quad \tfrac{12}{5}-\tfrac{1}{5}x_1\le x_2<4.\end{gathered}\tag{F.41}$$

$$\begin{gathered}\frac{x_1+\lambda_1\varepsilon+\frac{27}{8}(3-\varepsilon)+\frac{3}{8}(x_2+\lambda_2\varepsilon)-12}{\mu_1\varepsilon}-0=\frac{8x_1+3x_2-15}{64\varepsilon}\to\infty,\ \text{if } \varepsilon\to 0,\\ \text{for } \textcircled{A},\quad \tfrac{15}{8}-\tfrac{3}{8}x_2\le x_1<12,\quad x_2<\tfrac{12}{5}-\tfrac{1}{5}x_1.\end{gathered}\tag{F.42}$$

$$\begin{gathered}0-0=0\\ \text{for } \textcircled{A},\quad x_1<\tfrac{15}{8}-\tfrac{3}{8}x_2<12,\quad x_2<\tfrac{12}{5}-\tfrac{1}{5}x_1.\end{gathered}\tag{F.43}$$

$$\dot{V}_{\mathrm{S2S}}=\begin{cases}\infty & \text{for } \textcircled{A}, x_1\ge 12, x_2\ge 5,\\ \infty & \text{for } \textcircled{A}, \frac{104}{37}\le x_1<12, x_2\ge 5,\\ -\infty & \text{for } \textcircled{A}, x_1<\frac{104}{37}, x_2\ge 5,\\ \infty & \text{for } \textcircled{A}, \frac{15}{8}-\frac{3}{8}x_2\le\frac{5}{8}x_2-\frac{93}{296}\le x_1<12, 4\le x_2<5,\\ -\infty & \text{for } \textcircled{A}, \frac{15}{8}-\frac{3}{8}x_2\le x_1<\frac{5}{8}x_2-\frac{93}{296}\le 12, 4\le x_2<5,\\ \infty & \text{for } \textcircled{A}, x_1<\frac{15}{8}-\frac{3}{8}x_2<12, 4\le x_2<5,\\ \infty & \text{for } \textcircled{A}, \frac{15}{8}-\frac{3}{8}x_2\le\frac{5}{8}x_2-\frac{93}{296}\le x_1<12, \frac{12}{5}-\frac{1}{5}x_1\le x_2<4,\\ -\infty & \text{for } \textcircled{A}, \frac{15}{8}-\frac{3}{8}x_2\le x_1<\frac{5}{8}x_2-\frac{93}{296}<12, \frac{12}{5}-\frac{1}{5}x_1\le x_2<4,\\ \infty & \text{for } \textcircled{A}, x_1<\frac{15}{8}-\frac{3}{8}x_2<12, \frac{12}{5}-\frac{1}{5}x_1\le x_2<4,\\ \infty & \text{for } \textcircled{A}, \frac{15}{8}-\frac{3}{8}x_2\le x_1<12, x_2<\frac{12}{5}-\frac{1}{5}x_1,\\ 0 & \text{for } \textcircled{A}, x_1<\frac{15}{8}-\frac{3}{8}x_2<12, x_2<\frac{12}{5}-\frac{1}{5}x_1.\end{cases}\tag{F.44}$$

F.2.2 Mode *B*

In case the server starts in $\textcircled{B}$, the server can only switch to $\textcircled{A}$. Implementing the buffer contents during setup in (F.45), the candidate Lyapunov function derivative if the server starts in $\textcircled{B}$ is derived.

$$\lim_{\varepsilon\to 0}\frac{V\left(\textcircled{B},3-\varepsilon,x_1(t+\varepsilon),x_2(t+\varepsilon)\right)-V\left(\textcircled{A},1,x_1(t),x_2(t)\right)}{\varepsilon}.\tag{F.45}$$

$$\begin{gathered}\frac{x_1+\lambda_1\varepsilon+3(1-\varepsilon)-15}{\mu_1\varepsilon}+\frac{x_2+\lambda_2\varepsilon+(1-\varepsilon)-1}{\mu_2\varepsilon}-\frac{x_1+9-9}{\mu_1\varepsilon}-\frac{x_2+3-8}{\mu_2\varepsilon}=\frac{-68}{72\varepsilon}\to-\infty,\ \text{if } \varepsilon\to 0,\\ \text{for } \textcircled{B},\quad x_1\ge 12,\quad x_2\ge 5.\end{gathered}\tag{F.46}$$

$$\begin{gathered}\frac{x_2+\lambda_2\varepsilon+\frac{8}{5}(1-\varepsilon)+\frac{1}{5}(x_1+\lambda_1\varepsilon)-4}{\mu_2\varepsilon}-\frac{x_1+9-9}{\mu_1\varepsilon}-\frac{x_2+3-8}{\mu_2\varepsilon}=\frac{13-\frac{37}{8}x_1}{45\varepsilon}\to\infty,\ \text{if } \varepsilon\to 0,\\ \text{for } \textcircled{B},\quad x_1<\tfrac{104}{37},\quad x_2\ge 5.\end{gathered}\tag{F.47}$$

$$\begin{gathered}\frac{x_2+\lambda_2\varepsilon+\frac{8}{5}(1-\varepsilon)+\frac{1}{5}(x_1+\lambda_1\varepsilon)-4}{\mu_2\varepsilon}-\frac{x_1+9-9}{\mu_1\varepsilon}-\frac{x_2+3-8}{\mu_2\varepsilon}=\frac{13-\frac{37}{8}x_1}{45\varepsilon}\to-\infty,\ \text{if } \varepsilon\to 0,\\ \text{for } \textcircled{B},\quad \tfrac{104}{37}\le x_1<12,\quad x_2\ge 5.\end{gathered}\tag{F.48}$$

$$\begin{gathered}\frac{x_1+\lambda_1\varepsilon+3(1-\varepsilon)-15}{\mu_1\varepsilon}+\frac{x_2+\lambda_2\varepsilon+(1-\varepsilon)-1}{\mu_2\varepsilon}-\frac{x_1+\frac{3}{8}x_2-\frac{15}{8}}{\mu_1\varepsilon}=\frac{\frac{37}{9}x_2-81}{64\varepsilon}\to-\infty,\ \text{if } \varepsilon\to 0,\\ \text{for } \textcircled{B},\quad x_1\ge 12,\quad x_2<5.\end{gathered}\tag{F.49}$$

$$\begin{gathered}\frac{x_2+\lambda_2\varepsilon+\frac{8}{5}(1-\varepsilon)+\frac{1}{5}(x_1+\lambda_1\varepsilon)-4}{\mu_2\varepsilon}-\frac{x_1+\frac{3}{8}x_2-\frac{15}{8}}{\mu_1\varepsilon}=\frac{\frac{37}{9}x_2-\frac{296}{45}x_1-\frac{31}{15}}{64\varepsilon}\to\infty,\ \text{if } \varepsilon\to 0,\\ \text{for } \textcircled{B},\quad \tfrac{15}{8}-\tfrac{3}{8}x_2\le x_1<12,\quad 4\le\tfrac{93}{185}+\tfrac{8}{5}x_1\le x_2<5.\end{gathered}\tag{F.50}$$

$$\begin{gathered}\frac{x_2+\lambda_2\varepsilon+\frac{8}{5}(1-\varepsilon)+\frac{1}{5}(x_1+\lambda_1\varepsilon)-4}{\mu_2\varepsilon}-\frac{x_1+\frac{3}{8}x_2-\frac{15}{8}}{\mu_1\varepsilon}=\frac{\frac{37}{9}x_2-\frac{296}{45}x_1-\frac{31}{15}}{64\varepsilon}\to-\infty,\ \text{if } \varepsilon\to 0,\\ \text{for } \textcircled{B},\quad \tfrac{15}{8}-\tfrac{3}{8}x_2\le x_1<12,\quad 4\le x_2<\tfrac{93}{185}+\tfrac{8}{5}x_1<5.\end{gathered}\tag{F.51}$$

$$\begin{gathered}\frac{x_2+\lambda_2\varepsilon+\frac{8}{5}(1-\varepsilon)+\frac{1}{5}(x_1+\lambda_1\varepsilon)-4}{\mu_2\varepsilon} - 0 = \frac{5x_2+x_1-12}{45\varepsilon} \to \infty,\ \text{if } \varepsilon \to 0,\\ \text{for } \textcircled{B},\quad x_1 < \tfrac{15}{8} - \tfrac{3}{8}x_2 < 12,\quad 4 \le x_2 < 5.\end{gathered} \tag{F.52}$$

$$\begin{gathered}\frac{x_2+\lambda_2\varepsilon+\frac{8}{5}(1-\varepsilon)+\frac{1}{5}(x_1+\lambda_1\varepsilon)-4}{\mu_2\varepsilon} - \frac{x_1+\frac{3}{8}x_2-\frac{15}{8}}{\mu_1\varepsilon} = \frac{\frac{37}{9}x_2-\frac{296}{45}x_1-\frac{31}{15}}{64\varepsilon} \to \infty,\ \text{if } \varepsilon \to 0,\\ \text{for } \textcircled{B},\quad \tfrac{15}{8} - \tfrac{3}{8}x_2 \le x_1 < 12,\quad \tfrac{12}{5} - \tfrac{1}{5}x_1 \le \tfrac{93}{185} + \tfrac{8}{5}x_1 \le x_2 < 5.\end{gathered} \tag{F.53}$$

$$\begin{gathered}\frac{x_2+\lambda_2\varepsilon+\frac{8}{5}(1-\varepsilon)+\frac{1}{5}(x_1+\lambda_1\varepsilon)-4}{\mu_2\varepsilon} - \frac{x_1+\frac{3}{8}x_2-\frac{15}{8}}{\mu_1\varepsilon} = \frac{\frac{37}{9}x_2-\frac{296}{45}x_1-\frac{31}{15}}{64\varepsilon} \to -\infty,\ \text{if } \varepsilon \to 0,\\ \text{for } \textcircled{B},\quad \tfrac{15}{8} - \tfrac{3}{8}x_2 \le x_1 < 12,\quad \tfrac{12}{5} - \tfrac{1}{5}x_1 \le x_2 < \tfrac{93}{185} + \tfrac{8}{5}x_1 < 5.\end{gathered} \tag{F.54}$$

$$\begin{gathered}\frac{x_2+\lambda_2\varepsilon+\frac{8}{5}(1-\varepsilon)+\frac{1}{5}(x_1+\lambda_1\varepsilon)-4}{\mu_2\varepsilon} - 0 = \frac{5x_2+x_1-12}{45\varepsilon} \to \infty,\ \text{if } \varepsilon \to 0,\\ \text{for } \textcircled{B},\quad x_1 < \tfrac{15}{8} - \tfrac{3}{8}x_2 < 12,\quad \tfrac{12}{5} - \tfrac{1}{5}x_1 \le x_2 < 5.\end{gathered} \tag{F.55}$$

$$\begin{gathered}0 - \frac{x_1+\frac{3}{8}x_2-\frac{15}{8}}{\mu_1\varepsilon} = \frac{-x_1-\frac{3}{8}x_2+\frac{15}{8}}{8\varepsilon} \to -\infty,\ \text{if } \varepsilon \to 0,\\ \text{for } \textcircled{B},\quad \tfrac{15}{8} - \tfrac{3}{8}x_2 \le x_1 < 12,\quad x_2 < \tfrac{12}{5} - \tfrac{1}{5}x_1.\end{gathered} \tag{F.56}$$

$$\begin{gathered}0 - 0 = 0,\\ \text{for } \textcircled{B},\quad x_1 < \tfrac{15}{8} - \tfrac{3}{8}x_2 < 12,\quad x_2 < \tfrac{12}{5} - \tfrac{1}{5}x_1.\end{gathered} \tag{F.57}$$

$$\dot{V}_{\mathrm{S2S}} = \begin{cases} -\infty & \text{for } \textcircled{B}, x_1 \ge 12, x_2 \ge 5,\\ \infty & \text{for } \textcircled{B}, x_1 < \frac{104}{37}, x_2 \ge 5,\\ -\infty & \text{for } \textcircled{B}, \frac{104}{37} \le x_1 < 12, x_2 \ge 5,\\ -\infty & \text{for } \textcircled{B}, x_1 \ge 12, x_2 < 5,\\ \infty & \text{for } \textcircled{B}, \frac{15}{8} - \frac{3}{8}x_2 \le x_1 < 12, 4 \le \frac{93}{185} + \frac{8}{5}x_1 \le x_2 < 5,\\ -\infty & \text{for } \textcircled{B}, \frac{15}{8} - \frac{3}{8}x_2 \le x_1 < 12, 4 \le x_2 < \frac{93}{185} + \frac{8}{5}x_1 < 5,\\ \infty & \text{for } \textcircled{B}, x_1 < \frac{15}{8} - \frac{3}{8}x_2 < 12, 4 \le x_2 < 5,\\ \infty & \text{for } \textcircled{B}, \frac{15}{8} - \frac{3}{8}x_2 \le x_1 < 12, \frac{12}{5} - \frac{1}{5}x_1 \le \frac{93}{185} + \frac{8}{5}x_1 \le x_2 < 5,\\ -\infty & \text{for } \textcircled{B}, \frac{15}{8} - \frac{3}{8}x_2 \le x_1 < 12, \frac{12}{5} - \frac{1}{5}x_1 \le x_2 < \frac{93}{185} + \frac{8}{5}x_1 < 5,\\ \infty & \text{for } \textcircled{B}, x_1 < \frac{15}{8} - \frac{3}{8}x_2 < 12, \frac{12}{5} - \frac{1}{5}x_1 \le x_2 < 5,\\ -\infty & \text{for } \textcircled{B}, \frac{15}{8} - \frac{3}{8}x_2 \le x_1 < 12, x_2 < \frac{12}{5} - \frac{1}{5}x_1,\\ 0 & \text{for } \textcircled{B}, x_1 < \frac{15}{8} - \frac{3}{8}x_2 < 12, x_2 < \frac{12}{5} - \frac{1}{5}x_1. \end{cases} \tag{F.58}$$

Combining the expressions of the candidate Lyapunov function at switching instants for switching to setup and switching between setups, and present these derivative in reduced form, gives respectively (F.59) and (F.60). These expression are compared to (4.3) in Chapter 4. The option that results in the minimum derivative value, presents the control action to perform in that part of the domain.

$$\dot{V}_{\mathrm{P2S}} = \begin{cases} \infty & \text{for } Ⓐ, x_1 \geq 15, \\ \infty & \text{for } Ⓐ, \frac{40}{37} \leq x_1 \leq 15, x_2 \geq 5, \\ 0 & \text{for } Ⓐ, x_1 < \frac{40}{37}, x_2 \geq 5, \\ \infty & \text{for } Ⓐ, \frac{15}{8} - \frac{3}{8}x_2 \leq \frac{5}{8}x_2 - \frac{605}{296} \leq x_1 < 15, 4 - \frac{1}{5}x_1 \leq x_2 < \frac{37}{40}x_1 + 4 < 5, \\ -\infty & \text{for } Ⓐ, \frac{15}{8} - \frac{3}{8}x_2 \leq x_1 < \frac{5}{8}x_2 - \frac{605}{296} < 15, 4 - \frac{1}{5}x_1 \leq x_2 < \frac{37}{40}x_1 + 4 < 5, \\ \infty & \text{for } Ⓐ, x_1 < \frac{15}{8} - \frac{3}{8}x_2 < 15, 4 - \frac{1}{5}x_1 = x_2 < \frac{37}{40}x_1 + 4 < 5, \\ -\infty & \text{for } Ⓐ, x_1 < \frac{15}{8} - \frac{3}{8}x_2 < 15, 4 - \frac{1}{5}x_1 \leq x_2 < \frac{37}{40}x_1 + 4 < 5, \\ \infty & \text{for } Ⓐ, \frac{15}{8} - \frac{3}{8}x_2 \leq x_1 < 15, \frac{37}{40}x_1 + 4 \leq x_2 = 5, \\ -\infty & \text{for } Ⓐ, 0 < x_1 < 15, \frac{37}{40}x_1 + 4 \leq x_2 < 5, \\ \infty & \text{for } Ⓐ, x_1 = 0, 4 \leq x_2 < 5, \\ \infty & \text{for } Ⓐ, \frac{15}{8} - \frac{3}{8}x_2 \leq x_1 < 15, x_2 < 4 - \frac{1}{5}x_1, \\ 0 & \text{for } Ⓐ, x_1 < \frac{15}{8} - \frac{3}{8}x_2, x_2 < 4 - \frac{1}{5}x_1, \\ \infty & \text{for } Ⓑ, x_2 \geq 8, \\ \infty & \text{for } Ⓑ, 12 - \frac{3}{8}x_2 \leq x_1 < 12 + \frac{5}{8}x_2, 4 \leq x_2 < 8, \\ -\infty & \text{for } Ⓑ, 12 - \frac{3}{8}x_2 \leq 12 + \frac{5}{8}x_2 \leq x_1, 4 \leq x_2 < 8, \\ \infty & \text{for } Ⓑ, x_1 < 12 - \frac{3}{8}x_2, 0 < \frac{12}{5} - \frac{1}{5}x_1 \leq x_2 < 8, \\ -\infty & \text{for } Ⓑ, 12 - \frac{3}{8}x_2 \leq x_1, 0 \leq x_2 < \frac{12}{5} - \frac{1}{5}x_1 < 8, \\ 0 & \text{for } Ⓑ, x_1 < 12 - \frac{3}{8}x_2, x_2 < \frac{12}{5} - \frac{1}{5}x_1 < 8. \end{cases} \tag{F.59}$$

$$\dot{V}_{\mathrm{S2S}} = \begin{cases} \infty & \text{for } 🅐, x_1 \geq \frac{104}{37}, x_2 \geq 5, \\ -\infty & \text{for } 🅐, x_1 < \frac{104}{37}, x_2 \geq 5, \\ \infty & \text{for } 🅐, \frac{15}{8} - \frac{3}{8}x_2 \leq \frac{5}{8}x_2 - \frac{93}{296} \leq x_1 < 12, \frac{12}{5} - \frac{1}{5}x_1 \leq x_2 < 5, \\ -\infty & \text{for } 🅐, \frac{15}{8} - \frac{3}{8}x_2 \leq x_1 < \frac{5}{8}x_2 - \frac{93}{296} < 12, \frac{12}{5} - \frac{1}{5}x_1 \leq x_2 < 5, \\ \infty & \text{for } 🅐, x_1 < \frac{15}{8} - \frac{3}{8}x_2 < 12, \frac{12}{5} - \frac{1}{5}x_1 \leq x_2 < 5, \\ \infty & \text{for } 🅐, \frac{15}{8} - \frac{3}{8}x_2 \leq x_1 < 12, x_2 < \frac{12}{5} - \frac{1}{5}x_1, \\ 0 & \text{for } 🅐, x_1 < \frac{15}{8} - \frac{3}{8}x_2 < 12, x_2 < \frac{12}{5} - \frac{1}{5}x_1, \\ -\infty & \text{for } 🅑, x_1 \geq \frac{104}{37}, x_2 \geq 5, \\ \infty & \text{for } 🅑, x_1 < \frac{104}{37}, x_2 \geq 5, \\ -\infty & \text{for } 🅑, x_1 \geq 12, x_2 < 5, \\ \infty & \text{for } 🅑, \frac{15}{8} - \frac{3}{8}x_2 \leq x_1 < 12, \frac{12}{5} - \frac{1}{5}x_1 \leq \frac{93}{185} + \frac{8}{5}x_1 \leq x_2 < 5, \\ -\infty & \text{for } 🅑, \frac{15}{8} - \frac{3}{8}x_2 \leq x_1 < 12, \frac{12}{5} - \frac{1}{5}x_1 \leq x_2 < \frac{93}{185} + \frac{8}{5}x_1 < 5, \\ \infty & \text{for } 🅑, x_1 < \frac{15}{8} - \frac{3}{8}x_2 < 12, \frac{12}{5} - \frac{1}{5}x_1 \leq x_2 < 5, \\ -\infty & \text{for } 🅑, \frac{15}{8} - \frac{3}{8}x_2 \leq x_1 < 12, x_2 < \frac{12}{5} - \frac{1}{5}x_1, \\ 0 & \text{for } 🅑, x_1 < \frac{15}{8} - \frac{3}{8}x_2 < 12, x_2 < \frac{12}{5} - \frac{1}{5}x_1. \end{cases} \tag{F.60}$$

Appendix G. Undefined Control Action

This appendix lists the control actions to be performed, in specific parts of the domain in the two direction system. The domain parts in which no conclusions could be made regarding the control action to perform, by examining the candidate Lyapunov function derivative. The solution presented in this appendix provides a time efficient solution regarding convergence of the system. Although it results in a time efficient solution with respect to the convergence rate, it unnecessarily complicates the control actions expression. Another option is that the server continues in the current mode, which results in a less complex description of control actions and provides a stabilizing controller as well. The research objective was to find a controller that is easy to implement, because of that the results presented in this appendix are disregarded.
Figure G.1 to Figure G.4, visualize the part of mode $\boldsymbol{A}$ and mode $\boldsymbol{B}$, in which the control action is undefined. These parts of the domain are unaccounted for in the derivation presented in [1], ergo this problem does not occur in [1].

G.1 Undefined Domain in Processing

In parts of the domain in Ⓐ and Ⓑ both server options, continue processing in the current mode and perform setup of the subsequent mode, result in $\dot{V}(s, x_1, x_2) = 0$. Therefore, no conclusion can be made by evaluating the candidate Lyapunov function derivative, regarding the control action to perform in these parts of the domain. Figure G.1 and Figure G.2 are graphical representations of the domain parts in respectively Ⓐ and Ⓑ, in which the control action is undetermined.

Although the choice of control action does not result in different candidate Lyapunov function values, the difference between both actions might appear in the time it takes the system to stabilize the system in optimal periodic behaviour. Figure G.1 shows the part of the domain of mode $\boldsymbol{A}$ where the control action is yet to be defined. The time it takes to obtain the desired periodic behaviour, is set as a decision variable. By minimizing the decision variable, the control action is designed such that the system converges as quickly as possible.

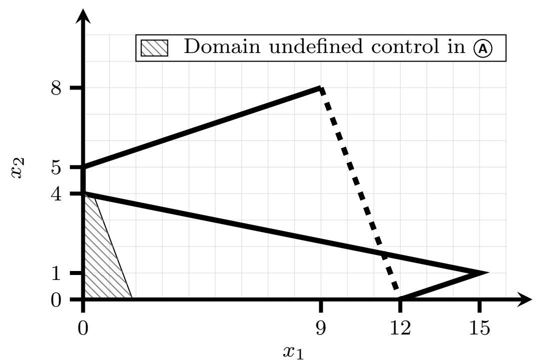

Figure G.1: $\dot{V}\left(Ⓐ, x_1, x_2\right) = 0$.

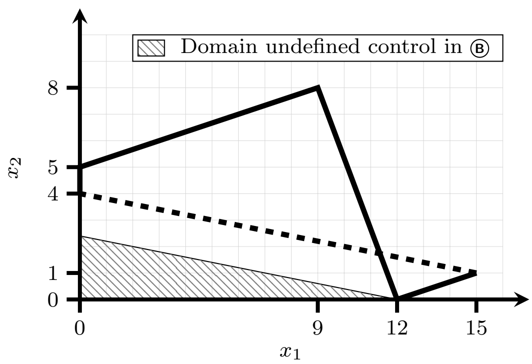

Figure G.2: $\dot{V}\left(Ⓑ, x_1, x_2\right) = 0$.

The time it takes to obtain the desired periodic behaviour via continuing processing in mode $\boldsymbol{A}$ is referred to as $t_{Ⓐ}$. In case $x_2 = 4 - \frac{1}{5}x_1$ the system is in periodic behaviour immediately when the server continues in Ⓐ. For all other values of x_2 in this part of the domain, periodic behaviour is obtained when $x_2 = 4$. With x_2 increases with rate λ_2, combined with the previous statements this gives an expression of $t_{Ⓐ}$, (G.1).
The time it takes to obtain the desired periodic behaviour in mode $\boldsymbol{A}$, by switching to setup in mode $\boldsymbol{B}$, is referred to as $t_{Ⓐ,s}$. If after setup the system values are on the dashed line illustrated in Figure G.1, the system is in periodic behaviour after $\sigma_{🅑} = 3$. In all other parts of the domain, the time it takes to obtain periodic behaviour equals the time it takes x_1 to equal 12. This gives an expression of the time it takes to obtain optimal periodic behaviour in case the system switches to 🅑, $t_{Ⓐ,s}$ namely (G.2).
If $t_{Ⓐ,s} < t_{Ⓐ}$, it is time efficient to switch to setup of mode $\boldsymbol{B}$, else the server should continue in Ⓐ. Therefore, if (G.3) or (G.3) holds, the server should switch to setup of mode $\boldsymbol{B}$.

$$t_{Ⓐ} = \begin{cases} 0 & \text{if }, x_2 = 4 - \frac{1}{5}x_1, \\ \frac{4-x_2}{\lambda_2} = 4 - x_2 & \text{if }, x_2 \neq 4 - \frac{1}{5}x_1. \end{cases} \tag{G.1}$$

$$t_{Ⓐ,s} = \begin{cases} \sigma_{🅑} & \text{if }, x_2 = 5 - \frac{8}{3}x_1, \\ \sigma_{🅑} + \frac{12-\left(x_1+\sigma_{🅑}\lambda_1\right)}{\lambda_1} = 4 - \frac{1}{3}x_1 & \text{if }, x_2 \neq 5 - \frac{8}{3}x_1. \end{cases} \tag{G.2}$$

$$x_1 > \frac{1}{3} \text{ for } x_1 \leq 15, x_2 = 5 - \frac{8}{3}x_1 \leq 4 - \frac{1}{5}x_1. \tag{G.3}$$

$$x_1 > 3x_2 \text{ for } x_1 \leq 15, x_2 < 5 - \frac{8}{3}x_1 \leq 4 - \frac{1}{5}x_1. \tag{G.4}$$

Similarly a control action can be defined in the undetermined domain part in Ⓑ. In Figure G.2, the part of the domain where control actions have yet to be determined is hashed. The time it takes to obtain the desired periodic behaviour if the server continues in Ⓑ, is referred to as $t_{Ⓑ}$. If the server continues in Ⓑ and $x_1 = 12 - \frac{3}{8}x_2$ then the system is immediately in optimal periodic behaviour. In all other parts of the domain, the time it takes to obtain optimal periodic behaviour equals the time it takes for x_1 to become 12. This gives the expression for $t_{Ⓑ}$, (G.5).
The travel time from the starting values to the optimal periodic behaviour is referred to as $t_{Ⓑ,s}$, if the server switches to mode $\boldsymbol{A}$. If after setup is performed the system values are on the dashed line illustrated in Figure G.2, the system is in periodic behaviour after $\sigma_{🅐} = 1$ time units. For all other values, periodic behaviour is obtained when x_2 equals 4. Both statements combined result in the expression of $t_{Ⓑ,s}$, (G.6).
If $t_{Ⓑ,s} < t_{Ⓑ}$ it is time efficient to switch to setup of mode $\boldsymbol{A}$. Hence, if (G.7) or (G.8) holds, the server should switch to setup of mode $\boldsymbol{A}$.

$$t_{Ⓑ} = \begin{cases} 0 & \text{if }, x_1 = 12 - \frac{3}{8}x_2, \\ \frac{12-x_1}{\lambda_1} = 4 - \frac{1}{3}x_1 & \text{if }, x_1 \neq 12 - \frac{3}{8}x_2. \end{cases} \tag{G.5}$$

$$t_{Ⓑ,s} = \begin{cases} \sigma_{🅐} & \text{if }, x_1 = 12 - 5x_2, \\ \sigma_{🅐} + \frac{4-\left(x_2+\sigma_{🅐}\lambda_2\right)}{\lambda_2} = 4 - x_2 & \text{if }, x_1 \neq 12 - 5x_2, \end{cases} \tag{G.6}$$

$$x_2 > \frac{1}{5}, \text{ for } x_1 \leq 15, x_2 = \frac{12}{5} - \frac{1}{5}x_1 \leq 32 - \frac{8}{3}x_1. \tag{G.7}$$

$$x_2 > \frac{1}{3}x_1, \text{ for } x_1 \leq 15, x_2 < \frac{12}{5} - \frac{1}{5}x_1 \leq 32 - \frac{8}{3}x_1. \tag{G.8}$$

G.2 Undefined Domain in Setup

The final part is when the server starts in setup at the first time instant. A visualisation of this domain part in 🅐 and 🅑, is presented in Figure G.3 respectively Figure G.4. The domain parts in 🅐 and 🅑, in which the control action is undefined are equal.
The same procedure as used when the server is in processing of a mode, is used if the server starts in setup. This results in the most time efficient control action, regarding the convergence time.
The time it takes to obtain the desired periodic behaviour if the server continues in mode $\boldsymbol{A}$, is referred to as $t_{\text{🅐}}$. If x_0 is exactly $\sigma_{\text{🅐}}$ and the values of x_1 and x_2 are on the boundary, $x_1 = 12 - 5x_2$, periodic behaviour is obtained when setup is finished. In all other cases, the system is in periodic behaviour when $x_2 = 4$. Which gives the expression for $t_{\text{🅐}}$, (G.9).
The time it takes to obtain the desired periodic behaviour in mode $\boldsymbol{A}$, if the server switches to setup in mode $\boldsymbol{B}$, is referred to as $t_{\text{🅐},s}$. Switching to 🅑 means the complete setup time needs to be executed. Hence, if the x_1 and x_2 are on the boundary, $x_2 = 5 - \frac{8}{3}x_1$, the system is in periodic behaviour immediately after setup. In all other cases periodic behaviour is obtained when $x_1 = 12$, which yields the expression for $t_{\text{🅐},s}$, (G.10).
If $t_{\text{🅐},s} < t_{\text{🅐}}$, it is time efficient to switch to setup of mode $\boldsymbol{B}$, else the server should continue in Ⓐ. Therefore, if (G.11) or (G.12) holds, the server should switch to setup of mode $\boldsymbol{B}$.

$$t_{\text{🅐}} = \begin{cases} 1 & \text{if } x_0 = 1, x_1 = 12 - 5x_2, \\ x_0 + \frac{4-(x_2+x_0\lambda_2)}{\lambda_2} = 4 - x_2 & \text{if } x_0 < 1, x_1 \leq 12 - 5x_2 \text{ or } x_0 = 1, x_1 < 12 - 5x_2. \end{cases} \tag{G.9}$$

$$t_{\text{🅐},s} = \begin{cases} \sigma_{\text{🅑}} & \text{if } x_2 = 5 - \frac{8}{3}x_1, \\ \sigma_{\text{🅑}} + \frac{12-\left(x_1+\sigma_{\text{🅑}}\lambda_1\right)}{\lambda_1} = 4 - \frac{1}{3}x_1 & x_2 \neq 5 - \frac{8}{3}x_1. \end{cases} \tag{G.10}$$

$$x_1 > 3x_2 \text{ for } x_1 \leq 15, x_2 < 5 - \frac{8}{3}x_1 \leq \frac{12}{5} - \frac{1}{5}x_1. \tag{G.11}$$

$$x_2 > 1, \text{ for } x_1 \leq 15, x_2 = 5 - \frac{8}{3}x_1 \leq \frac{12}{5} - \frac{1}{5}x_1. \tag{G.12}$$

Similarly a control action can be defined in the undetermined domain part in 🅑. Ihe part of the domain where control actions have yet to be determined is hashed in Figure G.2. The time it takes to obtain the desired periodic behaviour if the server continues in Ⓑ, is referred to as $t_{\text{🅑}}$. If $x_0 = 3$ and for x_1 and x_2 the expression $x_2 = 5 - \frac{8}{3}x_2$ holds, the system is in periodic behaviour immediately after setup. In all other situations the system is in periodic behaviour when $x_1 = 12$. This gives the expression for $t_{\text{🅑}}$, (G.13).
The travel time from the starting values to the optimal periodic behaviour, if the server switches

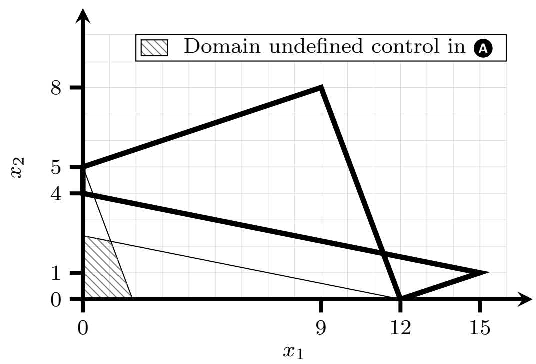

Figure G.3: $\dot{V}\left(\text{🅐}, x_1, x_2\right) = 0$.

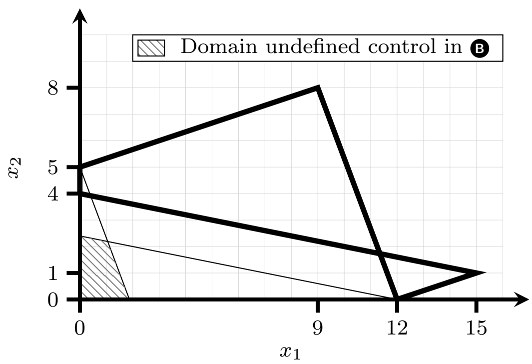

Figure G.4: $\dot{V}\left(\text{🅑}, x_1, x_2\right) = 0$.

to mode $\boldsymbol{A}$, is referred to as $t_{\textcircled{B},s}$, (G.14). In this case if $x_2 = 12 - 5x_2$ holds, the system is in periodic behaviour when setup is finished. For all other values of x_1 and x_2 the system is in periodic behaviour when $x_2 = 4$.
If $t_{\textcircled{B},s} < t_{\textcircled{B}}$ it is time efficient to switch to setup of mode $\boldsymbol{A}$. Hence, if (G.15), (G.16) or (G.17) holds, the server should switch to setup of mode $\boldsymbol{A}$.

$$t_{\textcircled{B}} = \begin{cases} 3 & \text{if } x_0 = 3, x_2 = 5 - \frac{8}{3}x_2, \\ x_0 + \frac{12-(x_1+x_0\lambda_1)}{\lambda_1} = 4 - \frac{1}{3}x_1 & \text{if } x_0 < 3, x_2 \leq 5 - \frac{8}{3}x_2 \text{ or } x_0 = 3, x_2 < 5 - \frac{8}{3}x_2. \end{cases} \tag{G.13}$$

$$t_{\textcircled{B},s} = \begin{cases} \sigma_{\textcircled{A}} & \text{if } x_2 = 12 - 5x_2, \\ \sigma_{\textcircled{A}} + \frac{4-\left(x_2+\sigma_{\textcircled{A}}\lambda_2\right)}{\lambda_2} = 4 - x_2 & x_2 \neq 12 - 5x_2. \end{cases} \tag{G.14}$$

$$3x_2 > 1, \text{ for } x_1 \leq 15, x_2 < 5 - \frac{8}{3}x_1 < \frac{12}{5} - \frac{1}{5}x_1. \tag{G.15}$$

$$x_2 < 1, \text{ for } x_1 \leq 15, x_2 = 5 - \frac{8}{3}x_1 \leq \frac{12}{5} - \frac{1}{5}x_1. \tag{G.16}$$

$$\text{for } x_1 = 12 - 5x_2, x_2 \leq 5 - \frac{8}{3}x_1 \leq \frac{12}{5} - \frac{1}{5}x_1. \tag{G.17}$$

Finding an explicit expression for the time efficient control action is already quite extensive in this two direction intersection example. When the number of flows is increased, determining control actions in the undefined areas of the domain by minimizing the convergence time becomes even more cumbersome. Therefore determining the control actions as such is not feasible.
The part of the domain is referred to as undefined because both possible control actions lead to the desired steady-state behaviour. This means it is a possibility for the server to continue in the current mode, it might only lead to an increase in the convergence time. The research objective is finding a control policy that is easy to implement, hence the latter solution is chosen in case the control action is undetermined.

Appendix H. Complex Control Actions

The control actions derived in Section 4.1 listed in (4.6), and (4.7), are extensive. The control actions are more complex than the ones presented in [1]. Although the candidate Lyapunov function is not defined in the entire domain in [1], this is not the source of the complex control action in the domain parts referred to as $\mathscr{D}^{\mathrm{V}}$ and $\mathscr{D}^{\mathrm{V}}$. These domain parts are taken into account in [1], the only difference is the definition of the candidate Lyapunov function in setup. This appendix is added to proof that some of the complexity of the control actions, is due to the definition taken as mean extra work in the system during setup.
In [1] the candidate Lyapunov function is defined as (H.1). Notice that some changes are made in the expression regarding the variables, to match the definitions used throughout this thesis.

$$V = \begin{cases} \frac{1}{8}\left(x_1 - 15 + 3x_0\right) + \frac{1}{9}\left(x_2 - 1 + x_0\right) & \text{for B}, \\ \frac{1}{8}\left(x_1 - 15\right) + \frac{1}{9}\left(x_2 - 1\right) & \text{for A}, x_1 \geq 15, \\ \min\left(\frac{1}{9}\left(\frac{1}{5}x_1 + x_2 - 4\right), \frac{1}{8}x_1 + \frac{1}{9}\left(x_2 - 5\right)\right) & \text{for A}, x_1 \leq 15, x_2 \geq 5, \\ \min\left(\frac{1}{9}\left(\frac{1}{5}x_1 + x_2 - 4\right), \frac{1}{8}x_1\right) & \text{for A}, x_1 \leq 15, x_2 \leq 5, \\ \frac{1}{8}\left(x_1 - 9 + 3x_0\right) + \frac{1}{9}\left(x_2 - 8 + x_0\right) & \text{for A}, \\ \frac{1}{8}\left(x_1 - 9\right) + \frac{1}{9}\left(x_2 - 8\right) & \text{for B}, x_2 \geq 8, \\ \frac{1}{8}\left(x_1 + \frac{3}{8}x_2 - 12\right) & \text{for B}, x_2 \leq 8. \end{cases} \tag{H.1}$$

In mode $\boldsymbol{A}$ the candidate Lyapunov function derivative at switching instants is based on the definition of the derivative used throughout this thesis, (H.2). This equation uses the definition of the candidate Lyapunov function in setup. The candidate Lyapunov function is now redefined to match (H.1), this expression is taken to compute $\dot{V}\left(s, x_1, x_2\right)$ in mode $\boldsymbol{A}$ in case the server switches to setup of mode $\boldsymbol{B}$. The definition of the candidate Lyapunov function (H.1) is equal to (3.23) for corresponding parts of the domain, apart from the cases where $x_0 > 0$. Meaning the candidate Lyapunov derivative if the server continues in its current mode is equal and can be found in (4.3).

$$\lim_{\varepsilon \to 0} \frac{V\left(\text{B}, 3 - \varepsilon, x_1\left(t + \varepsilon\right), x_2\left(t + \varepsilon\right)\right) - V\left(\text{A}, 0, x_1\left(t\right), x_2\left(t\right)\right)}{\varepsilon}. \tag{H.2}$$

$$\mathscr{D}^{\mathrm{I}}, \mathscr{D}^{\mathrm{II}} : \quad \frac{x_1+\lambda_1\varepsilon+-9+3(3-\varepsilon)}{\mu_1\varepsilon} + \frac{x_2+\lambda_2\varepsilon-8+3-\varepsilon}{\mu_2\varepsilon} - \frac{x_1-15}{\mu_1\varepsilon} - \frac{x_2-1}{\mu_2\varepsilon} = \frac{167}{72\varepsilon} \to \infty, \text{ if } \varepsilon \to 0 \ , \quad \text{for A}, \quad x_1 \geq 15. \tag{H.3}$$

$$\mathscr{D}^{\mathrm{III_a}} : \quad \frac{x_1+\lambda_1\varepsilon+-9+3(3-\varepsilon)}{\mu_1\varepsilon} + \frac{x_2+\lambda_2\varepsilon-8+3-\varepsilon}{\mu_2\varepsilon} - \frac{\frac{1}{5}x_1+x_2-4}{\mu_2\varepsilon} = \frac{\frac{37}{5}x_1-8}{72\varepsilon} \to \infty, \text{ if } \varepsilon \to 0 \ , \quad \text{for A}, \quad \frac{40}{37} < x_1 \leq 15, \quad x_2 \geq 5. \tag{H.4}$$

$$\mathscr{D}^{\mathrm{IV}} : \quad \frac{x_1+\lambda_1\varepsilon+-9+3(3-\varepsilon)}{\mu_1\varepsilon} + \frac{x_2+\lambda_2\varepsilon-8+3-\varepsilon}{\mu_2\varepsilon} - \frac{x_1}{\mu_1\varepsilon} - \frac{x_2-5}{\mu_2\varepsilon} = 0 \ , \quad \text{for A}, \quad x_1 \leq \frac{40}{37}, \quad x_2 \geq 5. \tag{H.5}$$

$$\mathscr{D}^{\mathrm{III_b}} : \quad \frac{x_1+\lambda_1\varepsilon+-9+3(3-\varepsilon)}{\mu_1\varepsilon} + \frac{x_2+\lambda_2\varepsilon-8+3-\varepsilon}{\mu_2\varepsilon} - \frac{\frac{1}{5}x_1+x_2-4}{\mu_2\varepsilon} = \frac{\frac{37}{5}x_1-8}{72\varepsilon} \to \infty, \text{ if } \varepsilon \to 0 \ , \quad \text{for A}, \quad x_1 \leq 15, \quad 4 - \frac{1}{5}x_1 \leq x_2 \leq \frac{37}{40}x_1 + 4 < 5. \tag{H.6}$$

$$\mathscr{D}^{\mathrm{V}} : \quad \frac{x_1+\lambda_1\varepsilon+-9+3(3-\varepsilon)}{\mu_1\varepsilon} + \frac{x_2+\lambda_2\varepsilon-8+3-\varepsilon}{\mu_2\varepsilon} - \frac{x_1}{\mu_1\varepsilon} = \frac{x_2-5}{9\varepsilon} \to -\infty, \text{ if } \varepsilon \to 0 \ , \quad \text{for A}, \quad x_1 \leq 15, \quad 4 - \frac{1}{5}x_1 < \frac{37}{40}x_1 + 4 < x_2 < 5. \tag{H.7}$$

Due to the fact that there is only one expression for the candidate Lyapunov function if the system is in setup, the number of derived expressions decreased. Furthermore, the results are less complex as can easily be concluded from the derivation presented above. This yields the candidate Lyapunov function derivative in case the server switches to setup, (H.8). Now comparing these results to the corresponding domain parts in (3.23), the control actions if the server is in Ⓐ are determined. This results in expression (H.9).

$$\dot{V} = \begin{cases} \infty & \text{for Ⓐ}, x_1 \geq 15, \\ \infty & \text{for Ⓐ}, \frac{40}{37} < x_1 \leq 15, x_2 \geq 5, \\ 0 & \text{for Ⓐ}, x_1 \leq \frac{40}{37}, x_2 \geq 5, \\ \infty & \text{for Ⓐ}, x_1 \leq 15, 4 - \frac{1}{5}x_1 \leq x_2 \leq \frac{37}{40}x_1 + 4 < 5, \\ -\infty & \text{for Ⓐ}, x_1 \leq 15, 4 - \frac{1}{5}x_1 < \frac{37}{40}x_1 + 4 < x_2 < 5. \end{cases} \tag{H.8}$$

$$(u_0, u_1, u_2) = \begin{cases} (\text{Ⓐ}, \mu_1, 0) & \text{if } m = \boldsymbol{A}, x_0 = 0, x_1 > 15, \\ (\text{Ⓐ}, \mu_1, 0) & \text{if } m = \boldsymbol{A}, x_0 = 0, 0 < x_1 \leq 15, x_2 \geq 5, \\ (\text{Ⓑ}, 0, 0) & \text{if } m = \boldsymbol{A}, x_0 = 0, x_1 = 0, x_2 \geq 5, \\ (\text{Ⓐ}, \mu_1, 0) & \text{if } m = \boldsymbol{A}, x_0 = 0, 0 < x_1 \leq 15, 4 - \frac{1}{5}x_1 \leq x_2 \leq \frac{37}{40}x_1 + 4 < 5, \\ (\text{Ⓐ}, \lambda_1, 0) & \text{if } m = \boldsymbol{A}, x_0 = 0, x_1 = 0, 4 - \frac{1}{5}x_1 \leq x_2 \leq \frac{37}{40}x_1 + 4 < 5, \\ (\text{Ⓑ}, 0, 0) & \text{if } m = \boldsymbol{A}, x_0 = 0, x_1 \leq 15, 4 - \frac{1}{5}x_1 < \frac{37}{40}x_1 + 4 < x_2 < 5. \end{cases} \tag{H.9}$$

The results presented in this appendix show that a different definition of the candidate Lyapunov function in setup results in different control actions to be performed by the server. A similar derivation could be given for mode $\boldsymbol{B}$, however this is straightforward from the results presented on mode $\boldsymbol{A}$ and therefore omitted from this appendix.

A decrease in complexity showed, a switch to setup in $\mathscr{D}^{\mathrm{III_b}}$ does not show. However, the setup in the domain part referred to as $\mathscr{D}^{\mathrm{V}}$ is (although simplified) still required based on these derivations. This does not show in the controller presented in [1]. Therefore, it is concluded that the definition of the candidate Lyapunov in setup is the source of the vast majority of the complex control actions, but it is not the source of all complexity in the control action expressions. Furthermore, it should be noted that the definition for the candidate Lyapunov function in setup used in [1] significantly shortens all derivations. This might be useful if the convergence speed is of less importance.

Appendix I. Intersection Data

This appendix lists the data used in the case studies of Section 5.4 and includes the computation of the buffer contents in optimal behaviour.

I.1 A2N279 Intersection

The intersection consists of six flows, similar to the previously discussed example. With the exception that in this case the data is not chosen such that the example results are simple and straightforward. Although technically the number of vehicles is an integer number, in case the control policy is derived the arrival rates are assumed to be constant which combined with the fixed time schedule effective red and green times might result in real numbers. These resulting numbers are not rounded to integers, it is assumed that this results in the most accurate control policy. The arrival and process rates corresponding the intersection flows are listed in Table I.1. The fixed time schedule is presented in Figure I.1. An example that explains in detail how the optimal buffer contents are computed is given in Section I.2.

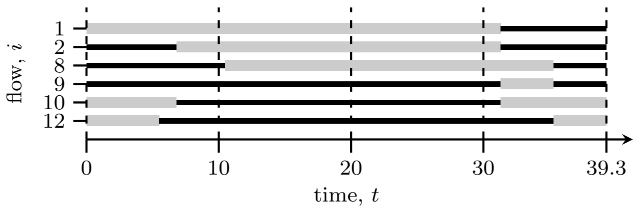

Figure I.1: Fixed time schedule, A2N279 intersection

Table I.1: Arrival and process rates of A2N279 intersection

i	$3600\lambda_i$	$3600\mu_i$
1	2254	3230
2	1189	3800
8	2467	5700
9	127	1805
10	434	1615
12	643	3610

I.2 's Gravendijkwal Intersection

The intersection is considered a large intersection, it consists of 29 flows of which the fixed time schedule is given in Figure I.2. The data corresponding to the flows, the arrival and process rates are listed in Table I.2. The computation of the optimal buffer content based on the flow data is presented in this section and the effective red and green times given in the fixed time schedule. The optimal buffer content is shown in six figures opposed to one in the previous discussed examples, for ease of reading.

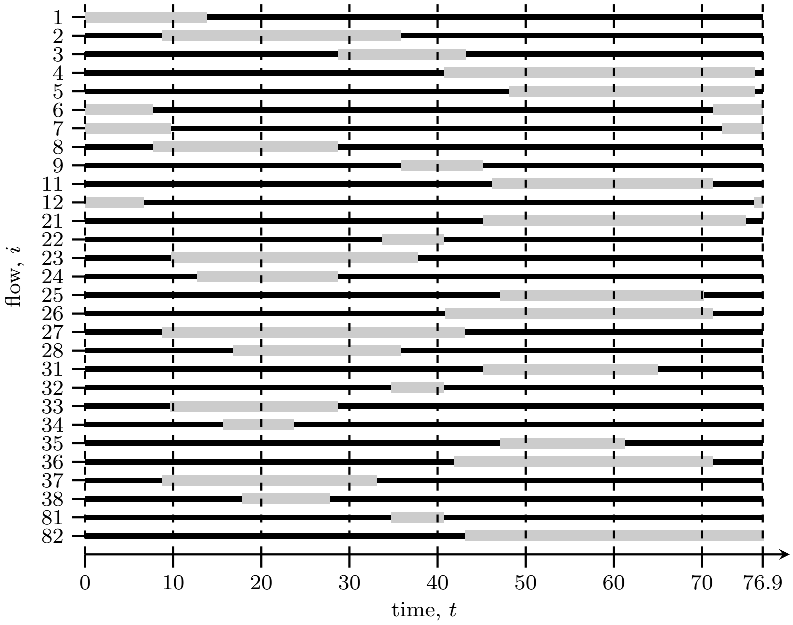

Figure I.2: Fixed time schedule, 's Gravendijkwal intersection

Assuming the buffer is empty when the red time of a flow starts, the content of buffer i increases with λ_i during red time. When buffer i is served, x_i is processed at rate μ_i, until $x_i = 0$ then buffer i is served at rate λ_i. For instance, the red time of flow 1 during a period equals $76.9 - 13.8 = 63.1$ time units. The green time of flow 1 equals 13.8 time units. So after red time, $x_1 = \lambda_1 * 63.1 = \frac{250}{3600} * 63.1 = 4.3$, during green time the x_1 is processed at $\mu_1 - \lambda_1$ which means that the buffer is emptied in $\frac{4.3*3600}{1560-250} = 12$ time units. Buffer 1 then proceeds with a slow mode, in which $x_1 = 0$ for 2.8 time units. Computing these values for each flow in the system gives the optimal buffer contents as function of time, which is given in Figure I.3 to Figure I.6. Although 4.3 vehicles are impossible, it is assumed to be accurate to derive a control policy as the constant rates are an assumption it is noted that in practice this situation will not occur.
The conflict matrix Σ is large for the 's Gravendijkwal intersection, therefore the conflict matrix is presented in a condensed form in Table I.3.

Table I.2: Arrival and process rates of ’s Gravendijkwal intersection

i	$3600\lambda_i$	$3600\mu_i$
1	250	1560
2	400	1900
3	500	3800
4	300	1700
5	1200	3800
6	250	1900
7	400	3400
8	400	2000
9	200	1900
11	1200	5360
12	100	1800
21	14	5000
22	14	5000
23	8	5000
24	8	5000
25	10	5000
26	10	5000
27	10	5000
28	10	5000
31	50	9999
32	50	9999
33	50	9999
34	50	9999
35	50	9999
36	50	9999
37	50	9999
38	50	9999
81	40	5000
82	40	5000

Table I.3: Conflicts and clearance time, ’s Gravendijkwal Intersection

i	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*11*	*12*	*21*	*22*	*23*	*24*	*25*	*26*	*27*	*28*	*31*	*32*	*33*	*34*	*35*	*36*	*37*	*38*	*81*	*82*
1					0				0			0							3	2							4		2
2					0	0			0	2	1	0					5			1					6				0
3					1	2	4	3		3	1	2			7					2			8						0
4								0			0		4	0							6	2						5	
5	1	1	0					0	0		0			0					4			0					5		
6		1	0					0	1	4				2			8					2			9				
7			0							0					3	1							6	2					
8			0	2	1	1				0	0		5			0					6			0				6	
9	5	4			3	0				1	3					2			8					2			9		
11		0	0			0	1	0	0						3			0					5			2			
12		0	1	3	3			1	0				7					2			8					2		7	
21	2	2	1																										
22				0				0			0																		
23				3	2	0																							
24			0				0			0																			
25							2	1	0																				
26		0				0																							
27										3	0																		
28	0				0				0																				
31	12	8	12																										
32				0				0			0																		
33				12	8	12																							
34			5				8			6																			
35							11	7	11																				
36		0				0																							
37										13	13																		
38	11				8				8																				
81				0				0			0																		
82	0	1	2																										

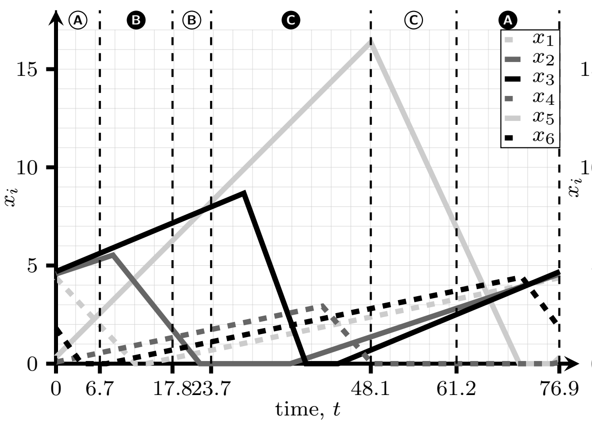

Figure I.3: Optimal buffer contents, 1–6.

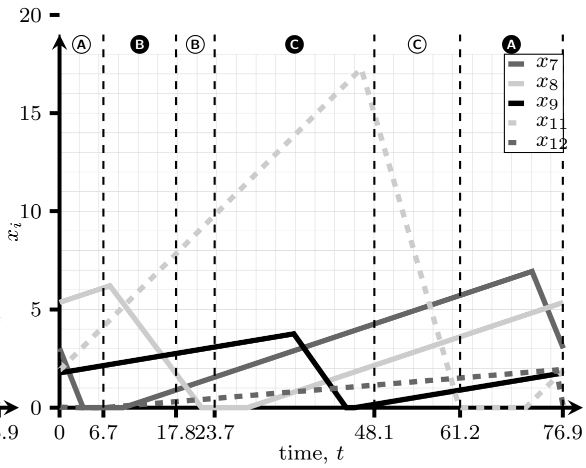

Figure I.4: Optimal buffer contents, 7–9,11–12.

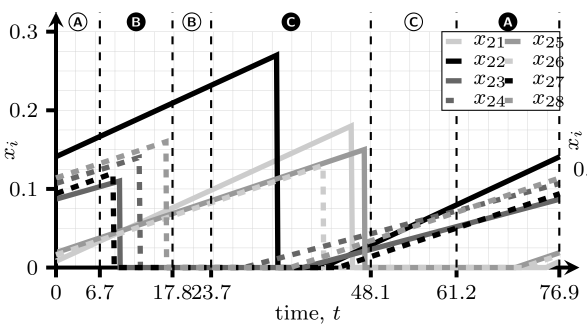

Figure I.5: Optimal buffer contents, 21–28.

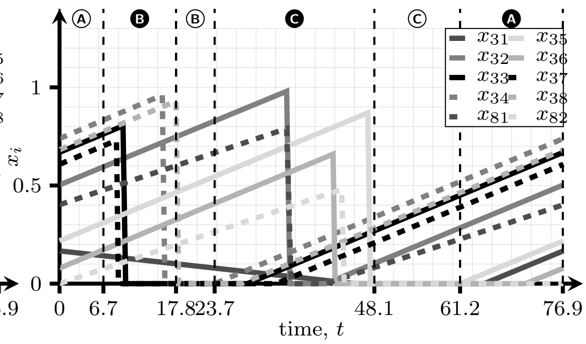

Figure I.6: Optimal buffer contents, 31–38,81–82.

Table I.4: Active Conflicts 's Gravendijkwal Intersection

i	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*11*	*12*	*21*	*22*	*23*	*24*	*25*	*26*	*27*	*28*	*31*	*32*	*33*	*34*	*35*	*36*	*37*	*38*	*81*	*82*
1					0				0			0							0	0							1		0
2					0	0			1	0	0	0					1			0					**1**				0
3					0	0	0	0		**1**	0	1			0					1			0						1
4								0			1		0	0							0	0						0	
5	**1**	0	0					0	0		1			0					0			0					0		
6		1	0					1	0	0				**1**			0					**1**			0				
7			0							0					1	0							**1**	0					
8			1	0	0	0				0	0		1			0					**1**			0				**1**	
9	0	0			**1**	0				1	0					1			0					1			0		
11		0	0			1	**1**	0	0						0			0					0			0			
12		0	0	0	0			1	0				0					**1**			0					**1**		0	
21	**1**	0	0																										
22				**1**				0			0																		
23				**1**	0	0																							
24			**1**				0			0																			
25							**1**	0	0																				
26		0				**1**																							
27										**1**	0																		
28	0				0				**1**																				
31	**1**	0	0																										
32				**1**				0			0																		
33				**1**	0	0																							
34			**1**				0			0																			
35							**1**	0	0																				
36		0				**1**																							
37										**1**	0																		
38	0				0				**1**																				
81				**1**				0			0																		
82	**1**	0	0																										