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Stability Analysis for Fluid Limit Models of Multiclass Queueing Networks

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Where innovation starts

Acknowledgment

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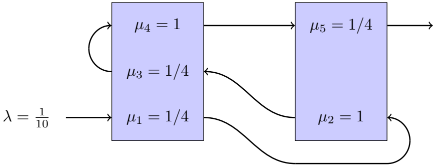
Contribution

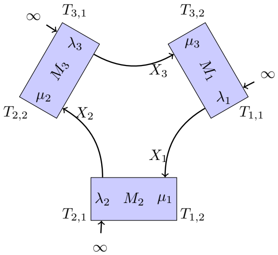
We present a method (finite time algorithm) for describing solutions of a fluid limit model as differential inclusion.
This leads to a graph that can be used for analyzing stability of the fluid limit model.

Multiclass queueing networks

Dai,Hasenbein,Vande Vate (2004)



Push-pull ring



- Head-of-the-line (HL)
- Work conserving (non-idling)
- Service of a class can be prohibited depending on the (non-)presence of customers of certain classes, e.g. Static Buffer Priority discipline (SBP)

Introduction

Key result: Dai (1995)

Consider a HL queueing network under some given policy. Assume that the associated fluid model for the network is stable. Then under certain technical assumptions the queueing network is stable.

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Our problem of interest

When is an associated fluid model stable?

Problem

Consider the following set of signals

$$\mathcal{B} = \left\{ \begin{bmatrix} X(t) \\ T(t) \end{bmatrix} \;\middle|\; \begin{array}{cc} 0 \leq X(t) = X(0) + \alpha t + FT(t) & T(0) = 0 \\ T(t) \text{ non-decreasing} & G[T(t) - T(s)] \leq \beta(t-s) \\ 0 = \int_0^t X_i(s)\,\mathrm{d}\,T_j(s) & \end{array} \right\}$$

When does it hold that all signals $X(t) \in \mathcal{B}$ converge to 0 in finite time?

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When does it hold that all signals $X(t) \in \mathcal{B}$ converge to 0 in finite time?

Some remarks

- Think of $T(t)$ here as $[T(t)', Y(t)']'$ or $[T(t)', T^+(t)']'$
- Think of F as $[R'|0]'$ with input-output-matrix $R = (I-P)^{-1}\text{diag}(\mu)$
- G used for modeling constituency, as well as equality constraints

Problem

Consider the following set of signals

$$\mathcal{B} = \left\{ \begin{bmatrix} X(t) \\ T(t) \end{bmatrix} \middle| \begin{array}{l} 0 \le X(t) = X(0) + \alpha t + FT(t) \qquad T(0) = 0 \\ T(t) \text{ non-decreasing} \qquad G[T(t) - T(s)] \le \beta(t-s) \\ 0 = \int_0^t X_i(s)\, \mathrm{d}\, T_j(s) \end{array} \right\}$$

When does it hold that all signals $X(t) \in \mathcal{B}$ converge to 0 in finite time?

Additional assumptions

- $X(t)$ piecewise linear on countable partition of intervals
- rank conditions involving α, β, F, and G

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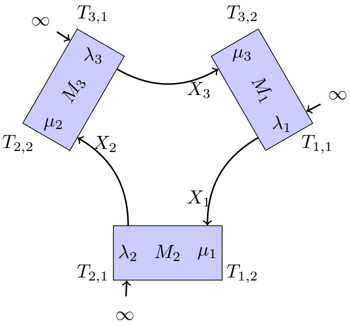
Example 1: Push-pull ring

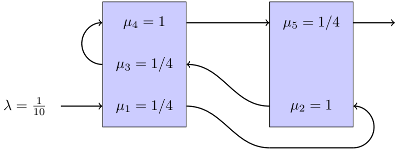
See also Weiss et al. (Session 3.11, yesterday)

$$
\begin{aligned}
X_i(t) &= X_i(0) + \lambda_i T_{i,1}(t) - \mu_i T_{i,2}(t)\\
t &= T_{i,1}(t) + T_{i-1,2}(t)\\
0 &= \int_0^t X_i(s)\,\mathrm{d}\,T_{i+1,1}(s)\\
0 &\leq X_i(t)\\
T_{i,j}(t) &\ \text{non-decreasing}\\
T_{i,j}(0) &= 0
\end{aligned}
$$



Examples

Example 2: Dai, Hasenbein, Vande Vate (2004)



$$
\begin{aligned}
X_1(t) &= X_1(0) + \lambda t - \mu_1 T_1(t) \\
X_i(t) &= X_i(0) + \mu_{i-1} T_{i-1}(t) - \mu_i T_i(t) \\
T_1^+(t) &= t - T_1(t) \\
T_3^+(t) &= t - T_1(t) - T_3(t) \\
T_4^+(t) &= t - T_1(t) - T_3(t) - T_4(t) \\
T_5^+(t) &= t - T_5(t) \\
T_2^+(t) &= t - T_5(t) - T_2(t) \\
& T_i(t), T_i^+(t) \text{ non-decreasing}
\end{aligned}
$$

$$
\begin{aligned}
0 &= T_i(0) = T_i^+(0) \\
0 &\le X_i(t) \\
0 &= \int_0^t X_1(s)\, \mathrm{d}\, T_1^+(s) \\
0 &= \int_0^t (X_1 + X_3)(s)\, \mathrm{d}\, T_3^+(s) \\
0 &= \int_0^t (X_1 + X_3 + X_4)(s)\, \mathrm{d}\, T_4^+(s) \\
0 &= \int_0^t X_5(s)\, \mathrm{d}\, T_5^+(s) \\
0 &= \int_0^t (X_2 + X_5)(s)\, \mathrm{d}\, T_2^+(s)
\end{aligned}
$$

Preliminaries

Some standard observations

- For $s \leq t$: $0 \leq T_i(t) - T_i(s) \leq t - s$, so solutions in $\mathcal{B}$ are Lipschitz continuous
- In particular they are absolutely continuous
- Therefore differentiable almost everywhere

Definition

Points t where all time derivatives exist are called regular points.

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Definition

Points t where all time derivatives exist are called regular points.

Remark

Since $X(t)$ piecewise linear on countable union of intervals, we can define derivatives at non-regular points by taking limits from the right.

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Approach

- Rewrite $X(t) \in \mathcal{B}$ as a differential inclusion:

$$\dot{X}(t) \in S_{X(t)} \subset \mathcal{S} \tag{1}$$

where $S_{X(t)}$ denotes set, depending on $X(t)$ and $\mathcal{S}$ is a finite set.

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  - Dynamics for regular points
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  - Dynamics for regular points
  - Dynamics for non-regular points
- Derive graph with possible transitions
- Stability analysis of the differential inclusion (1) by means of the graph

Mode-dynamics for regular points

Partition state space into regions

Define $L(t) = (1_{\{X_1(t)>0\}}, \dots, 1_{\{X_n(t)>0\}}) \in \{0,1\}^n$.
We refer to $L(t) = (\ell_1, \dots, \ell_n)$ as the mode of the system at time t.

Goal

Derive mode-dynamics for regular points (i.e. regular modes).

Definition

Region is union of (regular) modes with same dynamics.

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Mode-dynamics for regular points

Example 1: Push-pull ring

Recall equations

$$\dot{X}_i(t) = \lambda_i \dot{T}_{i,1}(t) - \mu_i \dot{T}_{i,2}(t) \qquad 0 = X_i(t)\dot{T}_{i+1,1}(t)$$
$$1 = \dot{T}_{i,1}(t) + \dot{T}_{i-1,2}(t) \qquad 0 \leq \dot{T}_{i,j}(t), X_i(t)$$

During mode: two cases

$$X_i(t) > 0 : \quad \dot{T}_{i+1,1}(t) = 0$$
$$X_i(t) = 0, \text{ i.e. } \dot{X}_i(t) = 0 : \quad \lambda_i \dot{T}_{i,1}(t) - \mu_i \dot{T}_{i,2}(t) = 0$$

For each mode: 6 linear equations with 6 unknown $\dot{T}_{i,j}(t)$.

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For each mode: 6 linear equations with 6 unknown $\dot{T}_{i,j}(t)$.

Solution needs to satisfy $0 \leq \dot{T}_{i,j}(t)$ for mode to be regular.

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Mode-dynamics for regular points

Example 1: Push-pull ring ($\lambda_i > \mu_i$)

Regular modes (5):

$L(t) = (1,1,1)$: $\dot{X}(t) = [-\mu_1, -\mu_2, -\mu_3]'$

$L(t) = (0,1,1)$: $\dot{X}(t) = [0, \lambda_2 - \mu_2, -\mu_3]'$

$L(t) = (1,0,1)$: $\dot{X}(t) = [-\mu_1, 0, \lambda_3 - \mu_3]'$

$L(t) = (1,1,0)$: $\dot{X}(t) = [\lambda_1 - \mu_1, -\mu_2, 0]'$

$L(t) = (0,0,0)$: $\dot{X}(t) = [0, 0, 0]'$

Result: 5 possible directions of movement.

Non-regular modes (3):

$L(t) = (1,0,0)$

$L(t) = (0,1,0)$

$L(t) = (0,0,1)$

Mode-dynamics for regular points

Example 2: Dai, Hasenbein, Vande Vate (2004)

Along the same lines we obtain

- 16 regular modes
- 16 non-regular modes

Some modes have same direction of movement.

Result: 11 possible directions of movement.

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Mode-dynamics for regular points

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- 16 non-regular modes

Some modes have same direction of movement.

Result: 11 possible directions of movement.

Remark

Mode $L(t) = (0, 0, 0, 1, 0)$ is regular: $\dot{X}(t) = \left(0, 0, 0, -\frac{1}{10}, 0\right)$.

Mode-dynamics for regular points

Two problems

- Dynamics for non-regular modes?
- Non-unique direction of movement is a challenge

Next step

Need to determine dynamics for non-regular points.

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Some observations

- So far, two options considered:
  - $X_i(t) > 0$
  - $X_i(t) = 0$ and $\dot{X}_i(t) = 0$

  For mode-dynamics in regular points this suffices.

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  For mode-dynamics in regular points this suffices.
- For non-regular points, a third case needs to be considered:
  - $X_i(t) = 0$ and $\dot{X}_i(t) > 0$

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Dynamics for non-regular points

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- So far, two options considered:
  - $X_i(t) > 0$
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  For mode-dynamics in regular points this suffices.
- For non-regular points, a third case needs to be considered:
  - $X_i(t) = 0$ and $\dot{X}_i(t) > 0$
- Extra condition: $X_i(t)\dot{T}_j(t) = 0$ implies $\dot{X}_i(t)\dot{T}_j(t) = 0$

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$$1 = \dot{T}_{i,1}(t) + \dot{T}_{i-1,2}(t) \qquad 0 = \dot{X}_i(t)\dot{T}_{i+1,1}(t)$$
$$0 \leq \dot{T}_{i,j}(t) \qquad 0 \leq X_i(t)$$

For each of the buffers consider three cases

$$X_i(t) > 0 : \quad \dot{T}_{i+1,1}(t) = 0$$
$$X_i(t) = 0 \text{ and } \dot{X}_i(t) = 0 : \quad \lambda_i \dot{T}_{i,1}(t) - \mu_i \dot{T}_{i,2}(t) = 0$$
$$X_i(t) = 0 \text{ and } \dot{X}_i(t) > 0 : \quad \dot{T}_{i+1,1}(t) = 0$$

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Dynamics for non-regular points

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Solution needs to satisfy $\dot{T}_{i,j}(t) \geq 0$ and case conditions for feasibility.

Dynamics for non-regular points

Example 1: Push-pull ring ($\lambda_i > \mu_i$)

$L = (0, \cdot, 1)$: $\dot{X} = (0, \lambda_2 - \mu_2, -\mu_3)'$

$L = (\cdot, 1, 0)$: $\dot{X} = (\lambda_1 - \mu_1, -\mu_2, 0)'$

$L = (1, 0, \cdot)$: $\dot{X} = (-\mu_1, 0, \lambda_3 - \mu_3)'$

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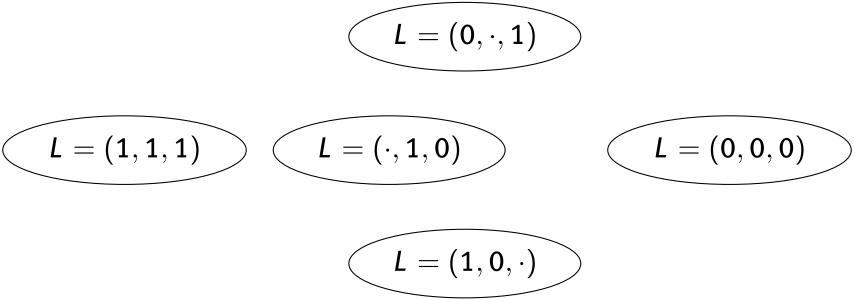

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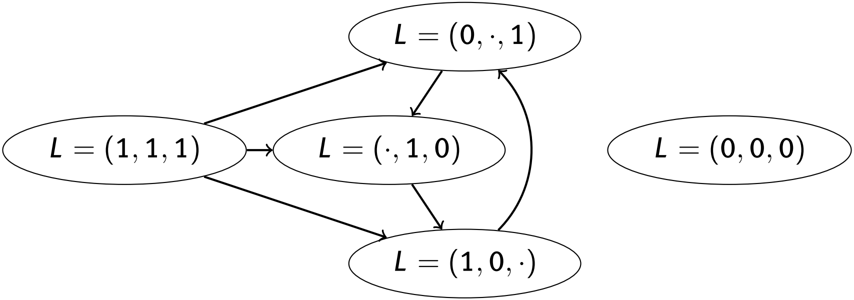

Dynamics for non-regular points

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$L = (0, \cdot, 1)$: $\dot{X} = (0, \lambda_2 - \mu_2, -\mu_3)'$ $L = (1, 1, 1)$: $\dot{X} = (-\mu_1, -\mu_2, -\mu_3)'$

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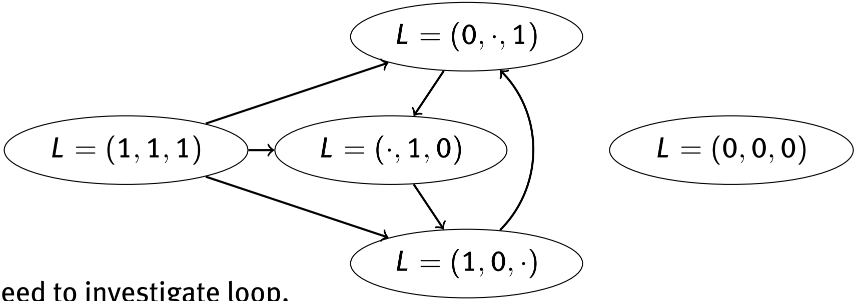


Need to investigate loop.

Dynamics for non-regular points

Example 1: Push-pull ring ($\lambda_i > \mu_i$)

Recall dynamics

$L(t) = (0, \cdot, 1)$: $\dot{X} = (0, \lambda_2 - \mu_2, -\mu_3)'$

$L(t) = (\cdot, 1, 0)$: $\dot{X} = (\lambda_1 - \mu_1, -\mu_2, 0)'$

$L(t) = (1, 0, \cdot)$: $\dot{X} = (-\mu_1, 0, \lambda_3 - \mu_3)'$

Consider Lyapunov function (define $\rho_i = \lambda_i/\mu_i$)

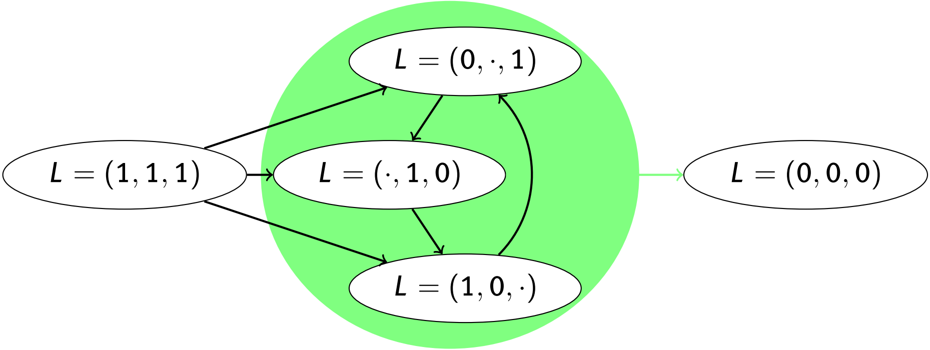
$$V = [1 + \rho_2(\rho_3 - 1)]\frac{x_1}{\mu_1} + [1 + \rho_3(\rho_1 - 1)]\frac{x_2}{\mu_2} + [1 + \rho_1(\rho_2 - 1)]\frac{x_3}{\mu_3}$$

Along any of the three modes we obtain:

$$\dot{V} = \prod_{i=1}^{3}(\rho_i - 1) - 1$$

Example 1: Push-pull ring ($\lambda_i > \mu_i$)

Resulting graph for $\prod_{i=1}^{3}(\rho_i - 1) < 1$:



For $\prod_{i=1}^{3}(\rho_i - 1) > 1$ we have instability.

Dynamics for non-regular points

Example 2: Dai, Hasenbein, Vande Vate (2004)

Resulting dynamics

1: $L(t) = (1, \cdot, \cdot, \cdot, 1)$: $\dot{X} = [-3/20, 1/4, 0, 0, -1/4]'$

2: $L(t) = (0, \cdot, 1, \cdot, 1)$: $\dot{X} = [0, 1/10, -3/20, 3/20, -1/4]'$

3: $L(t) = (0, 1, 0, 1, 0)$: $\dot{X} \in S_{(0,1,0,1,0)}$

4: $L(t) = (0, \cdot, 0, 1, 1)$: $\dot{X} = [0, 1/10, 0, -3/5, 7/20]'$

5: $L(t) = (0, \cdot, 0, 0, 1)$: $\dot{X} = [0, 1/10, 0, 0, -1/4]'$

6: $L(t) = (1, 1, \cdot, \cdot, 0)$: $\dot{X} = [-3/20, -3/4, 1, 0, 0]'$

7: $L(t) \in \{(0, 1, 1, \cdot, 0), (0, 1, \cdot, 0, 0)\}$: $\dot{X} = [0, -9/10, 17/20, 3/20, 0]'$

8: $L(t) = (1, 0, \cdot, \cdot, 0)$: $\dot{X} = [-3/20, 0, 1/4, 0, 0]'$

9: $L(t) = (0, 0, 1, \cdot, 0)$: $\dot{X} = [0, 0, -1/20, 3/20, 0]'$

10: $L(t) = (0, 0, 0, 1, 0)$: $\dot{X} \in S_{(0,0,0,1,0)}$

11: $L(t) = (0, 0, 0, 0, 0)$: $\dot{X} = [0, 0, 0, 0, 0]'$

Dynamics for non-regular points

Example 2: Dai, Hasenbein, Vande Vate (2004)

Two interesting modes:

3:$L(t) = (0, 1, 0, 1, 0)$:

$$\dot{X}(t) \in \left\{ [0, -\frac{9}{10}, \frac{17}{20}, \frac{3}{20}, 0]', [0, \frac{1}{150}, 0, -\frac{2}{15}, 0]', [0, \frac{1}{10}, 0, -\frac{3}{5}, \frac{7}{20}]' \right\}$$

10:$L(t) = (0, 0, 0, 1, 0)$:

$$\dot{X}(t) \in \left\{ [0, 0, 0, -\frac{1}{10}, 0]', [0, \frac{1}{150}, 0, -\frac{2}{15}, 0]', [0, \frac{1}{10}, 0, -\frac{3}{5}, \frac{7}{20}]' \right\}$$

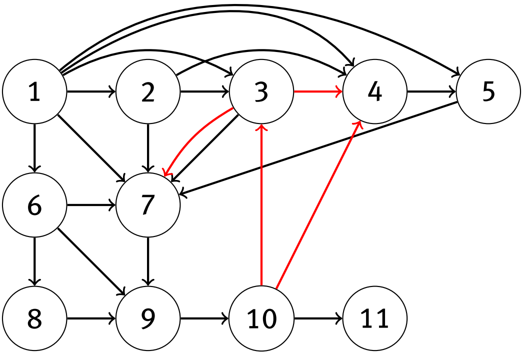
Remark

Notice: for mode 10 not two possible trajectories, but three.

Dynamics for non-regular points

Example 2: Dai, Hasenbein, Vande Vate (2004)

Resulting graph:



Need to investigate loops (3-)4-5-7-9-10: Unstable

Obtaining stable dynamics

Assume that $\mathcal{B}$ contains both stable and unstable trajectories. Can we remove the unstable trajectories?

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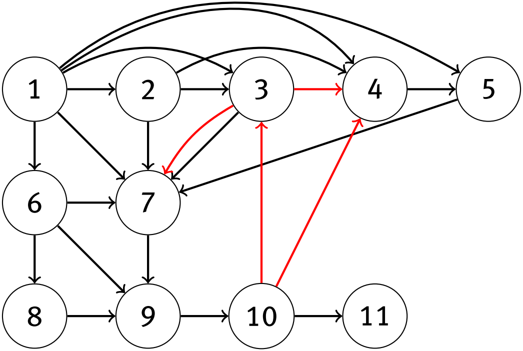

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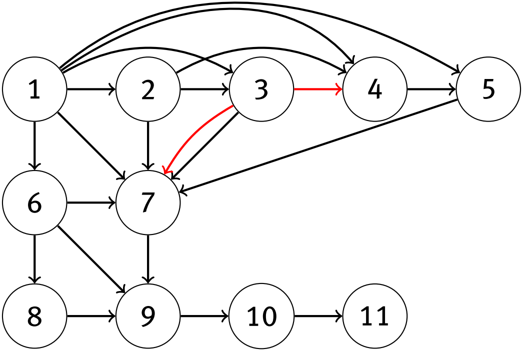

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Obtaining stable dynamics

Modified policy:
Machine B starts a job of type two whenever both $x_3 = 0$ and $x_2 > 0$.

Obtaining stable dynamics

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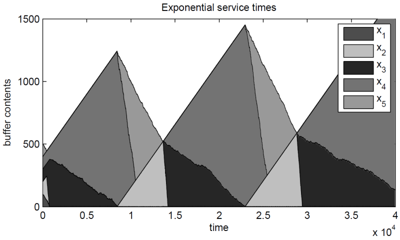
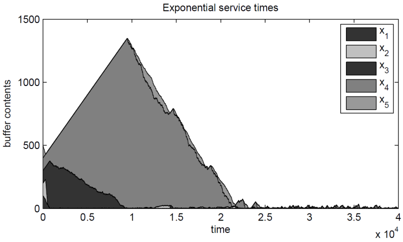
Machine B starts a job of type two whenever both $x_3 = 0$ and $x_2 > 0$.

Illustration by simulation

Original SBP policy



Modified policy



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- Unstable solutions can be eliminated by modifying policy (on set of measure zero)